https://doi.org/10.15407/ujpe69.4.247

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# LOW-ENERGY n–p AND n–d SCATTERINGS WITH THE DENG–FAN POTENTIAL

*In any first approach toward a nuclear structure problem, one presumes the nucleons to be elementary particles. The failure or success of this approach may then instruct us something about the significance of sub-nuclear degrees of freedom. The Deng–Fan potential, although extensively used in molecular dynamics to reproduce several observables for the atomic-atomic and atomic-molecular interactions, is parametrized for nuclear systems to fit low-energy observables. By exploiting the variable phase approach (VPA) to potential scattering, phase parameters, cross-sections and analyzing powers are estimated for the nucleon–nucleon and nucleon–nucleus systems. Our results show good concurrence with the earlier theoretical and experimental data within this simple model of interaction.*

*Keywords*: Deng–Fan potential, variable phase approach, scattering phase parameters, cross-section, analyzing power, n–p and n–d systems.

## 1. Introduction

In the domain of non-relativistic quantum scattering theory [1], understanding the energy spectra and wave functions of a quantum system under various potentials is an interesting subject, as one can gather all the necessary information about the system under consideration. In 1957, Deng and Fan [2] proposed a diatomic potential to define the molecular vibrational spectrum. The Deng–Fan potential is a generalized Morse potential (GMP) [3], as it satisfies the correct physical boundary conditions at $r = 0$ and $r = \infty$ what the generic Morse potential fails to do. Thus, the Deng–Fan potential is consistent with the quantum needs and can be a good choice for studying quantum physical systems. Many authors have investigated this potential via different quantum mechanical wave equations [2–11] by utilizing several standard approximation prescriptions to the solution in both relativistic and non-relativistic domains. Refs. [2–3] studied this potential for the S-wave case. This potential was considered by Hassanabadi *et al.* [4] who obtained an ansatz for a quantum mechanical solution applying a Pekeris-type approximation. Dong [5] used a proper approximation to the centrifugal term, and Oluwadare [6] applied the Nikiforov–Uvarov method for solving the Klein–Gordon equation. Yazarloo [7] and Dong [8] have obtained approximate bound and scattering state solutions to the Schrödinger equation in all partial waves with approximation to the centrifugal term for the potential under consideration. The Deng–Fan potential is a multiparameter exponential-type potential. Various other exponential-type potentials have been treated for their approximate analytic solutions in a number of publications [12–25].

Thus, for theoretical physicists, the Deng–Fan potential is already an interesting choice in the context of its application to the molecular dynamics [2–11]. However, we will apply this potential for the two-particle nuclear scattering with judicious exploitation of the variable phase approach (VPA) [26]. Several authors have computed quantum mechanical scattering phase shifts through VPA with various types of potentials [27–35]. Relatively recently, Behera *et al.* [31] studied the nucleon-nucleon and alpha-nucleon elastic scatterings for the motion in the Manning-Rosen potential, whereas Sahoo *et al.* [32] studied the

Citation: Khirali B., Swain B., Laha S., Laha U. Low energy n–p and n–d scattering with Deng–Fan potential. *Ukr. J. Phys.* **69**, No. 4, 247 (2024). https://doi.org/10.15407/ujpe69.4.247.

Цитування: Хіралі Б., Свейн Б., Лаха С., Лаха У. Низькоенергетичне n–p та n–d розсіювання з потенціалом Денг–Фана. *Укр. фіз. журн.* **69**, №4, 247 (2024).

nucleon-nucleon scattering for F- and G-partial waves using the Hulthén potential both by the proper utilization of the variable phase method.

The fundamental concept of nuclear physics is the interaction between two nucleons. The conventional objective of nuclear physics is to comprehend the properties of atomic nuclei in terms of the exposed interaction between pairs of nucleons. With the inception of quantum chromo dynamics (QCD), it became comprehensible that the nucleon-nucleon (NN) interaction is not elemental. On the other hand, even today, in any first approach toward a nuclear structure problem, one presumes the nucleons to be elementary particles. The failure or success of this approach may then instruct us something about the significance of sub-nuclear degrees of freedom. The NN interaction has been inspected by a large number of physicists all over the world for the past 90 years. It is empirically the best known piece of strong interactions. Although, in the light of QCD, meson theory is not supposed as elementary anymore, the meson exchange thought continues to symbolize the best working model for a quantitative nucleon-nucleon potential.

The present paper is an analysis of (n–p) and (n–d) scattering experiments below 50 MeV. At low energies, only a few partial waves are expected to be important in the nuclear scattering, a fact that simplifies the phase-shift analysis. Unfortunately, the only accurate measurements in this energy range are angular distribution measurements. Hence the present paper is primarily a study of the restrictions imposed on phase-shift solutions by the single requirement that a good least-squares fit be obtained to the angular distribution. When the analysis is carried out using $^1$So, $^3P_0$, $^3P_1$, $^3P_2$ and $^1D_2$ nuclear phase shifts, it is noticed that equally good fits to the angular distribution are obtained which are within the permissible error limits. Here, we consider a simple minded potential model without any mixing parameters to treat nucleon-nucleon and nucleon-nucleus systems.

