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MINISUPERSPACE QUANTUM COSMOLOGY IN THE MADELUNG–BOHM FORMALISM

An analogy between non-relativistic quantum mechanics in the Madelung formulation and quantum geometrodynamics in the case of the maximally symmetric space is drawn. The equations equivalent to the continuity equation and the hydrodynamic Euler equation describing the evolution of the velocity introduced for the case of hypothetical fluid flow characterizing the cosmological system are obtained. It is shown that the perfect nature of the fluid is broken by the quantum Bohm potential. The quantum potential is calculated in the semi-classical approximation for different forces acting in the system both in standard quantum mechanics and in the minisuperspace model of quantum geometrodynamics. The explicit dependences of the cosmic scale factor on the conformal time, which account for the quantum additive, are found for the empty space with spatial curvature and for a spatially flat universe with dust and radiation.
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1. Introduction

Madelung's paper [1] on quantum theory in the hydrodynamical form can be viewed as one of the most seminal in the scope of quantum theory. It became the starting point for many works on the relationship between the newly emerged quantum theory based on the Schrödinger equation [2] and the well-established classical theories of mechanics and electrodynamics, as well as for the attempts to interpret quantum theory itself. There was an intense discussion about the issue and the necessity of finding a consistent formulation of quantum theory in the 1950s [3–10]. Madelung's formalism played an important role in introducing the idea of an ensemble of classical trajectories in the further development of quantum mechanics. Specifically, Dirac's idea [11], realized in Feynman's path integral formulation of quantum mechanics, can be discerned in Madelung's hydrodynamic approach.

In the last two decades, there has been renewed the interest in the Madelung–Bohm representation of wavefunctions [12]. The elements of the Madelung approach can be seen in the quantum Hamilton–Jacobi formalism [13, 14] which provides an alternative approach to non-relativistic quantum mechanics and can be used, in particular, in computational method for one-dimensional scattering problems [15, 16]. The three-dimensional quantum Hamilton–Jacobi equation in the case of separated variables describes physical states not detected by the Schrödinger wave function [17, 18].

Classical mechanics is widely believed to emerge from quantum mechanics as its limiting case [19]. The question of the correct transition from the quantum

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description to the classical one is actual and still open. The resolution of this question may appear to be more feasible in the context of the Madelung–Bohm approach [20]. On the other hand, this approach allows one to describe the interaction of classical and quantum systems, when classical motion is not regarded as an approximation of quantum mechanics [21].

Since the Madelung equations transform the non-relativistic time-dependent Schrödinger equation into hydrodynamic equations of non-classical virtual Eulerian fluid [22], one can analyze the properties of the Madelung fluid, such as entropy [23], and establish the relation between the Fisher information and the thermodynamic-like internal energy of this fluid [24].

The exact mathematical relationship between the Madelung equations and the Schrödinger equation is still a matter of debate [25, 26]. It was argued that the consistent mathematical theory for this system of equations cannot be developed without the imposition of an additional quantization condition [5, 27, 28]. Some approaches claim that the Madelung equations are physically more fundamental than the Schrödinger equation. However, if we consider the Madelung equations as an 'unrolling' of the Schrödinger equation into the form, in which the contribution of quantum effects is explicitly singled out (without considering the reverse procedure of recovering the Schrödinger equation), then certain questions about the status of the Madelung equations disappear. The proportionality of quantum effects to $\hbar^2$ makes it possible to develop the WKB method.

As mentioned above, the Madelung–Bohm approach remains a guideline for many current studies in quantum theory, in particular with regard to its connection with classical theory. The objective of this paper is not to provide a comprehensive review of all existing studies in this area of research. The article consists of two parts: the first relates to quantum mechanics, and the second one to quantum geometrodynamics. The brief description of the Madelung–Bohm approach given in Sect. 2 is intended to present some of its key points, with a view to a further application in quantum geometrodynamics. In order to find out the specific dependences of a quantum Bohm potential on a spatial variable, several specific cases of potentials in which a quantum particle moves are considered. When studying the properties of the Bohm potential, we focus on the semi-classical approximation with respect to $\hbar^2$. Then the Bohm potential can be calculated directly from the potential field in which the particle moves. In Sect. 3, we consider the homogeneous and isotropic quantum cosmological system (universe). In this case, we can draw direct parallels with quantum theory in $(1+1)$ dimensions. The equations of quantum geometrodynamics for a system filled with a homogeneous scalar field and a perfect fluid are reduced to the set of equations of the hydrodynamic form. Analogs of the quantum Bohm potential for the cases of radiation dominance, matter dominance, for a cosmological model with spatially flat geometry filled with radiation and dust, and for the empty space with spatial curvature are calculated in the semi-classical approximation. The impact of the quantum potential on the expansion of the universe is studied here. In Section 4, we discuss the results obtained in the previous sections.

2. Madelung–Bohm Approach in $(1 + 1)$ Dimensions

2.1. Basic equations

The reason for the choice of $(1 + 1)$ dimensions is induced by the observation that the quantum geometrodynamics for maximally symmetric spacetime reduces to a problem in $(1 + 1)$ dimensions. This allows us to directly draw parallels between quantum mechanics and quantum cosmology.

In non-relativistic quantum theory, the basic equations [1, 2]

$$\Psi = \psi\, e^{-\frac{i}{\hbar} W t}, \tag{1}$$

$$-\frac{\hbar^2}{2m}\, \partial_x^2 \psi + [V(x) - W]\, \psi = 0 \tag{2}$$

describe the motion of a particle of mass m along the x-axis under the action of the potential $V(x)$. Here, $\psi = \psi(x)$, $\Psi = \Psi(x,t)$, t is time, and W is a total energy. Since the phase of Ψ depends on t linearly, Eqs. (1) and (2) may be rewritten as a single equation,

$$i\hbar \partial_t \Psi = \left[-\frac{\hbar^2}{2m}\, \partial_x^2 + V(x)\right] \Psi. \tag{3}$$

This equation is known as the time-dependent Schrödinger equation. We will refrain from generalizing Eq. (3) to the case where the potential function depends on time, because then it has no solution in the form (1).

