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N.A. SUSULOVSKA

Ivan Franko National University of Lviv,
Professor Ivan Vakarchuk Department for Theoretical Physics
(12, Drahomanov Str., Lviv 79005, Ukraine; e-mail: n.a.susulovska@gmail.com)

GEOMETRIC MEASURE OF ENTANGLEMENT OF QUANTUM GRAPH STATES PREPARED WITH CONTROLLED PHASE SHIFT OPERATORS

We will consider graph states generated by the action of controlled phase shift operators on a separable state of a multiqubit system. The case where all the qubits are initially prepared in arbitrary states is investigated. We will obtain the geometric measure of entanglement of a qubit with the remaining system in graph states represented by arbitrary weighted graphs and will establish its relationship with state parameters. For two-qubit graph states, the geometric measure of entanglement is also quantified on IBM's simulator Qiskit Aer and quantum processor ibmq lima based on auxiliary mean spin measurements. The results of quantum computations verify our analytic predictions.

Keywords: geometric measure of entanglement, multiqubit graph states, weighted graph, quantum computer.

1. Introduction

Over the past decades, a lot of efforts have been directed toward theorizing and implementing a variety of practical schemes and algorithms, which could leverage the amazing potential of quantum mechanics (as an example, see [1–7] and references therein). Studies of the quantum entanglement, indisputably considered one of the fundamental quantum-mechanical features [8, 9], took a central role in this endeavour and soon gave rise to quantum computing and quantum communications. Manifesting itself in peculiar long-range correlations, which result in nonfactorable states of composite systems, the quantum entanglement is viewed as an indispensable resource in a range of applications. For instance, this phenomenon lies in the foundation of quantum teleportation [1, 2], quantum cryptography [3] and allows the unprecedented capabilities of quantum computers [4]. These devices operate with superpositions of quantum states in high-dimensional Hilbert spaces and harness the power of entanglement to solve complex and often classically intractable computational problems [5–7]. In this light, exploring entangled quantum states and their physical properties, as well as the ways to efficiently prepare such states on a quantum computer, is of crucial importance.

Recently, graph states have received a considerable amount of attention due to their high degree and persistence of the entanglement [10–13]. These multipartite quantum states are widely used in areas such as quantum error correction [14–16], quantum metrology [17, 18], and quantum machine learning [19]. As for the latter, graph states appear, for instance, as a

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resource in the hybrid qGAN model, being produced by a single layer of a variational circuit constituting a quantum generator [20, 21]. With regards for such broad applicability, quantifying the entanglement of graph states becomes an important task, which has been considered in a number of studies both analytically and on the basis of quantum computations (for instance, see [21–27] and references therein). Various entanglement measures have been adopted for this purpose. For instance, in [22], the authors utilized the definition of negativity and performed a computationally heavy state tomography procedure to detect the full entanglement of graph states associated with ring graphs. In another range of studies [21, 24–29], the entanglement of different multiqubit states was determined as the distance between an entangled target state and the nearest separable state. This quantity, first introduced in [30], is known as the geometric measure of entanglement and can be formally represented by the expression

$$E(|\psi\rangle) = \min_{\{|\psi_s\rangle\}} (1 - |\langle\psi|\psi_s\rangle|^2), \quad (1)$$

where $|\psi\rangle$, $\{|\psi_s\rangle\}$ denote an entangled target state and a set of separable states, respectively, $d^2_{\mathrm{FS}}(|\psi\rangle, |\psi_s\rangle) = 1 - |\langle\psi|\psi_s\rangle|^2$ is a squared Fubini–Study distance between $|\psi\rangle$ and $|\psi_s\rangle$. In [31] similar geometric considerations were extended to derive a distance-based entanglement measure for hybrid systems of qudits.

In general, when estimating the entanglement on a quantum computer, selecting an entanglement measure, which can be directly connected to some easily measurable observable is a huge benefit. It was shown in [32] that, in order to calculate the geometric measure of entanglement of a spin 1/2, which can represent a qubit, with an arbitrary quantum system in a pure state $|\psi\rangle$ it is enough to obtain the mean value of this spin. This useful property is reflected in the following relation:

$$E(|\psi\rangle) = \frac{1}{2}(1 - |\langle\boldsymbol{\sigma}\rangle|), \quad (2)$$

where

$$|\langle\boldsymbol{\sigma}\rangle| = |\langle\psi|\boldsymbol{\sigma}|\psi\rangle| = \left(\langle\sigma^x\rangle^2 + \langle\sigma^y\rangle^2 + \langle\sigma^z\rangle^2\right)^{1/2}. \quad (3)$$

Here, σ^x, σ^y, σ^z are Pauli operators.