The content of the present approach is that a function which satisfies the Ricatti equation/phase equation has, at each point, the meaning of the phase-shift of the wave function for scattering by the potential at that point. This helps us in the investigation of the different regions of the potential in producing the phase shift. VPA [26] is more effective with short range potentials scattering and, therefore, is a suitable choice for the nuclear scattering studies toward the calculation of the scattering phase shifts for quantum mechanical systems with local [27–33], as well as nonlocal [34, 35] potentials. The findings of a similar research utilizing the Deng–Fan potential [2] are presented in this paper. To judge the merit of our approach, model calculations are presented for real systems, namely,for neutron-proton (n–p) and neutron-deuteron (n–d). For the systems under consideration, we shall compute phase parameters, angular distributions, total elastic scattering cross sections, and analysing powers to compare them with more advanced calculations up to partial waves $\ell = 2$. In Section 2, we present our methodology, and Section 3 is devoted to the results and discussion. We conclude in Section 4.

## 2. Methodology

The unreformed Deng–Fan potential [4, 7] has the form of $\frac{v_1}{(e^{\alpha r}-1)} + \frac{v_2}{(e^{\alpha r}-1)^2}$. We modify it to another convenient form which reads in $\ell$th partial wave as

$$V_\ell(r) = V_N(r) = v_1 \frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + v_2 \frac{e^{-2\alpha r}}{(1-e^{-\alpha r})^2} + \frac{\ell(\ell+1)}{r^2}, \quad (1)$$

where $\alpha$ is the inverse range parameter with dimension of fm$^{-1}$, and $v_1$ and $v_2$ are the strength parameters with dimension of fm$^{-2}$. The phase shifts for the potential given by Eq. (1) are computed by applying the standard prescription, the variable phase approach, to the potential scattering [26]. The VPA is an alternative prescription to calculate phase parameters for quantum mechanical problems without solving the standard wave equation. The VPA is based on the separation of the radial wave function of the Schrödinger equation into an amplitude part $A_\ell(r;k)$ and an oscillating part with a variable phase $\delta_\ell(r;k)$. This amounts to separating out the two effects of the potential which manifest themselves in distorting the wave function and in generating the scattering phases [34, 35]. The function $\delta_\ell(r;k)$, termed as the phase function, has, at each point, the meaning of the phase shift of the wave function for scattering by the potential truncated at a distance $r$. A completely cut off potential will not produce any phase shift. Thus, the phase function $\delta_\ell(r;k) = 0$.

For a local potential $\delta_\ell(k,r)$ satisfies a first-order non-linear differential equation [26] written as

$$\delta'_\ell(r;k) = -k^{-1}V_\ell(r)\left[\cos\delta_\ell(r;k)\hat{j}_\ell(kr) - \right.$$
$$\left. - \sin\delta_\ell(r;k)\hat{\eta}_\ell(kr)\right]^2, \quad (2)$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati–Bessel functions [36]. For $\ell = 0,\ 1\,\&\,2$, the phase equations yield

$$\delta'_0(r;k) = -k^{-1}V_0(r)\ [\sin(\delta_0(r;k) + kr)]^2, \quad (3)$$

$$\delta'_1(r;k) = -\frac{V_1(r)}{k^3r^2}\Big[\sin(\delta_1(r;k) + kr) -$$
$$- kr\ \cos(\delta_1(r;k) + kr)\Big]^2 \quad (4)$$

and

$$\delta'_2(r;k) = -k^{-1}V_2(r)\left[\left(\frac{3}{k^2r^2} - 1\right)\times\right.$$
$$\left.\times\sin(\delta_2(r;k) + kr) - \frac{3}{kr}\cos(\delta_2(r;k) + kr)\right]^2. \quad (5)$$

The quantity $k$ is the center of mass momentum and, in turn, is related to the center of mass energy by the relation $k = \sqrt{2\mathrm{mE}}/\hbar$. The scattering phase shift $\delta_\ell(r;k)$ is obtained by solving the phase equation from the origin to the asymptotic region with the initial condition $\delta_\ell(0;k) = 0$. During the solution of the phase equation, $\delta_\ell(r;k)$ is up surged by the potential as one moves away from the origin to its asymptotic value as soon as one moves out of the range of the potential. Obviously, $\delta_\ell(k) = \lim_{r\to\infty}\delta_\ell(r;k)$. Having the phase function $\delta_\ell(r;k)$, one can easily determine the amplitude function $A_\ell(r;k)$.

## 3. Results and Discussion

We shall parametrize the nuclear Deng–Fan potential, given in Eq. (1), to obtain the standard phase parameters [37, 38] of different states of the (n–p) and (n–d) systems by numerically solving the differential equations (3)–(5) from the origin to the asymptotic region. As the function $\delta_\ell(r;k)$ generates the phase shift at each point, the step size of the variable $r$ in calculating accumulation of phase within the range of the interaction is very crucial. Thus, to have proper phase parameters, one has to optimize the step size judiciously. The parameters for the (n–p) and (n–d) systems for different states along with the optimized step sizes are given in Table 1.