The continuity equation follows from Eq. (3),

$$\partial_t\rho + \partial_x J = 0, \tag{4}$$

where $\rho = |\Psi|^2$ is the probability density for finding a particle at some point in space, and

$$J = \frac{\hbar}{2im}\left(\Psi^*\partial_x\Psi - \Psi\partial_x\Psi^*\right) \tag{5}$$

is the density of the probability current of a particle at point x and at time t.

We will look for the solution of Eq. (3) in polar form

$$\Psi(x,t) = A(x,t)\, e^{\frac{i}{\hbar}S(x,t)}, \tag{6}$$

where the amplitude A and the phase S are real-valued functions of their arguments. Substituting Eq. (6) into Eq. (3) yields two equations [1],

$$m\,\partial_t A^2 + \partial_x\left(A^2\partial_x S\right) = 0, \tag{7}$$

$$\partial_t S + \frac{1}{2m}(\partial_x S)^2 + V = Q, \tag{8}$$

where

$$Q = \frac{\hbar^2}{2m}\frac{\partial_x^2 A}{A} \tag{9}$$

is the quantum Bohm potential [1] [3]. The potential Q can be considered as an additional source generated by variations in the amplitude A in space. In the formal limit $\hbar^2 \to 0$, we have $Q = 0$, and Eq. (8) in this limit resembles the Hamilton–Jacobi equation, if the phase S is viewed as a classical action [11, 29]. By analogy with classical mechanics, we define the velocity u by the expression

$$u = \frac{\partial_x S}{m}. \tag{10}$$

Substitution of Eq. (6) into Eq. (5) gives $J = \rho u$, and Eq. (4) turns into a continuity equation, as in hydrodynamics,

$$\partial_t\rho + \partial_x(\rho u) = 0. \tag{11}$$

This equation is Eq. (7) in new notations. It describes the conservation law for a fluid of density ρ and current ρu.

Equation (8) can be interpreted as the quantum Hamilton–Jacobi equation, although there have been discussions [6, 7] about its meaning that continue until now [25, 26]. From this equation, with regard for Eq. (9), it follows a quantum analog of Euler's equation, which determines the velocity u,

$$(\partial_t + u\partial_x)\, mu = -\partial_x(V - Q). \tag{12}$$

Equations (11) and (12) describe the motion of the fluid and include the additional force $\frac{1}{m}\partial_x Q$ caused by the quantum effects contained in the original Schrödinger equation. After setting $\hbar = 0$ and interpreting ρ as the matter density, these equations become equations of classical hydrodynamics, where the quantity u is the velocity of a fluid at any given point x of space at time t, without consideration of which the particle occupies the position x. When $\hbar \neq 0$, one can only talk about the velocity u of a hypothetical fluid with the density ρ, where ρ and u are purely quantum characteristics, and their connection with any physical substance is questionable. In this case, a hypothetical fluid is not perfect, since the additional force of quantum nature produces non-linear effects in the behavior of the velocity u.

2.2. Stationary states

For the potential $V(x)$ which does not depend on time, the Hamiltonian on the right-hand side of Eq. (3) does not depend on time explicitly. In that case, the phase in Eq. (6) can be written as: $S(x,t) = S_0(x) - Wt$. Then the wave function (1) describes a special case of the so-called stationary state of a particle with energy W in an external field $V(x)$.

The solution of Eq. (2) for the stationary state can be written as

$$\psi(x) = A(x)\, e^{\frac{i}{\hbar}S_0(x)}, \tag{13}$$

where $A(x)$ and $S_0(x)$ are real-valued functions which do not depend on time. Then we have

$$\partial_t A = 0 \quad \text{and} \quad \partial_t S = -W. \tag{14}$$

The velocity (10) for the stationary flow of a hypothetical fluid is

$$u = \frac{\partial_x S_0}{m}. \tag{15}$$

Substituting Eq. (1) into (5) and taking Eqs. (13) and (14) into account, we find that the probability current

[1] This potential is usually defined with a minus sign. We leave the plus sign, as it appears when deriving Eq. (8).

density $J = \rho u = \text{const}$. For the complex-valued wave functions, the amplitude in Eq. (13) can be written as

$$A = \frac{\text{const}}{\sqrt{u}}. \tag{16}$$

Equation (8) reduces to a non-linear equation for the velocity u,

$$\frac{mu^2}{2} + V - Q = W, \tag{17}$$

where

$$Q = \frac{\hbar^2}{2m}\left[\frac{3}{4}\left(\frac{\partial_x u}{u}\right)^2 - \frac{1}{2}\frac{\partial_x^2 u}{u}\right]. \tag{18}$$

The gradient of Q is

$$\partial_x Q = \frac{\hbar^2}{2m}\left[2\frac{\partial_x u\,\partial_x^2 u}{u^2} - \frac{3}{2}\left(\frac{\partial_x u}{u}\right)^3 - \frac{1}{2}\frac{\partial_x^3 u}{u}\right]. \tag{19}$$

Let us note that the complex-valued state $\psi(x)$ is, in general, a state of a continuous spectrum for which the energy is given, as for example in the scattering of a particle on a target. The so-called plane-wave approximation is often used, in which

$$\psi(x) = C\, e^{\frac{i}{\hbar}px}, \tag{20}$$

where $p = mu$ is a momentum and C is a normalization factor. This wave function is normalized either by the condition $J = u$, so that $C = 1$ or by the condition $J = 1$, for which $C = 1/\sqrt{u}$, as in Eq. (16) (normalization to a delta function [29]).