Similarly, the author of work [33] determined that von Neumann entanglement entropy of the partial traces in bipartite systems of two-level atoms can be obtained basing solely on the mean spin value corresponding to one of the atoms.

It should be stressed that the mean value of spin in an arbitrary pure state can be straightforwardly measured on a quantum computer according to the protocol described in [28]. Therefore, the quantity of entanglement associated with the corresponding quantum state can be estimated on the basis of such measurements. In recent years, many studies have exploited this idea to detect the geometric measure of entanglement of graph states with various structures. For instance, in [24] evolutionary graph states of spin systems with Ising interaction were considered. The authors of work [26] studied graph states generated by the action of controlled phase shift operators on the initial multiqubit state corresponding to the uniform superposition over the computational basis. Paper [27] also focused on graph states prepared with the help of controlled phase shift operators, in this case, starting from the state of the system, in which all of the qubits are in arbitrary identical states. These studies concluded that the geometric measure of the entanglement of an arbitrary qubit with other qubits in respective graph states depends on the degree of the corresponding graph vertex. This important result established a connection between a physical property of the quantum state and a geometric property of the graph used to describe it.

In the present study, we will elaborate on the findings presented in [27] and revisit the problem of entanglement quantification for a class of multiqubit graph states generated by the action of controlled phase shift operators. We will examine a more general case where the system of qubits is initially prepared in an arbitrary separable state. In addition, parameters of controlled phase shift operators corresponding to different graph edges take independent arbitrary values. This means that weighted graphs are used to represent quantum states under investigation. As a result of the analytic considerations, a general expression for the geometric measure of entanglement of an arbitrary qubit with the rest of the system in a state corresponding to an arbitrary weighted graph is derived. We examine how this quantity depends on the initial state of the multiqubit system, as well as the parametrization of the entangling controlled phase shift operators. Furthermore, the geometric measure of entanglement for a selection of graph states is quantified both on IBM's quantum simulator *Qiskit Aer* and real quantum backend *ibmq lima*.

The paper is organized as follows. Section 2 is devoted to the analytic derivation of the geometric measure of entanglement of a qubit with other qubits in a graph state associated with an arbitrary weighted graph and its subsequent analysis. In Section 3, we detect the geometric measure of entanglement of two-qubit graph states on IBM's simulator *Qiskit Aer* and quantum computer *ibmq lima* [34] and discuss the obtained results. Conclusions are presented in Section 4.

2. Analytical Consideration of the Geometric Measure of Entanglement of Graph States

Consider an arbitrary multiqubit state characterized by a general structure

$$|\psi_G\rangle = \prod_{(i,j)\in E} U_{ij}|\psi_{\text{init}}\rangle, \tag{4}$$

where $|\psi_{\text{init}}\rangle$ is an initial separable state of the system and U_{ij} represents a two-qubit entangling unitary acting on states of qubits q_i, q_j. We can establish a one-to-one mapping between the class of quantum states (4) and a set of graphs $G(V,E)$. In this context, the set of graph vertices V represents qubits, whereas the set of graph edges E is associated with two-qubit operators acting on their initial states. Each unitary operator U_{ij} can be written in the exponential form

$$U_{ij} = \mathrm{e}^{\mathrm{i}\phi_{ij}H_{ij}}, \tag{5}$$

where H_{ij} is the Hermitian operator, ϕ_{ij} is a scalar parameter.

In the present study, let us begin by considering a system of N qubits in the initial separable state

$$|\psi_{\text{init}}\rangle = \prod_{k\in V} |\psi(\alpha_k,\theta_k)\rangle, \tag{6}$$

where

$$|\psi(\alpha_k,\theta_k)\rangle = \cos\frac{\theta_k}{2}\,|0\rangle + \mathrm{e}^{\mathrm{i}\alpha_k}\sin\frac{\theta_k}{2}\,|1\rangle \tag{7}$$

is an arbitrary one-qubit state, $0 \leq \alpha_k < 2\pi$, $0 \leq \theta_k \leq \pi$, $k \in V = \{0,...,N-1\}$. Conveniently, state (7) can be prepared with the help of the parameterized rotation operators $RY(\theta_k)$, $RZ(\alpha_k)$ acting on state $|0\rangle$ (accurate to the phase factor)

$$|\psi(\alpha_k,\theta_k)\rangle = \mathrm{e}^{-\mathrm{i}\frac{\alpha_k}{2}} RZ(\alpha_k)RY(\theta_k)|0\rangle, \tag{8}$$

here $RY(\theta_k) = \exp(-\mathrm{i}\theta_k\sigma_k^y/2)$ and $RZ(\alpha_k) = \exp(-\mathrm{i}\alpha_k\sigma_k^z/2)$.