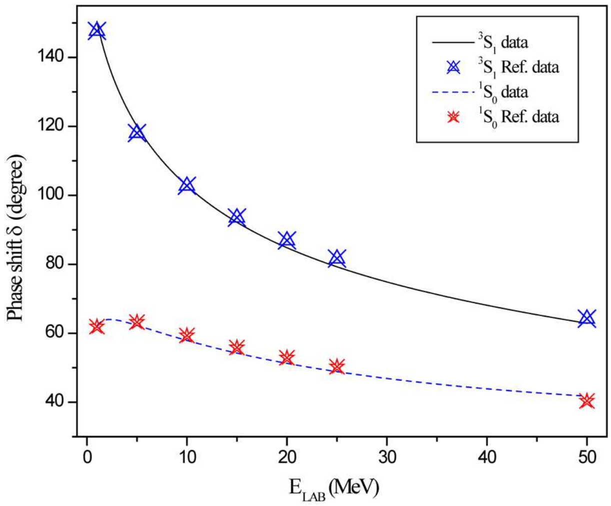


***Fig. 1.*** (n–p) scattering phase shifts (S-wave) as a function of laboratory energy. The standard data are from Ref. [37]

For the numerical computation, we use $\hbar^2/2m = 41.47\,\mathrm{MeV\,fm^2}$ and $31.1025\,\mathrm{MeV\,fm^2}$ for (n–p) and (n–d) systems, respectively, where $m$ is the reduced mass of the respective systems. From Fig. 1, we observe that our parameters for the $^3\mathrm{S}_1$ and $^1\mathrm{S}_0$ states of (n–p) system reproduce the correct phase parameters up to a laboratory energy of 50 MeV. Our results are in conformity with those of Pérez *et al.* [37]. Similarly, the P- and D-wave phase shifts are also in

*Table 1.* **List of parameters for the potential**

| Systems | States | $\alpha$ (fm$^{-1}$) | $v_1$ (fm$^{-2}$) | $v_2$ (fm$^{-2}$) | Step size |
|---|---|---|---|---|---|
| n–p | $^1\mathrm{S}_0$ | 0.868 | –0.7174 | –0.0032 | 0.0100 |
| | $^3\mathrm{S}_1$ | 0.868 | –1.8950 | 0.8000 | 0.0085 |
| | $^1\mathrm{P}_1$ | 0.756 | –2.1500 | 2.9000 | 0.0100 |
| | $^3\mathrm{P}_0$ | 0.756 | –2.9600 | 2.2000 | 0.0100 |
| | $^3\mathrm{P}_1$ | 0.756 | –2.3650 | 3.5500 | 0.0100 |
| | $^3\mathrm{P}_2$ | 0.756 | –2.7530 | 1.4000 | 0.0100 |
| | $^1\mathrm{D}_2$ | 0.350 | –1.4500 | 0.0050 | 0.0471 |
| | $^3\mathrm{D}_1$ | 0.800 | –0.8000 | 5.0000 | 0.0008 |
| | $^3\mathrm{D}_2$ | 0.400 | –1.6000 | 0.0500 | 0.0010 |
| | $^3\mathrm{D}_3$ | 0.350 | –1.4000 | 0.0050 | 0.0950 |
| n–d | $1/2^{(+)}$ | 0.860 | –2.500 | 2.00 | 0.1590 |
| | $1/2^{(-)}$ | 0.868 | –2.480 | –1.65 | 0.0032 |
| | $3/2^{(-)}$ | 0.340 | –2.770 | 0.50 | 0.900 |
| | $3/2^{(+)}$ | 0.830 | –0.415 | 2.00 | 0.0098 |

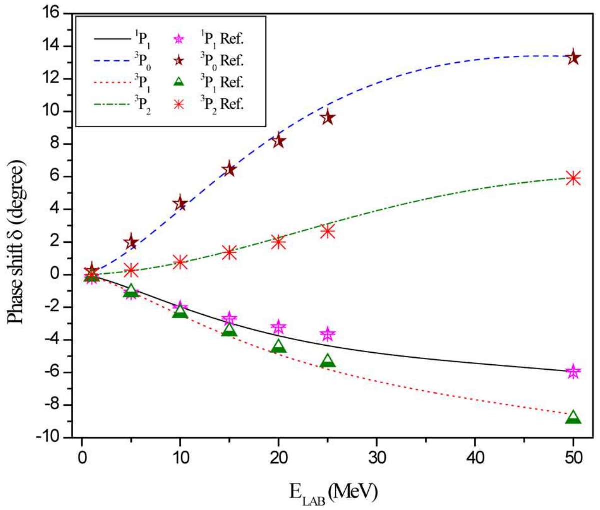


***Fig. 2.*** (n–p) scattering phase shifts (P-wave) as a function of laboratory energy. The standard data are from Ref. [37]