2.3. Semi-classical Bohm potential

From dimensional reasons, it follows that

$$Q(x) = \frac{\hbar^2}{2mx^2}\,\lambda(x), \tag{21}$$

where $\lambda(x)$ is a dimensionless coupling function of x. As we will see later, the functions $Q(x)$ or $\lambda(x)$ can tend to a constant under certain conditions. We will calculate these functions in the semi-classical approximation. Since $Q \sim \hbar^2$, then, in this approximation, the quantum addition can be neglected in Eq. (17), and we can set $u = \sqrt{\frac{2}{m}(W - V)}$. This allows us to calculate the quantum potential (18) explicitly as

$$Q = \frac{\hbar^2}{2m}\left[\frac{5}{16}\left(\frac{\partial_x V}{W - V}\right)^2 + \frac{1}{4}\frac{\partial_x^2 V}{W - V}\right]. \tag{22}$$

Generally speaking, if the solution to Eq. (3) is known in an explicit form, then the quantum Bohm potential can be restored from Eq. (9). But then the answer will only be of a informative nature, since the quantum problem is assumed to be already solved, and Eq. (8) is reduced to the identity. Applying the fact that the quantum potential is proportional to $\hbar^2$, and, according to Eq. (8), it is a summand added to other summands in which the Planck constant is absent, allows us to find the Bohm potential using perturbation theory and the explicit form of the potential $V(x)$ only.

Consider some special cases of the potential $V(x)$.

2.3.1. Free fall of a particle

For the problem of the free fall of a particle of mass m over the Earth's surface, we have the gravitation potential in its standard form $V(x) = mgx$, where x is the height over the Earth's surface. Then the quantum potential (22) takes the form

$$Q = \frac{5}{32}\frac{\hbar^2 mg^2}{(W - mgx)^2}. \tag{23}$$

This equation can be reduced to Eq. (21) with

$$\lambda(x) = \frac{5}{16}\left(\frac{mgx}{W - mgx}\right)^2. \tag{24}$$

If $W \gg mgx$, then we have $\lambda(x) = \frac{5}{16}\left(\frac{x}{x_0}\right)^2$, where $x_0 \equiv \frac{W}{mg}$, and the quantum potential Q does not depend on $x < x_0$.

2.3.2. Particle in a static fluid

When a particle is immersed in a static fluid, it is subjected to the buoyant force which acts in the direction opposite to the gravitational force, so that the total potential is $V(x) = -(m - \rho_f V_d)gx$, where x is is the distance to the fluid surface, ρ_f is the density of the fluid, V_d is the volume of fluid displaced. Then the coupling function in Eq. (21) becomes

$$\lambda(x) = \frac{5}{16}\left(\frac{(m - \rho_f V_d)gx}{W + (m - \rho_f V_d)gx}\right)^2. \tag{25}$$

For a particle moving with energy $W \ll (m - \rho_f V_d)gx$, the constant $\lambda = \frac{5}{16}$ determines the quantum potential (21), so that it does not depend on the gravitational field g and fluid properties. In the opposite limit, $W \gg (m-\rho_f V_d)gx$, the situation is similar to the one discussed in subsection 2.3.1, so that the quantum potential Q is constant for small values of x.

2.3.3. Charged particle in an accelerating electrical field

When considering the one-dimensional motion of a particle of mass m and charge $-e$ in an accelerating electrical field $\mathcal{E}$, we have $V(x) = -e\mathcal{E}x$, and the quantum potential is

$$Q = \frac{5}{32}\frac{\hbar^2}{m}\left(\frac{e\mathcal{E}}{W + e\mathcal{E}x}\right)^2. \tag{26}$$

In the low-energy limit, $W \ll e\mathcal{E}x$, the quantum potential (21) is determined by the same value $\lambda = \frac{5}{16}$, as in Subsection 2.3.2. Here, the quantum potential Q does not depend on the electrical field $\mathcal{E}$.

2.3.4. Power potential

Let $V(x) = \Omega_n x^n$ (there is no summation over n in this expression), where the coupling constant Ω_n has a dimension of energy/length^n. Then the quantum potential takes the form (21) with

$$\lambda(x) = \frac{\Omega_n x^n}{W - \Omega_n x^n}\left[\frac{5}{16}n^2\frac{\Omega_n x^n}{W - \Omega_n x^n} + \frac{1}{4}n(n-1)\right]. \tag{27}$$

It should be pointed out that, at low energies, the parameters Ω_n of the potential $V(x)$ do not affect the quantum potential Q. Indeed, for the motion of a particle with the energy $W \ll \Omega_n x^n$, the quantum potential takes the form (21) with

$$\lambda = \frac{1}{4}n\left(\frac{1}{4}n + 1\right). \tag{28}$$

In the reverse limit $W \gg \Omega_n x^n$, the coupling function $\lambda(x)$ tends to

$$\lambda(x) = \frac{5}{16}n^2\left(\frac{x}{x_0}\right)^{2n} + \frac{1}{4}n(n-1)\left(\frac{x}{x_0}\right)^n, \tag{29}$$

where $x_0 \equiv (W/\Omega_n)^{1/n}$.

2.4. Exactly solvable problems

There is a class of problems for which the Bohm potential can be calculated exactly in the explicit form. If we restrict ourselves to considering only real-valued wave functions, then Eq. (5) yields that the probability current density $J = 0$. In that case, from the general expression $J = \rho u$, we find that $u = 0$ for such systems, since $\rho = A^2 \neq 0$. Further, from Eq. (17), we have a simple equation for Q:

$$Q = V - W, \tag{30}$$

and the Schrödinger equation for the real-valued amplitude A,

$$-\frac{\hbar^2}{2m}\,\partial_x^2 A(x) + [V(x) - W]\,A(x) = 0, \tag{31}$$

which follows from definition (9).