Subsequently, a graph state associated with a weighted graph of a predefined structure can be obtained by applying the controlled phase shift operator $CP_{ij}(\phi_{ij})$ to each pair of qubits q_i, q_j represented by vertices linked with an edge of weight ϕ_{ij}. This two-qubit operator is defined as $CP_{ij}(\phi_{ij}) = |0\rangle_{ii}\langle 0|\otimes I_j + |1\rangle_{ii}\langle 1|\otimes P_j(\phi_{ij})$, where I_j is an identity operator and $P_j(\phi_{ij}) = |0\rangle_{jj}\langle 0| + \mathrm{e}^{\mathrm{i}\phi_{ij}}|1\rangle_{jj}\langle 1|$ is a phase shift operator acting on qubit q_j, $0 \leq \phi_{ij} < 2\pi$, $(i,j)\in E$.

Note that we deal with weighted graphs and consider the case where all the phase shift parameters ϕ_{ij}, $(i,j)\in E$ take different values. The resulting graph state reads

$$|\psi_G\rangle = \prod_{(i,j)\in E} CP_{ij}(\phi_{ij}) \prod_{k\in V} |\psi(\alpha_k,\theta_k)\rangle, \tag{9}$$

where $|\psi(\alpha_k,\theta_k)\rangle$ is given by (7) and $CP_{ij}(\phi_{ij})$ acts on qubits q_i, q_j as a control and a target, respectively.

Here, we can resort to the exponential form of the controlled phase shift operator

$$CP_{ij}(\phi_{ij}) = \mathrm{e}^{\frac{\mathrm{i}\phi_{ij}}{4}(I_i-\sigma_i^z)(I_j-\sigma_j^z)}. \tag{10}$$

One can notice that our choice of entangling two-qubit operators allows us to simulate Ising interaction in systems of many spins equal to 1/2.

Let us analytically estimate the geometric measure of entanglement of an arbitrary qubit q_l with the remaining system in state (9) described by an arbitrary weighted graph. Note that we essentially study bipartite entanglement with one of the subsystems being represented by qubit q_l and the rest of the qubits constituting the second subsystem. As follows from (2), our objective can be achieved by calculating the mean value of the corresponding Pauli operator $\langle\boldsymbol{\sigma}_l\rangle$ in the graph state. Namely, one has to separately consider $\langle\sigma_l^x\rangle$, $\langle\sigma_l^y\rangle$, $\langle\sigma_l^z\rangle$. Hereafter, we use notation $\langle ...\rangle = \langle\psi_G|...|\psi_G\rangle$.

Accounting for (8) yields

$$\langle\sigma_l^x\rangle = \langle\psi_0|\prod_{q\in V}\mathrm{e}^{\mathrm{i}\frac{\theta_q}{2}\sigma_q^y}\mathrm{e}^{\mathrm{i}\frac{\alpha_q}{2}\sigma_q^z}\prod_{(j,k)\in E} CP^{+}_{jk}(\phi_{jk})\sigma_l^x \times \times \prod_{(m,n)\in E} CP_{mn}(\phi_{mn})\prod_{p\in V}\mathrm{e}^{-\mathrm{i}\frac{\alpha_p}{2}\sigma_p^z}\mathrm{e}^{-\mathrm{i}\frac{\theta_p}{2}\sigma_p^y}|\psi_0\rangle, \tag{11}$$

where $|\psi_0\rangle = |0\rangle^{\otimes V}$. To simplify this expression, we make use of identity (10) and take into consideration that the Pauli operators σ_l^x and σ_l^z anticommute; thus, obtaining:

$$\prod_{(j,k)\in E} CP^{+}_{jk}(\phi_{jk})\sigma_l^x \prod_{(m,n)\in E} CP_{mn}(\phi_{mn}) =$$
$$= \prod_{(j,k)\in E} \mathrm{e}^{-\mathrm{i}\frac{\phi_{jk}}{4}(I_j-\sigma_j^z)(I_k-\sigma_k^z)}\, \sigma_l^x \times$$
$$\times \prod_{(m,n)\in E} \mathrm{e}^{\mathrm{i}\frac{\phi_{mn}}{4}(I_m-\sigma_m^z)(I_n-\sigma_n^z)} =$$
$$= \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma_l^z \sum_{j\in N_G(l)} \phi_{jl}(I_j-\sigma_j^z)}\sigma_l^x, \tag{12}$$

where $N_G(l)$ denotes a set of vertices adjacent to the vertex l, known as its neighborhood. Eventually, we find

$$\langle\sigma_l^x\rangle = \sin\theta_l \,\mathrm{Re}\, z. \tag{13}$$

Here $z \in \mathbb{C}$ reads

$$z = \mathrm{e}^{-\mathrm{i}\left(\alpha_l + \frac{1}{2}\sum_{j\in N_G(l)} \phi_{jl}\right)} \times$$
$$\times \prod_{k\in N_G(l)} \left(\cos\frac{\phi_{kl}}{2} + \mathrm{i}\,\sin\frac{\phi_{kl}}{2}\cos\theta_k\right), \tag{14}$$

where $\sum_{j\in N_G(l)} \phi_{jl}$ is a weighted degree of the vertex denoting qubit q_l in the graph.

Performing similar mathematical transformations, we easily obtain the result for mean value $\langle\sigma_l^y\rangle$

$$\langle\sigma_l^y\rangle = -\sin\theta_l \,\mathrm{Im}\, z, \tag{15}$$

where z is given by the same expression (14).

Lastly,

$$\langle\sigma_l^z\rangle = \langle\psi_0|\mathrm{e}^{\mathrm{i}\frac{\theta_l}{2}\sigma_l^y}\,\sigma_l^z \mathrm{e}^{-\mathrm{i}\frac{\theta_l}{2}\sigma_l^y}|\psi_0\rangle = \cos\theta_l. \tag{16}$$

Eventually, to find the geometric measure of entanglement of qubit q_l with the rest of the qubits in state (9), we substitute mean values (13), (15), (16) into central expression (2) and obtain

$$E_l(|\psi_G\rangle) = \frac{1}{2}\left(1 - \left[\sin^2\theta_l \prod_{k\in N_G(l)} \left(\cos^2\frac{\phi_{kl}}{2} + \right.\right.\right.$$
$$\left.\left.\left. + \sin^2\frac{\phi_{kl}}{2}\cos^2\theta_k\right) + \cos^2\theta_l\right]^{1/2}\right). \tag{17}$$

As evident from (17), this entanglement measure depends on absolute values of a subset of parameters $\{\theta_m\}$ defining the initial state of the multiqubit system, which corresponds to a closed neighborhood $N_G[l]$ of vertex l representing the qubit under consideration (vertex l itself combined with a set of its adjacent vertices), $m \in N_G[l]$. It also depends on the absolute values of a set of parameters $\{\phi_{kl}\}$ passed to the controlled phase shift operators responsible for the generation of edges incident to vertex l, $k \in N_G(l)$.

Consider a special case of graph states (9) where all the qubits of the system are initially prepared in identical states (7) so that $\theta_m = \theta$, $m \in N_G[l]$. In addition assume that all the controlled phase shift operators associated with graph edges share the same parameter, hence, $\phi_{kl} = \phi$, $k \in N_G(l)$. Under these constraints (17) is reduced to

$$E_l(|\psi_G\rangle) = \frac{1}{2}\left(1 - \left[\sin^2\theta\left(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\cos^2\theta\right)^{n_l} + \right.\right.$$
$$\left.\left. + \cos^2\theta\right]^{1/2}\right), \tag{18}$$

which coincides with the expression obtained in [27]. Therefore, the results of the present study generalize our previous findings. Note that, in (18), the geometric measure of entanglement of qubit q_l explicitly depends on the degree of the corresponding graph vertex n_l, which is equal to the number of vertices in its neighborhood.