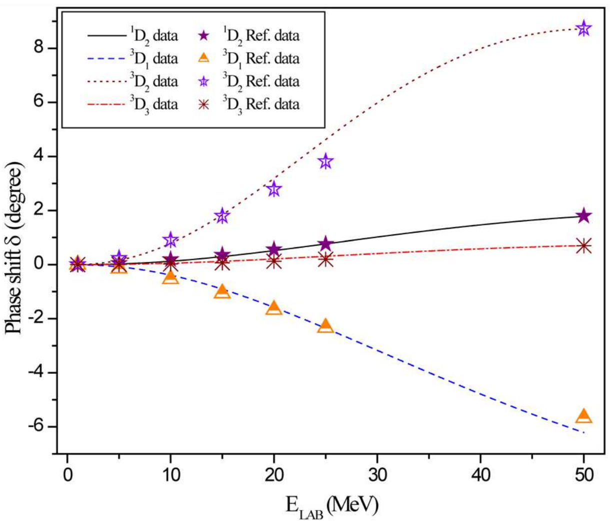


***Fig. 3.*** (n–p) scattering phase shifts (D-wave) as a function of laboratory energy. The standard data are from Ref. [37]

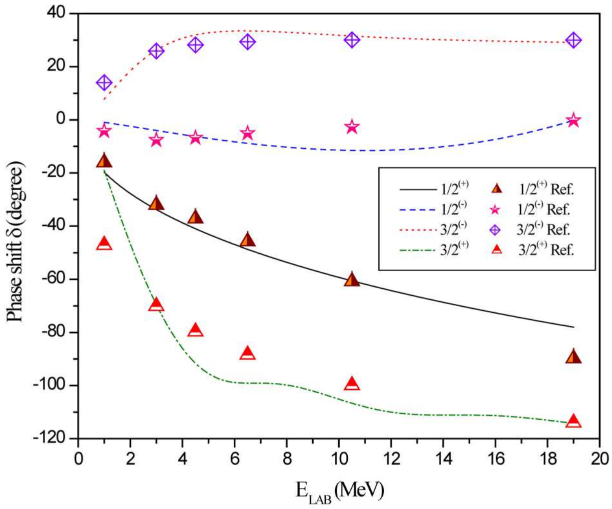


***Fig. 4.*** (n–d) scattering phase shifts as a function of laboratory energy. The standard data are from Ref. [38]

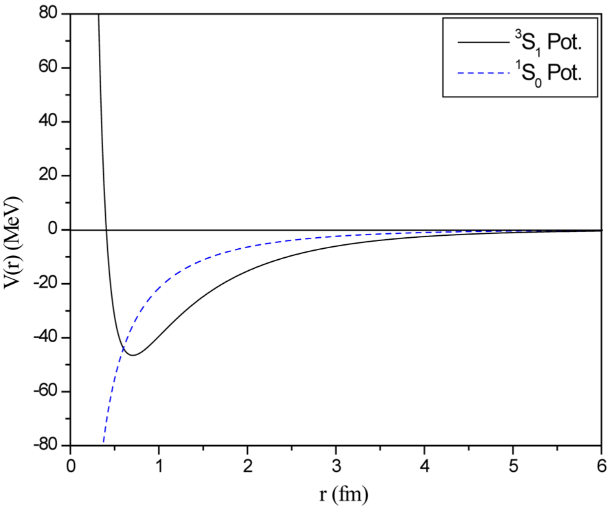


***Fig. 5.*** S-wave (n–p) potentials as a function of $r$

excellent agreement with Ref. 37, as shown in Figs. 2, 3. The n–d scattering phase parameters, portrayed in Fig. 4, reproduce reasonable agreement with those of Hüber *et al.* [38] except $3/2^{(+)}$ state. In that, our results show a slight oscillating character in the energy range 4–12 MeV.

For the Deng–Fan model,the scattering of the (n–p) and (n–d) systems, the associated potentials for different states are depicted in Figs. 5–8. Nuclear potentials are highly state-dependent, and the same has been shown by Figs. 5–7 for different S-, P- and D-states of the (n–p) system. The associated potentials for different states of the (n–d) system have also been displayed in Fig. 8.

The scattering cross-section is an effective area that quantifies the intrinsic rate a given event occurs during the scattering of a two-particle group. In the literature, the low-energy reliable data relating to (n–p) and (n–d) systems [39–62] exist for various nucleon-nucleon interaction models. However in low-energy collisions, the total cross-section comes out mostly from elastic scattering channel with insignificant involvement from the rest of the involved reaction chan-