Let us look at a few specific examples.

2.4.1. Harmonic oscillator

For the harmonic oscillator with the angular frequency ω, the potential is $V(x) = \frac{1}{2}m\omega^2 x^2$, and the energy eigenvalue equals $W = \hbar\omega\left(n + \frac{1}{2}\right)$, where n is any positive integer. The Bohm potential is

$$Q(x) = \frac{1}{2}m\omega^2 x^2 - \hbar\omega\left(n + \frac{1}{2}\right). \tag{32}$$

Equation (31) for $A(x)$ reduces to the Schrödinger equation for the wave function of the harmonic oscillator. It is worth to note that, for the ground state ($n = 0$), the zero-point oscillation energy is the difference between the potential V and the quantum Bohm potential,

$$\frac{1}{2}\hbar\omega = \frac{1}{2}m\omega^2 x^2 - Q_{n=0}(x). \tag{33}$$

2.4.2. Square potential well

Let the square potential well has the form: $V(x) = -V_0$, if $|x| < R$, and $V(x) = 0$, if $|x| > R$, where $V_0 > 0$, and $2R$ is the width of the well. Within the well, there are bound states of a particle with the energy $W = -\frac{\hbar^2\kappa^2}{2m}$, where $\kappa > 0$. The Bohm potential is

$$Q = \frac{\hbar^2}{2m}\left(\kappa^2 - k_0^2\right) \text{ at } |x| < R, \tag{34}$$

where $k_0^2 = \frac{2mV_0}{\hbar^2}$, and

$$Q = \frac{\hbar^2}{2m}\kappa^2 \text{ at } |x| > R. \tag{35}$$

The amplitude $A(x)$ is described by the Schrödinger equation for the square potential well.

3. Quantum Geometrodynamics in (1 + 1) Dimensions

3.1. Basic equations

A minisuperspace model with a finite number of degrees of freedom may provide a reasonable framework for addressing the problems of quantum gravity. In this paper, we consider the homogeneous and isotropic quantum cosmological system (further referred to as the universe), whose geometry is determined by the Robertson–Walker line element with the cosmic scale factor a. It is assumed that such a universe is originally filled with a uniform matter field ϕ and a reference perfect fluid. Following Dirac's scheme of canonical quantization, the basic equations describing such a quantum universe in Planck units can be reduced to the form [30–34]

$$(-i\partial_T - E)\,\Psi = 0, \tag{36}$$

$$\left(-i\partial_a^2 + \kappa a^2 - 2aH_\phi - E\right)\Psi = 0, \tag{37}$$

where $\Psi = \Psi(a, \phi; T)$ is a state vector, T is the rescaled conformal time connected with the proper time t by the differential equation $dt = adT$, and $dT = Nd\eta$, N is the lapse function whose choice is arbitrary [35], η is the "arc time". The parameter $\kappa = +1, 0, -1$ is the curvature parameter, and H_ϕ is the Hamiltonian of the matter field ϕ. For example, for the uniform scalar field,

$$H_\phi = \frac{a^3}{2}\rho_\phi, \quad \rho_\phi = -\frac{2}{a^6}\,\partial_\phi^2 + V(\phi), \tag{38}$$

where $\frac{a^3}{2} = \mathcal{V}$ is a proper volume, ρ_ϕ is the energy density, and $V(\phi)$ is the potential. All quantities will remain dimensionless.

The parameter E is determined by the energy density of relativistic matter (radiation) ρ_γ according to the equation $E = a^4\rho_\gamma$, and defines a matter reference frame used to track time T [30, 31, 36, 37]. Note that, in standard physical units, E and T have the dimensions: $[E]$ = Energy × Length, $[T]$ = Radians.

As for the physical content of Eqs. (36) and (37), the first one is a constraint equation that provides the conservation of the number of particles of the perfect fluid, the second one is the Hamiltonian constraint equation (an analog of the Wheeler–DeWitt equation).

Using Eq. (36), Eq. (37) can be rewritten as a time-dependent equation in $(2 + 1)$ dimensions

$$-i\partial_T\Psi = \left(-\partial_a^2 + \kappa a^2 - 2aH_\phi\right)\Psi. \tag{39}$$

Let us introduce a complete orthonormal set of functions $|\Phi_i\rangle$, $\sum_i |\Phi_i\rangle\langle\Phi_i| = 1$, $\langle\Phi_i|\Phi_k\rangle = \delta_{ik}$, that diagonalizes the Hamiltonian H_ϕ,

$$\langle\Phi_i|H_\phi|\Phi_k\rangle = M_i(a)\delta_{ik}, \tag{40}$$

where $M_i(a)$ is the mass-energy of matter in the universe in the discrete and/or continuous ith state, obtained after the averaging of H_ϕ with respect to the field $|\Phi_i\rangle$ in a proper volume $\mathcal{V}$. In the general case, the mass-energy of matter may depend on a. When H_ϕ describes the homogeneous scalar field, the matter turns into a barotropic fluid [33].

Expanding the vector Ψ in a series with respect to the functions $|\Phi_i\rangle$,

$$\Psi = \sum_i |\Phi_i\rangle\langle\Phi_i|\Psi\rangle, \tag{41}$$

and substituting it into Eq. (39), we pass to the equation in $(1 + 1)$ dimensions

$$-i\partial_T\psi = \left[-\partial_a^2 + U(a)\right]\psi, \tag{42}$$

where $\psi = \psi(a, T) \equiv \langle\Phi_i|\Psi\rangle$, and

$$U(a) = \kappa a^2 - 2aM(a) \tag{43}$$

plays the role of a potential. The state of matter index i will remain unchanged and may be omitted hereafter. Equation (42) is similar to the Schrödinger equation (3) except that the time variable enters it with the opposite sign due to the specifics of general relativity.