3. Investigating the Relation of the Geometric Measure of Entanglement of Graph States to the Mean Spin on a Quantum Device

In order to put our theoretical findings to the test, we examine two-qubit graph states of structure (9) on a quantum device and quantify their geometric measure of entanglement. Such states can be explicitly written as

$$|\psi_{G_2}\rangle = CP_{01}(\phi_{01})|\psi(0,\theta_0)\rangle|\psi(0,\theta_1)\rangle =$$
$$= CP_{01}(\phi_{01})\left(\cos\frac{\theta_0}{2}|0\rangle + \sin\frac{\theta_0}{2}|1\rangle\right) \times$$
$$\times \left(\cos\frac{\theta_1}{2}|0\rangle + \sin\frac{\theta_1}{2}|1\rangle\right) \tag{19}$$

and associated with a graph depicted in Fig. 1. Graph state (19) is determined by a set of three parameters, namely, θ_0, θ_1 defining the initial states of qubits in the bipartite system, and ϕ_{01} corresponding to the entangling gate parameter and, therefore, to the weight of the graph edge. Note that we set relative phases of the initial one-qubit states α_0, α_1 in (7) equal to zero, since they don't have any impact on the geometric measure of entanglement, according to expression (17).

On the basis of (8), we further obtain

$$|\psi_{G_2}\rangle = CP_{01}(\phi_{01})\mathrm{e}^{-\frac{\mathrm{i}\theta_1}{2}\sigma_1^y}\mathrm{e}^{-\frac{\mathrm{i}\theta_0}{2}\sigma_0^y}|00\rangle = \\ = CP_{01}(\phi_{01})RY_1(\theta_1)RY_0(\theta_0)|00\rangle, \qquad (20)$$

which shows that an arbitrary two-qubit graph state can be prepared by the consecutive action of two rotation RY gates and the controlled phase shift gate on the traditional initial state of a quantum register $|00\rangle$. See the protocol for generating quantum states of this structure corresponding to different weighted graphs on a gate-based quantum computer in Fig. 2.

For the purposes of this research, we aim to analyze how the geometric measure of entanglement of one qubit with another in graph state (19) is influenced by the choice of the initial separable state of the bipartite system, as well as the parameter of the entangling gate. Hence, two special cases are investigated in detail.

Firstly, we consider a subclass of graph states (19) with both parameters θ_0, θ_1 equal to $\pi/2$ and track the dependence of the geometric measure of entanglement on the parameter of the controlled phase shift gate ϕ_{01} as it runs in the interval $[0, 2\pi)$. Note that, in this case, the initial one-qubit states coincide with state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ (an eigenstate of Pauli-X operator corresponding to eigenvalue 1) and can be prepared with the help of Hadamard operators H. We have

$$|\psi_{G_2}(\phi_{01})\rangle = CP_{01}(\phi_{01})H_1H_0|00\rangle. \qquad (21)$$

According to (17), the geometric measure of entanglement of one qubit with another one in state (21) is reduced to

$$E(|\psi_{G_2}(\phi_{01})\rangle) = \frac{1}{2}\left(1 - \left|\cos\frac{\phi_{01}}{2}\right|\right). \qquad (22)$$

For the second subclass of graph states, we fix parameter ϕ_{01} equal to π. Since $CZ = CP(\pi)$, this

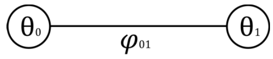

Fig. 1. A graph representing state (19)

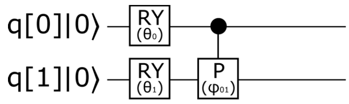

Fig. 2. Quantum protocol for preparing two-qubit graph states (19)

means in practice that the controlled-Z operators CZ are used to generate graph edges and the resulting states take the form

$$|\psi_{G_2}(\theta_0, \theta_1)\rangle = CZ_{01}RY_1(\theta_1)RY_0(\theta_0)|00\rangle. \qquad (23)$$

In this setting we can show the dependency of the geometric measure of entanglement on the initial state parameters by letting them variate independently in the interval $[0, \pi]$. Analytically, (17) yields

$$E(|\psi_{G_2}(\theta_0, \theta_1)\rangle) = \\ = \frac{1}{2}\left(1 - \left[\cos^2\theta_0 + \cos^2\theta_1 - \cos^2\theta_0\cos^2\theta_1\right]^{1/2}\right). \qquad (24)$$