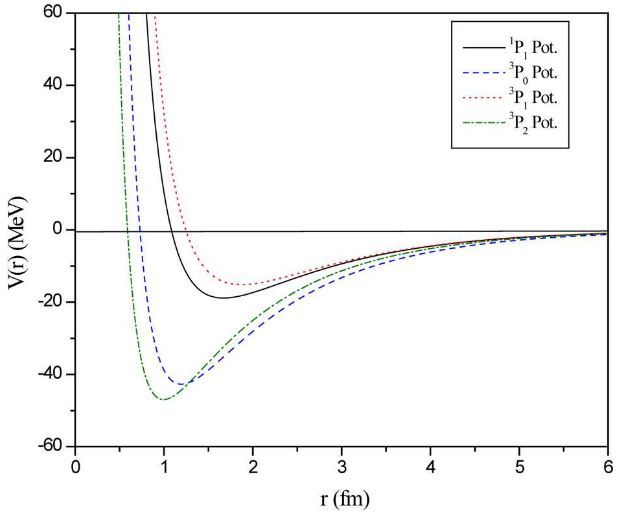


***Fig. 6.*** P-wave (n–p) potentials as a function of $r$

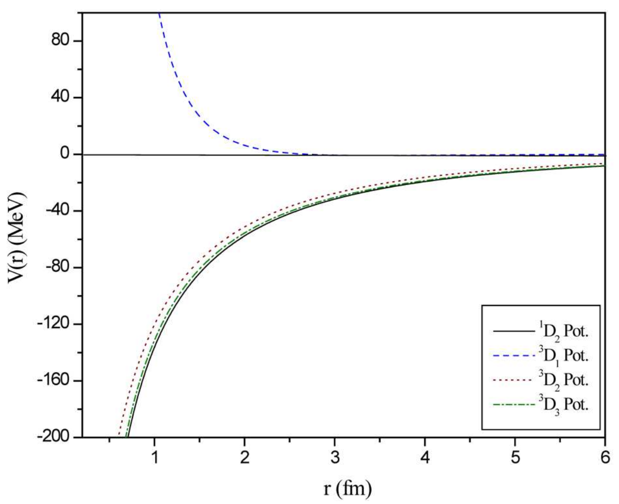


***Fig. 7.*** D-wave (n–p) potentials as a function of $r$

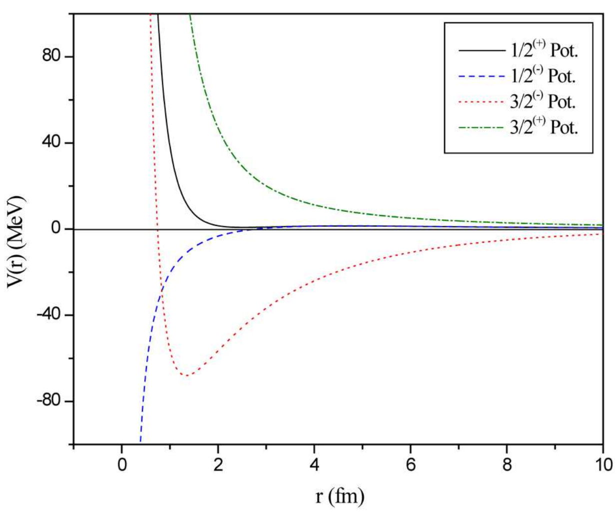


***Fig. 8.*** (n–d) potentials as a function of $r$

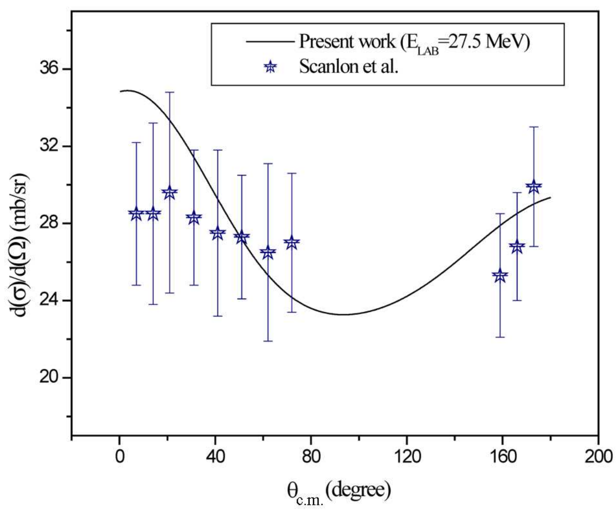


***Fig. 9.*** (n–p) differential scattering cross-section as a function of center of mass angle

nels. We aspire to investigate to what level our model calculations will be able to acquiesce realistic cross-section data in view of small incongruity between the results of our phase shift analysis and of other calculations. The scattering amplitude is expressed as

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\cos\theta)\,(\exp(2i\delta_\ell)-1). \quad (6)$$

The quantity $\delta_\ell$ is the nuclear phase shift. The differential scattering cross-section $\sigma(\theta)$ is given by $\sigma(\theta) = |f(\theta)|^2$. One may calculate the total scattering cross-section by integrating the differential cross-section $\sigma(\theta)$ over the entire solid angle and the angle integrated cross-section is

$$\sigma_T = \frac{4\pi}{k^2}\sum_{L=0}^{\infty}(2L+1)\,\sin^2\delta_\ell. \quad (7)$$