The solution to the equation will be found in the polar form:

$$\psi(a, T) = A(a, T)\,e^{iS(a,T)}, \tag{44}$$

where the amplitude A and the phase S are real functions. Then Eq. (42) reduces to two equations

$$-\frac{1}{2}\,\partial_T A^2 + \partial_a\left(A^2\partial_a S\right) = 0, \tag{45}$$

$$-\partial_T S + (\partial_a S)^2 + U = Q, \tag{46}$$

where

$$Q = \frac{\partial_a^2 A}{A} \tag{47}$$

is an analog of the quantum Bohm potential (9). It can be considered as an additional source generated by the amplitude of the wave function.

One may consider a test particle that moves along with an idealized homogeneous fluid that represents the matter in the universe smeared out. As a result of the expansion of the universe as a whole, a particle acquires the velocity consistent with Hubble's law. It is preferable to consider the cosmological system (universe) on its own as a "particle" or an infinitesimal volume element that moves with the velocity $v = \frac{da}{dT}$ in the minisuperspace. It can be shown [34] that the velocity v is connected with the derivative of the phase $\partial_a S$ by a simple relation

$$v = -\partial_a S. \tag{48}$$

Then Eqs. (45) and (46) take the form of hydrodynamic equations

$$\frac{1}{2}\partial_T \rho + \partial_a(\rho v) = 0, \tag{49}$$

$$\left(\frac{1}{2}\partial_T + v\partial_a\right) v = -\frac{1}{2}\partial_a(U - Q), \tag{50}$$

where $\rho = A^2 = |\psi|^2$ is the probability density, and ρv is the density of the probability current of an infinitesimal volume element at point a and at time T. Equations (49) and (50) are a continuity equation and a quantum analog of Euler's equation, respectively.

The generalized force

$$\mathcal{F} = -\frac{1}{2}\partial_a(U - Q) \tag{51}$$

can be calculated explicitly,

$$\mathcal{F} = -a\kappa + \frac{a^3}{2}(\rho_m - 3p_m) + \frac{1}{2}\partial_a Q, \tag{52}$$

where $\rho_m = \frac{M}{\mathcal{V}}$ is the matter energy density, $p_m = -\frac{dM}{d\mathcal{V}}$ is the pressure. The perfect nature of the fluid is broken by the quantum term. If this term is neglected, Eqs. (49) and (50) acquire only a formal resemblance to the equations of classical hydrodynamics.

The quantum term can be rewritten in 'standard' form as

$$\frac{1}{2}\partial_a Q = \frac{a^3}{2}(\rho_{\rm B} - 3p_{\rm B}), \tag{53}$$

where

$$\rho_{\rm B} = \frac{Q}{a^4},\; p_{\rm B} = w_{\rm B}\rho_{\rm B},\; w_{\rm B} = \frac{1}{3}\left(1 - \frac{d\ln Q}{d\ln a}\right). \tag{54}$$

Here, $w_{\rm B}$ is the equation-of-state (EoS) parameter for an additional source of quantum nature.

3.2. Stationary state

Substituting Eq. (44) into Eq. (36), we obtain the condition on the wave function $\psi \neq 0$,

$$\left(-i\frac{\partial_T A}{A} + \partial_T S - E\right)\psi = 0. \tag{55}$$

This leads to two equations,

$$\partial_T A = 0 \quad \text{and} \quad \partial_T S = E. \tag{56}$$

They are similar to Eq. (14) and describe a "stationary state" with a given parameter E. The requirement of "stationarity" is not additionally imposed on the quantum system, but it follows from the equations of quantum geometrodynamics (36) and (37). Then, from Eqs. (45) and (46), we come to the equations

$$\partial_a(\rho v) = 0, \tag{57}$$

$$v^2 + U - E - Q = 0. \tag{58}$$

From Eq. (57), we find the probability density

$$\rho = \frac{\text{const}}{v}, \tag{59}$$

so that the quantum potential takes the form

$$Q = \frac{3}{4}\left(\frac{\partial_a v}{v}\right)^2 - \frac{1}{2}\frac{\partial_a^2 v}{v}, \tag{60}$$

and its derivative with respect to a becomes

$$\partial_a Q = 2\frac{\partial_a v\, \partial_a^2 v}{v^2} - \frac{3}{2}\left(\frac{\partial_a v}{v}\right)^3 - \frac{1}{2}\frac{\partial_a^3 v}{v}. \tag{61}$$

A positive quantum potential leads to faster expansion of the universe.

3.3. Semi-classical examples

Since, in the standard physical units, the potential $Q \sim \hbar^2$ [32] (cf. Eq. (18)), then Eq. (58) can be solved via perturbation theory. In the Born approximation, we obtain

$$Q = \frac{5}{16}\left(\frac{\partial_a U}{E-U}\right)^2 + \frac{1}{4}\frac{\partial_a^2 U}{E-U}. \tag{62}$$

Let us take a look at a few specific examples.

3.3.1. Radiation dominance

For such a universe, $E \neq 0$, $M = 0$, and $\kappa = 0$. Then $U = 0$, and the quantum addition $Q = 0$. This result is consistent with the one obtained before in Ref. [34] under other assumptions.