Expression (2) suggests that the geometric measure of entanglement in multiqubit systems can be detected on a quantum device through the measurements of the mean spin. Namely, one has to obtain mean values of Pauli operators σ^x, σ^y, σ^z on the basis of quantum computations. This can be achieved, for instance, by following the approach presented in [28]. In our particular case, assume that we would like to estimate the geometric measure of entanglement of qubit q_0 in a certain graph state of a structure (19). It was shown that $\langle\sigma_0^x\rangle$, $\langle\sigma_0^y\rangle$, $\langle\sigma_0^z\rangle$ can be expressed in terms of probabilities defining the results of measurements performed on qubit q_0 in the computational basis $\{|0\rangle, |1\rangle\}$. We have

$$\langle\sigma_0^x\rangle = \langle\psi_{G_2}|\sigma_0^x|\psi_{G_2}\rangle = \langle\tilde{\psi}_{G_2}^y|\sigma_0^z|\tilde{\psi}_{G_2}^y\rangle = \\ = |\langle\tilde{\psi}_{G_2}^y|0\rangle|^2 - |\langle\tilde{\psi}_{G_2}^y|1\rangle|^2, \qquad (25)$$

$$\langle\sigma_0^y\rangle = \langle\psi_{G_2}|\sigma_0^y|\psi_{G_2}\rangle = \langle\tilde{\psi}_{G_2}^x|\sigma_0^z|\tilde{\psi}_{G_2}^x\rangle = \\ = |\langle\tilde{\psi}_{G_2}^x|0\rangle|^2 - |\langle\tilde{\psi}_{G_2}^x|1\rangle|^2, \qquad (26)$$

$$\langle\sigma_0^z\rangle = \langle\psi_{G_2}|\sigma_0^z|\psi_{G_2}\rangle = |\langle\psi_{G_2}|0\rangle|^2 - |\langle\psi_{G_2}|1\rangle|^2, \qquad (27)$$

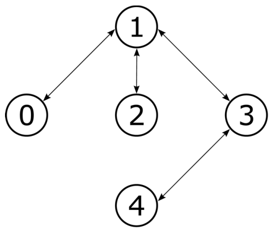

Fig. 3. Architecture of IBM's quantum computer *ibmq lima*. Arrows connect the qubits, to which CNOT gates can be directly applied. Each qubit can play a role of both a control and a target

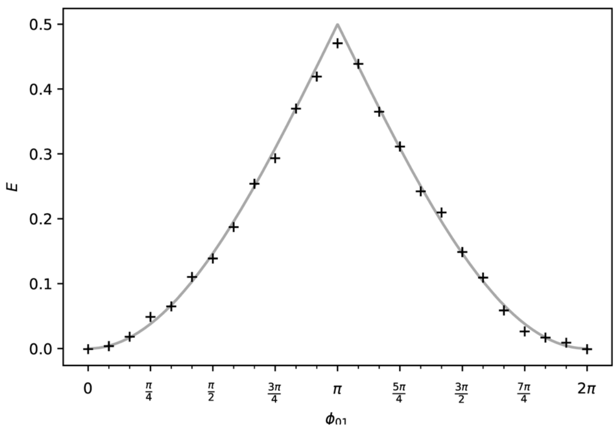

Fig. 4. Results of quantifying the geometric measure of entanglement of qubit q_0 with qubit q_1 in graph state (21) on IBM's simulator *Qiskit Aer* (marked with crosses) and analytical results (represented with a line)

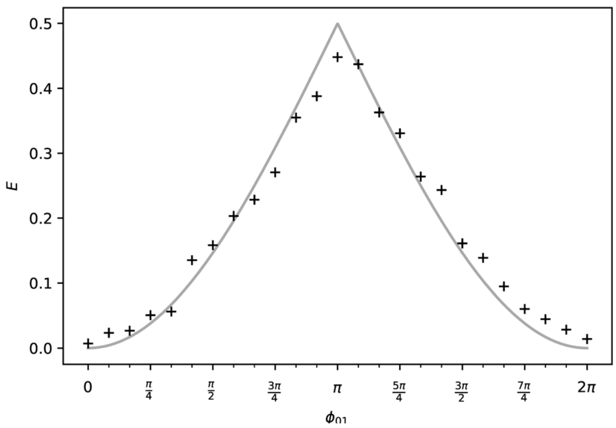

Fig. 5. Results of quantifying the geometric measure of entanglement of qubit q_0 with qubit q_1 in graph state (21) on IBM's quantum computer *ibmq lima* (marked with crosses) and analytic results (represented with a line)