Note that this integrated cross-section is sometimes called the total cross-section, because it is the total after integration over all angles. The elastic scattering of neutrons by proton has been investigated by a number of researchers [37, 51–61]. In the present text, we calculate differential and total scattering cross-

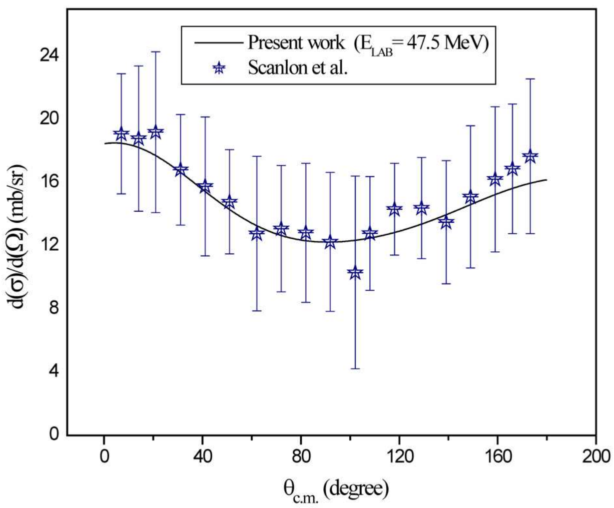


***Fig. 10.*** (n–p) differential scattering cross-section as a function of center of mass angle

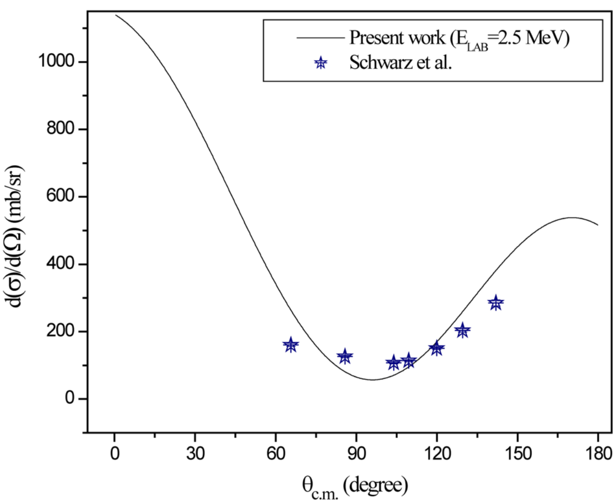


***Fig. 11.*** (n–d) differential scattering cross-section as a function of center of mass angle

sections for the (n–p) and (n–d) systems and compare them with the data [44, 45, 63] available in the literature by exploiting Eqs. (6), (7). The cross-section is distorted and characterized using analysing power $A_y$ and is proportional to the difference between left-right cross-sections. Analyzing powers have also been estimated following the prescription of Cooper *et al.* [64].

The differential scattering cross-sections are portrayed in Figs. 9–12 together with the standard results [45, 63] for the (n–p) and (n–d) systems.

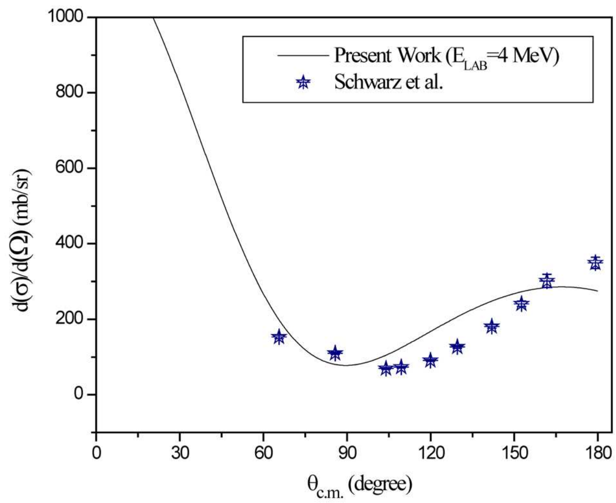


***Fig. 12.*** (n–d) differential scattering cross-section as a function of center of mass angle

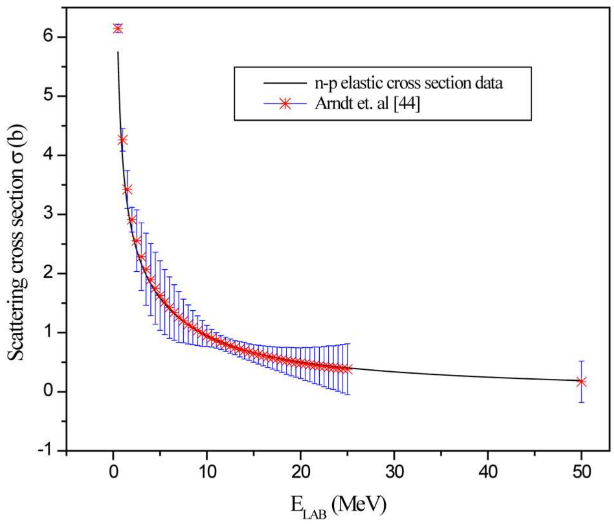


***Fig. 13.*** (n–p) total scattering cross-section as a function of laboratory energy

Our computed scattering cross-sections, using the obtained phase parameters, are in good conformity with Scanlon *et al.* [63] and Schwarz *et al.* [45]. The analyzing powers for the systems under discussion are also estimated and found them in good agreement with experimental data [65, 66]. Figures 9 and 10 show the differential scattering cross-sections for the (n–p) system at incident energies 27.5 and 47.5 MeV together with the Scanlon *et al.* [63]. The same are shown in Figs. 11, 12 for the (n–d) system for 2.5