3.3.2. Matter dominance

In the case, $E = 0$, $M \neq 0$, and $\kappa = 0$, we have $U = -2aM$, and

$$Q = \frac{5}{16}\frac{1}{a^2}, \tag{63}$$

if $M = \mathrm{const}$ (dust), and

$$Q = \frac{\lambda(a)}{a^2}, \tag{64}$$

where the coupling function is

$$\lambda(a) = \frac{5}{16}\left(1 + a\frac{\partial_a M}{M}\right)^2 + \frac{a}{4}\frac{\partial_a M}{M}\left[2 + a\frac{\partial_a^2 M}{\partial_a M}\right], \tag{65}$$

if $\partial_a M \neq 0$, and $\partial_a^2 M \neq 0$. The derivatives of M define the pressure

$$p_m = -\frac{2}{3a^2}\partial_a M \tag{66}$$

and its derivative

$$\partial_a p_m = \frac{4}{3a^3}\partial_a M - \frac{2}{3a^2}\partial_a^2 M. \tag{67}$$

The EoS parameters are

$$w_{\text{B}}^{\text{dust}} = 1 \text{ and } w_{\text{B}} = 1 - \frac{1}{3}\frac{d\ln\lambda}{d\ln a} \tag{68}$$

for the Bohm potentials (63) and (64), respectively.

3.3.3. Spatially flat universe with radiation and dust

For the universe with $\kappa = 0$, $E \neq 0$, and $M = \mathrm{const}$, we get $U = -2aM$ again, and the quantum potential is

$$Q = \frac{5}{4}\left(\frac{M}{E + 2aM}\right)^2. \tag{69}$$

It reduces to Eq. (63) in the case $E = 0$, and we have $Q = \frac{5}{4}\left(\frac{M}{E}\right)^2$ at the point $a = 0$, so that the quantum potential is non-singular.

The EoS parameter is

$$w_{\text{B}} = \frac{1}{3}\left(1 + \frac{4aM}{E + 2aM}\right). \tag{70}$$

3.3.4. Empty universe with spatial curvature

If $\kappa \neq 0$, $E = 0$, and $M = 0$, we obtain $U = \kappa a^2$ and

$$Q = \frac{3}{4}\frac{1}{a^2}, \ w_{\text{B}} = 1. \tag{71}$$

We see that the spatial curvature itself can generate quantum potential, even in the empty universe, with no dependence on the curvature parameter.

3.4. Quantum potential impact on expansion

Equation (58) can be rewritten in the form convenient for the comparison with general relativity

$$\left(\frac{v}{a^2}\right)^2 = \rho_{\text{mat}} + \rho_{\text{B}}, \tag{72}$$

where

$$\rho_{\text{mat}} = \frac{2M}{a^3} + \frac{E}{a^4} - \frac{\kappa}{a^2}, \quad \rho_{\text{B}} = \frac{Q}{a^4}, \tag{73}$$

ρ_{mat} is the total energy density with the contributions from matter with mass M, radiation, and the term accounting for the curvature of the space; ρ_{B} is the energy density stipulated by the quantum Bohm potential. As we can see, taking into account the quantum potential can affect the Hubble expansion rate $H = \frac{v}{a^2}$ [34]. Equation (72) can be viewed as the quantum analog of the first Friedmann equation, whereas Eq. (50) is the equation for acceleration $\frac{d^2a}{dT^2}$ with a quantum correction and it coincides with the corresponding equation of a homogeneous, isotropic model [32].

For illustrative purposes, we will do a few calculations.

In the simplest case of an empty universe, the quantum addition has a form (71). By integrating Eq. (58), we obtain the following relations,

$$e^{2T} = \left| a^2 + \sqrt{a^4 + \frac{3}{4}} \right|, \quad \text{for} \quad \kappa = 1, \tag{74}$$

$$\begin{aligned} &T = \frac{1}{2} \arcsin \frac{2}{\sqrt{3}} a^2, \qquad \text{if } a < \left(\frac{3}{4}\right)^{\frac{1}{4}}, \\ &e^{2iT} = \left| a^2 + \sqrt{a^4 - \frac{3}{4}} \right|, \ \text{if } a > \left(\frac{3}{4}\right)^{\frac{1}{4}} \\ &\text{for } \kappa = -1, \end{aligned} \tag{75}$$

For values $a \gg 1$, Eqs. (74) and (75) reduce to

$$\begin{aligned} a &= \frac{e^T}{\sqrt{2}} \quad \text{for } \kappa = 1, \\ a &= \frac{e^{iT}}{\sqrt{2}} \quad \text{for } \kappa = -1. \end{aligned} \tag{76}$$

The same expressions follow from the equations of general relativity, if we do not take quantum effects into account.

For a spatially flat universe filled with dust and radiation, the quantum additive has a form (69). Then from Eq. (58), it follows

$$\begin{aligned} \alpha T = &\frac{2\sqrt{x^3+1}}{x+1+\sqrt{3}} - \frac{\sqrt{3}-1}{3^{\frac{1}{4}}} F(\varphi, k) - \\ &- 2\sqrt{3}\, E(\varphi, k), \end{aligned} \tag{77}$$

where

$$\alpha = \left(\frac{4M}{5^{\frac{1}{4}}}\right)^{\frac{2}{3}}, \quad x = \frac{2aM + E}{\left(\frac{5}{4} M^2\right)^{\frac{1}{3}}}, \tag{78}$$

while $F(\varphi, k)$ and $E(\varphi, k)$ are elliptic integrals of the first and second kinds with

$$\varphi = \frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}, \quad k = \frac{\sqrt{2+\sqrt{3}}}{2}. \tag{79}$$

4. Discussion

In this paper, we draw an analogy between nonrelativistic quantum mechanics in the Madelung formulation and quantum geometrodynamics for a homogeneous and isotropic cosmological system. In quantum mechanics, one may pass from studying a particle moving in a force field to describing an ensemble of identical particles. The Schrödinger equation can be reduced to two hydrodynamic-type equations (11) and (12), one of which is the standard continuity equation, and the other is the Euler equation for the velocity of the fluid subjected to an additional force due to the quantum Bohm potential (9). We calculate this additional force for a number of example potentials that determine the motion of the fluid and obtain the coordinate dependence of the quantum potential.