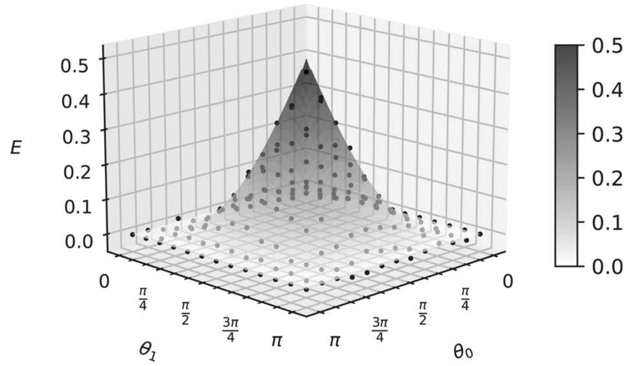

Fig. 6. Results of quantifying the geometric measure of entanglement of qubit q_0 with qubit q_1 in graph state (23) on IBM's simulator *Qiskit Aer* (marked with dots) and analytical results (represented with a continuous surface)

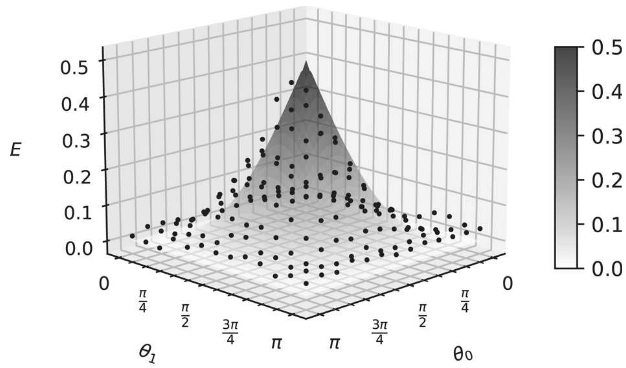

Fig. 7. Results of quantifying the geometric measure of entanglement of qubit q_0 with qubit q_1 in graph state (23) on IBM's quantum computer *ibmq lima* (marked with dots) and analytic results (represented with a continuous surface)

where

$$|\tilde{\psi}^x_{G_2}\rangle = \mathrm{e}^{-\mathrm{i}\pi\sigma^x_0/4}|\psi_{G_2}\rangle = RX_0(\pi/2)|\psi_{G_2}\rangle, \tag{28}$$

$$|\tilde{\psi}^y_{G_2}\rangle = \mathrm{e}^{\mathrm{i}\pi\sigma^y_0/4}|\psi_{G_2}\rangle = RY_0(-\pi/2)|\psi_{G_2}\rangle. \tag{29}$$

According to identities (28), (29), in order to detect mean values $\langle\sigma^x_0\rangle$, $\langle\sigma^y_0\rangle$ the state of qubit q_0 has to be rotated by angle $\pi/2$ around y and x axes, respectively, prior to the measurements in the computational basis.

In the present study, we prepare graph states (21), (23) with the help of quantum circuits generalized in Fig. 2 and estimate their entanglement according to the protocol described above on both IBM's simulator *Qiskit Aer* and real quantum backend *ibmq lima* [34]. The latter is a universal gate-based quantum processor consisting of 5 superconducting qubits connected according to the map in Fig. 3.

Note that, when preparing an arbitrary two-qubit state on a 5-qubit quantum computer, one has multiple choices of which physical qubits to utilize in the appropriate quantum circuit. Therefore, at the time of experiments calibration parameters of *ibmq lima* including the readout error, as well as one- and two-qubit gate errors were taken into account to minimize the cumulative error of computations.

To begin with, we quantify the geometric measure of entanglement of qubit q_0 with qubit q_1 in graph state $|\psi_{G_2}(\phi_{01})\rangle$ for various values of the controlled phase shift gate parameter $\phi_{01} \in [0, 2\pi]$ (see Figs. 4, 5). The obtained experimental dependency is then compared to the analytic one on the basis of expression (22). It's easy to see that, on the given interval, the geometric measure of entanglement reaches its maximum at $\phi_{01} = \pi$, and the respective experimental value is close to the theoretical prediction of 1/2. Note that this point on the plot corresponds to the graph state prepared by the action of the controlled-Z gate CZ

$$|\psi_{G_2}\rangle = CZ_{01}H_1H_0|00\rangle. \qquad (30)$$

Hereafter, the slight quantitative misalignment between the results of quantum computations and the theoretical ones can be explained by the errors inherent to the quantum hardware that were mentioned before. The geometric measure of entanglement takes its minimal value of 0 at points $\phi_{01} = 0,\, 2\pi$, which corresponds to the case where the controlled phase shift gate CP is reduced to the identity gate I, and the resulting two-qubit state is separable.