*Table 2.* **Neutron-proton elastic analyzing power at $E_{LAB} = 27.5$ MeV in comparison to experimental data of J. Wilczynski *et al.* [65]**

| $\theta_{c.m.}$ (deg) | $E_{LAB} = 27.5 \pm 1$ MeV | |
|---|---|---|
| | Analyzing power ($A_y$) | |
| | Present work | Wilczynski *et al.* [65] |
| 33.1 | 4.841904 | $5.60 \pm 0.029$ |
| 50.9 | 6.511997 | $8.11 \pm 0.039$ |
| 69.1 | 6.134383 | $7.74 \pm 0.034$ |
| 87.1 | 6.426498 | $7.68 \pm 0.039$ |
| 105.4 | 6.387566 | $5.42 \pm 0.027$ |
| 122.9 | 5.139937 | $3.60 \pm 0.019$ |
| 141.0 | 4.757877 | $2.37 \pm 0.004$ |
| 151.4 | 3.526850 | $2.25 \pm 0.007$ |

*Table 3.* **Neutron-proton elastic analyzing power data at $E_{LAB} = 50.0$ MeV in comparison to experimental data of J. Wilczynski *et al.* [65]**

| $\theta_{c.m.}$ (deg) | $E_{LAB} = 50.0 \pm 2$ MeV | |
|---|---|---|
| | Analyzing power ($A_y$) | |
| | Present work | Wilczynski *et al.* [65] |
| 33.1 | 15.241 | $12.15 \pm 1.24$ |
| 50.9 | 19.460 | $21.35 \pm 1.15$ |
| 69.1 | 20.097 | $23.21 \pm 1.12$ |
| 87.1 | 18.602 | $21.06 \pm 1.42$ |
| 105.4 | 16.379 | $14.53 \pm 1.45$ |
| 123.0 | 12.911 | $7.89 \pm 1.67$ |
| 141 | 8.389 | $4.94 \pm 0.51$ |
| 151.4 | 3.592 | $1.86 \pm 0.85$ |

and 4.0 MeV (laboratory energies) in conjunction with the results of Schwarz *et al.* [45]. We observe that the cross-section data vary uniformly with angle and consistently with energy. When the incident energy is higher, the cross-section monotonically drops, and the minimum for the (n–p) system, which is approximately 100°, then slowly shifts backward. The majority of our (n–p) data fit into the experimental error ranges. However, one must take it into account up to quite high energies in order to really calculate the cross-section via its partial-wave expansion. Conversely, we take into account energies up to 50 MeV, when a few lower partial waves are involved. The minimum for the (n–d) system comes at around 90°, as instead of 120° as measured by Schwarz *et al.* [45].

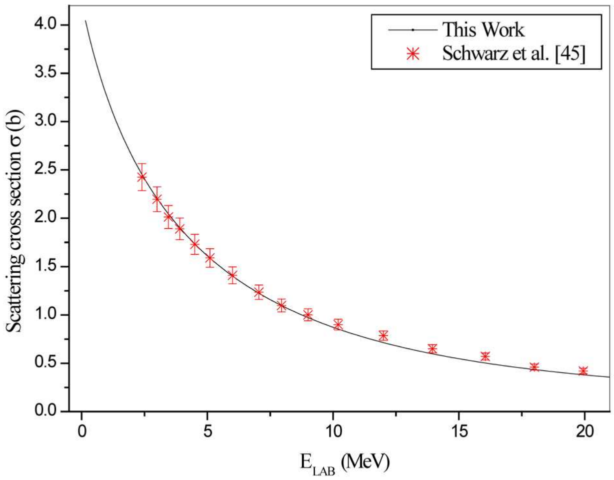


***Fig. 14.*** (n–d) total scattering cross-section as a function of laboratory energy

*Table 4.* **Neutron-deuteron elastic analyzing power at $E_{LAB} = 5.0$ MeV in comparison to experimental data of Tornow *et al.* [66]**

| $\theta_{c.m.}$ (deg) | $E_{LAB} = 5$ MeV | |
|---|---|---|
| | Analyzing power ($A_y$) | |
| | Present work | Tornow *et al.* [66] (Errors are about ±0.0035) |
| 34.1 | 0.007 | 0.013 |
| 45.3 | 0.009 | 0.021 |
| 53.3 | 0.012 | 0.025 |
| 61.5 | 0.017 | 0.025 |
| 70.2 | 0.027 | 0.032 |
| 77.1 | 0.047 | 0.043 |
| 85.6 | 0.124 | 0.051 |
| 91.3 | 0.335 | 0.061 |
| 97.0 | 0.623 | 0.075 |
| 103.5 | 0.199 | 0.082 |
| 107.4 | 0.100 | 0.086 |
| 115.2 | 0.038 | 0.092 |
| 117.6 | 0.030 | 0.091 |
| 120.3 | 0.024 | 0.084 |
| 123.1 | 0.021 | 0.074 |
| 128.0 | 0.014 | 0.067 |
| 131.4 | 0.011 | 0.057 |