The availability of a region for the fluid motion, where the quantum potential is non-vanishing, means that, according to Eq. (12), the motion of the fluid can differ considerably from that suggested by classical hydrodynamics. In particular, a quantum addition can be responsible for the diffraction pattern, when a particle is scattered on two slits [38]. Basically, the quantum potential can be given by Eq. (21), where the singularity at small coordinate values is singled out, and the dimensionless coupling function, strictly speaking, retains its dependence on the coordinate. From the considered examples, it can be concluded, in particular, that if the classical potential is negatively defined, then, for large values of the coordinate, the coupling function becomes constant, so that the quantum potential is inversely proportional to the square of the coordinate. For small values of the coordinate, the quantum potential itself can go to a constant. Only in the region, where the gradient of the quantum potential vanishes, it does not affect the velocity of the fluid.

In quantum geometrodynamics, we begin with Eqs. (36) and (37) that describe a homogeneous and isotropic universe. A state vector $\Psi(a, \phi; T)$ is considered as a function of the cosmic scale factor a that determines the geometry, a uniform scalar field ϕ responsible for matter in the universe, and a parameter T which describes the evolution of a system in conformal time. These two equations can be reduced to a single time-dependent equation (39) in $(2+1)$ dimensions. After the averaging over the states of the matter field, Eq. (39) is transformed into Eq. (42) in $(1+1)$ dimensions. Then, using a direct analogy with quantum mechanics in Madelung's formalism, the latter equation can be rewritten as two equations (45) and (46) for the amplitude of the wave function and its phase. The quantum potential (47) that emerges here is completely analogous to the quantum

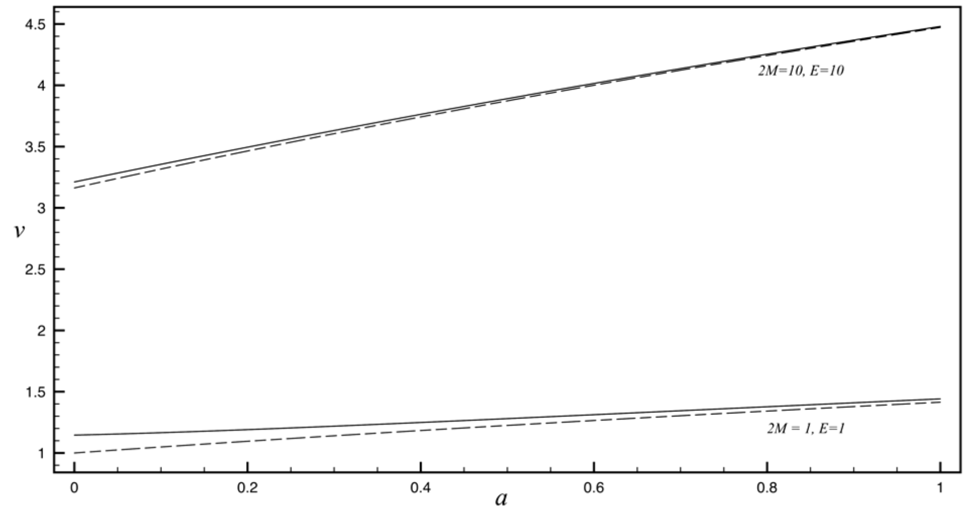

Fig. 1. The velocity v as a function on the scale factor a for different values of the parameters M and E. Dotted lines correspond to the cases $Q = 0$

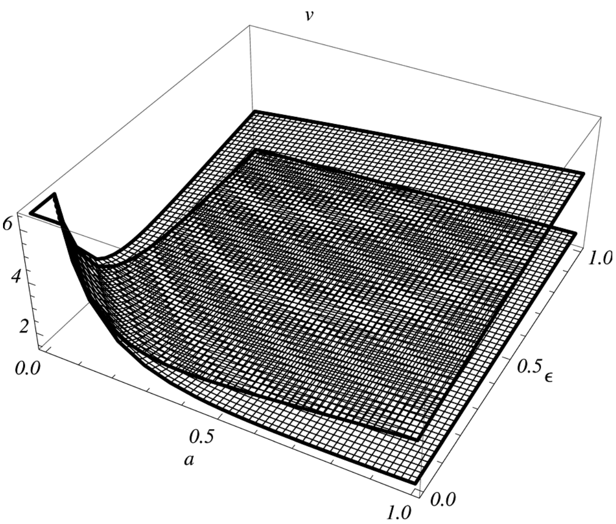

Fig. 2. The velocity v as a function on the scale factor a and the parameter $\varepsilon = \frac{E}{2M}$. The lower surface corresponds to the value $2M = 2$, and the upper surface to $2M = 10$

Bohm potential (9), except that the coordinate is substituted by the scale factor. In contrast to quantum mechanics, where the force field $V(x)$ is determined by the problem statement, in geometrodynamics the form of the potential function (43) is specific. After all the steps, Eqs. (45) and (46) can finally be reduced to two hydrodynamic-type equations (49) and (50).

In the hydrodynamic approach, the flow velocity $v = \frac{da}{dT}$ is the fundamental quantity through which the Hubble expansion rate $H = \frac{1}{a^2}\frac{da}{dT}$ is expressed.

The generalized force $\mathcal{F}$ (52), as in the case of quantum mechanics (see Eq. (12)), contains an additional force stipulated by the quantum potential gradient, which makes the corresponding equations (50) nonlinear with respect to the velocity field v. Similar to quantum mechanics, where nonlinear effects cause a diffraction pattern, the corresponding interference of classical trajectories should also occur in quantum geometrodynamics, but the consequences for observations remain unclear for now.

The transition to the so-called stationary states deserves a special mention. In contrast to quantum mechanics, the requirement of the state of the Universe being stationary is not imposed on the quantum system as an additional condition, but follows from the equations of quantum geometrodynamics themselves.