Similarly, the geometric measure of entanglement of qubit q_0 in graph state $|\psi_{G_2}(\theta_0, \theta_1)\rangle$ is obtained for various values of the initial state parameters $\theta_0, \theta_1 \in [0, \pi]$ (see Figs 6, 7). We show that the results of quantum computations are consistent with previously derived analytic expression (24). In particular, graph state (23) is maximally entangled, when both of the initial state parameters θ_0, θ_1 are equal to $\pi/2$. Considering that $RY(\pi/2) = XH$, and state $|+\rangle$ is an eigenstate of the operator Pauli-X associated with eigenvalue 1, this brings us back to the state (30), as expected. On the contrary, the graph state under consideration becomes separable (the geometric measure of entanglement goes to 0) if at least one of the initial state parameters θ_0, θ_1 is set equal to 0 or π. From the standpoint of quantum programming, to prepare such one-qubit states, the rotation RY gate is replaced in the quantum circuit by the identity gate I and the Pauli-Y gate (accurate to a phase factor), respectively.

As expected, in both Figs. 4, 5 and Figs. 6, 7, the results produced by a real quantum processor deviate from the theoretical results slightly more than the simulated ones due to the presence of noise. However, there is still a good agreement with our analytic predictions. Note that the associated computational errors are not significant, since we only consider two-qubit systems and all the quantum circuits executed for the purposes of this study are fairly shallow.

4. Conclusions

In this paper, a class of graph states (9) obtained as a result of the action of controlled phase shift operators on the initial separable state of a multiqubit system, in which all of the qubits are in arbitrary states, has been considered. We have derived an analytic expression (17) for the geometric measure of entanglement of an arbitrary qubit with other qubits in a graph state belonging to this class, which is described by an arbitrary weighted graph. This expression shows that the geometric measure of entanglement is related to the subset of absolute values of initial state parameters corresponding to the closed neighborhood of the vertex representing the qubit under consideration, as well as the set of absolute values of parameters passed to the controlled phase shift operators responsible for generating its incident edges.

In addition, the entanglement of two-qubit graph states (19) has been studied on the basis of quantum computations. Namely, we have proposed a protocol for preparing such quantum states (see Fig. 2) and detected the geometric measure of entanglement associated with their two special cases through auxiliary mean spin measurements on IBM's quantum simulator *Qiskit Aer* and quantum device *ibmq lima.* In the first case, both of the initial state parameters θ_0, θ_1 have been set equal to $\pi/2$ (21), which has allowed us to estimate the dependence of the entanglement on the parameter of the phase shift gate ϕ_{01}. Alternatively, in the second case, we have generated graph states with parameter ϕ_{01} equal to π (23) and analyzed how the choice of the initial separable bipartite state impacts the entanglement in the system. The results obtained on a quantum device in the course of this research are in a good agreement with analytic ones.

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Н. Сусуловська

ГЕОМЕТРИЧНА МІРА
ЗАПЛУТАНОСТІ КВАНТОВИХ ГРАФОВИХ СТАНІВ,
УТВОРЕНИХ ЗА ДОПОМОГОЮ ОПЕРАТОРІВ
КОНТРОЛЬОВАНОГО ЗСУВУ ФАЗИ

Розглядаються графові стани, утворені дією операторів контрольованого зсуву фази на факторизований стан багатокубітної системи. Досліджено випадок, коли кожен з кубітів початково приготований у довільному стані. Отримано геометричну міру заплутаності кубіта з іншими кубітами системи у графовому стані, що відповідає довільному зваженому графу, та встановлено її зв'язок з параметрами цього стану. Окрім цього, для двокубітних графових станів здійснено кількісну оцінку геометричної міри заплутаності на симуляторі *Qiskit Aer* та квантовому процесорі *ibmq lima* компанії IBM на основі вимірювань середнього значення спіна. Результати квантових обчислень узгоджуються з аналітичними оцінками.

Ключові слова: геометрична міра заплутаності, багатокубітні графові стани, зважений граф, квантовий комп'ютер.