*Table 5.* **Neutron-deuteron elastic analyzing power at $E_{\rm LAB} = 6.5$ MeV in comparison to experimental data of Tornow *et al.* [66]**

| $\theta_{\rm c.m.}$ (deg) | $E_{\rm LAB} = 6.5$ MeV | |
|---|---|---|
| | Analyzing power ($A_y$) | |
| | Present work | Tornow *et al.* [66] (Errors are about ±0.0035) |
| 30.6 | 0.0123 | 0.013 |
| 38.9 | 0.0148 | 0.020 |
| 46.6 | 0.0187 | 0.023 |
| 53.0 | 0.0233 | 0.032 |
| 61.3 | 0.0329 | 0.029 |
| 67.8 | 0.0449 | 0.037 |
| 74.9 | 0.0840 | 0.040 |
| 82.0 | 0.0878 | 0.045 |
| 91.1 | 0.1050 | 0.072 |
| 101.0 | 0.0846 | 0.087 |
| 105.6 | 0.0703 | 0.108 |
| 111.5 | 0.0549 | 0.115 |
| 117.3 | 0.0433 | 0.118 |
| 125.3 | 0.0326 | 0.099 |
| 134.9 | 0.0247 | 0.076 |
| 150.0 | 0.0183 | 0.026 |
| 155.0 | 0.0171 | 0.021 |

The analyzing powers for (n–p) and (n–d) systems at two different energies are presented in Tables 2–5 along with the experimental results [65, 66].

The total scattering cross-sections for both systems are portrayed in Figs. 13 and 14 along with the standard data [44, 45].

The cross-section calculations were performed including the contributions of S-, P- and D-waves. Our results for the total (n–p) and (n–d) cross-sections are in excellent agreement with those of Refs. [44, 45]. As the S-wave contribution to the total cross-section dominates over the higher partial wave involvement in the low-energy region, the overall agreement of our cross-section data with Ref. [44, 45] is noteworthy.

## 4. Conclusions

Our results for the elastic scattering phase shifts for different states of the (n–p) and (n–d) systems are in conformity with the standard data [37, 38]. Cross-sections and analysing powers are computed with consideration for the impacts of few lower partial waves like S-, P- and D-waves. A synchronized account of all theoretical and experimental omit it evidences over a wide range of the energy spectrum may provide more insights into the N–N interaction and, possibly, even the significance of the three-body forces for the nucleon-deuteron system. The difference in $A_y$ is thought to be greatly reduced by altering the short-range element of the LS force in the nucleon-nucleon potential. This implies that the off-energy-shell LS interaction has a major influence on $A_y$. This potential model needs to be refined, as indicated by the energy dependence of the difference in $A_y$. The Schrödinger equation for the effective potentials must first be solved in order to incorporate an electromagnetic potential into the current nuclear potential for the charged hadron scattering.

The (n–p) and (n–d) cross-section calculations with our simple minded potential model are in excellent agreement with those of earlier works with sophisticated potential models. In the recent time, physicists show much attention in probing for the exponential-type of potentials as they play a significant role in plasma, solid-state, atomic, and molecular physics. For the treatment of charged hadronic systems, one has to consider a combined interaction model: electromagnetic plus nuclear in origin. The electromagnetic part of the interaction is normally represented by the screened/cut off Coulomb interaction as the pure Coulomb potential has no existence in reality. This problem is in our active consideration with the Deng–Fan plus Hulthén potential to study, particularly, the proton-proton and proton-deuteron systems. The overall quality of the consistency between the theory and experiment is worth mentioning. Therefore, the present potential may turn out to be interesting to theoretical physicists.

We have no funding source for this work.

We have no conflict of interests in this paper.

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Received 01.08.23

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НИЗЬКОЕНЕРГЕТИЧНЕ n–p ТА n–d
РОЗСІЮВАННЯ З ПОТЕНЦІАЛОМ ДЕНГ–ФАНА

Потенціал Денг–Фана, який використовується в молекулярній динаміці, застосовано для опису n–p та n–d розсіювання в рамках методу фазових функцій. Знайдено перерізи і параметри фази розсіяння в узгодженні з іншими теоретичними роботами і експериментальними даними.

*Ключові слова*: потенціал Денг–Фана, метод фазових функцій, параметри фази розсіяння, переріз, поляризаційна асиметрія, n–p і n–d системи.