The explicit forms of the quantum potential in the leading Born approximation are given in Eqs. (63)–(71) for the universes dominated by one or another type of matter.

The impact of the quantum potential Q on the expansion of the universe is demonstrated in the form of explicit solutions (74)–(76) of Eq. (58) in the case of an empty universe with different spatial curvatures. The solution for a spatially flat universe filled with dust and radiation is given by Eq. (77).

The flow velocity v as a function on a is shown in Fig. 1 for the spatially flat universe filled with dust and radiation. The contribution of the quantum potential (69) to the evolution of the universe is noticeable only at small values of a. It is worth noting that, in this model, the flow velocity is not singular at the point $a = 0$. A three-dimensional picture for the function $v = v(a, \varepsilon)$, where $\varepsilon = \frac{E}{2M}$, is shown in Fig. 2[2]. This figure is for illustrative purposes only. Each point on the surfaces corresponds to one particular universe with specific parameter values. The set of universes has a singularity $v \to \infty$, when $a = 0$ and $\varepsilon \to 0$. For comparison, Fig. 3 represents the case where there are no quantum effects.

When the universe is filled with dust and radiation, from Eqs. (72) and (69), we obtain

$$v = a^2 \sqrt{\rho_m \left(1 + \frac{\rho_\gamma}{\rho_m}\right) + \frac{5}{16} \frac{1}{a^6} \left(1 + \frac{\rho_\gamma}{\rho_m}\right)^{-2}}. \qquad (80)$$

At the epoch, when matter dominates over radiation and $\frac{\rho_\gamma}{\rho_m} \ll 1$, the flow velocity v is

$$v = a^2 \sqrt{\rho_m + \frac{5}{16} \frac{1}{a^6}}. \qquad (81)$$

At the epoch, when radiation prevails over matter and $\frac{\rho_\gamma}{\rho_m} \gg 1$, we have

$$v = a^2 \sqrt{\rho_\gamma + \frac{5}{16} \frac{1}{a^6} \left(\frac{\rho_m}{\rho_\gamma}\right)^2}. \qquad (82)$$

In the region of values of a, where quantum effects can be neglected in Eq. (81), we pass to the classical limit, where general relativity is applicable. When matter dominates over radiation, we get

$$\frac{v}{a^2} = \sqrt{\rho_m}, \qquad (83)$$

where $\frac{v}{a^2} = H$ is a Hubble expansion rate.

If we introduce a cosmological constant Λ with energy density ρ_Λ into the model, Eq. (83) takes the form

$$\frac{v}{a^2} = \sqrt{\rho_m + \rho_\Lambda}. \qquad (84)$$

For the universe with $\rho_\Lambda \approx \rho_m$, comparing Eqs. (83) and (84), we see that the density associated with the cosmological constaint increases the expansion rate by a factor of $\sqrt{2}$.

[2] Different values ε for a constant contribution of non-relativistic matter M correspond to universes with different radiation densities.

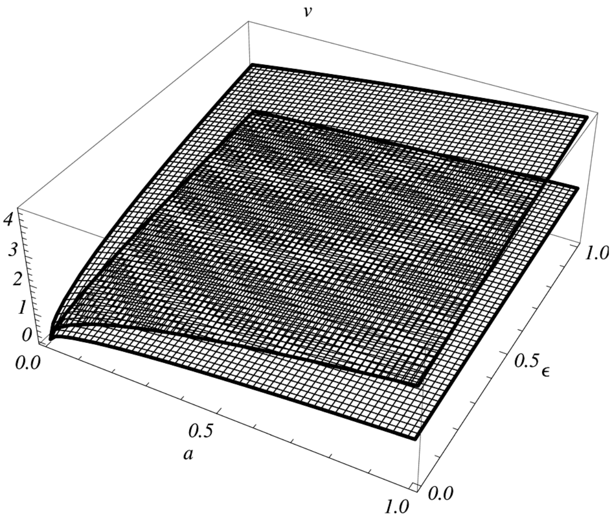

Fig. 3. The velocity v as a function of the scale factor a and the parameter $\varepsilon = \frac{E}{2M}$ for different values of M and $Q = 0$. See the caption to Fig. 2

The case of the dominance of radiation over matter requires a more careful analysis. It is possible that the quantum term in Eq. (82) cannot be neglected, despite the fact that it is proportional to the second power of a small value $\frac{\rho_m}{\rho_\gamma}$, and everything depends on how closely we approach the initial cosmological singularity $a = 0$. From Eq. (82), it follows that both terms under the square root can be estimated as being proportional to a^{-4}, and v tends to a constant.

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МІНІСУПЕРПРОСТОРОВА КВАНТОВА КОСМОЛОГІЯ У ФОРМАЛІЗМІ МАДЕЛУНГА–БОМА

Проведено аналогію між нерелятивістською квантовою механікою у формулюванні Маделунга та квантовою геометродинамікою для випадку максимально симетричного простору. Отримано рівняння, еквівалентні рівнянню неперервності та гідродинамічному рівнянню Ейлера, що описують еволюцію швидкості, введеної для випадку гіпотетичного потоку рідини, що характеризує космологічну систему. Показано, що ідеальна природа рідини порушується квантовим потенціалом Бома. Квантовий потенціал обчислено в напівкласичному наближенні для різних сил, що діють у системі, як у стандартній квантовій механіці, так і в мінісуперпросторовій моделі квантової геометродинаміки. Отримано явні залежності космічного масштабного фактора від конформного часу, які враховують квантову складову, для пустого простору з просторовою кривизною та для просторово плаского Всесвіту з пилом і випромінюванням.

Ключові слова: квантова теорія, гідродинаміка, космологія, геометродинаміка.