A.M. PAVLYUK

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143, Ukraine; e-mail: pavlyuk@bitp.kiev.ua)

HOMFLY POLYNOMIAL INVARIANTS
OF TORUS KNOTS AND BOSONIC (q, p)-CALCULUS[1]

PACS 02.10.Kn, 02.20.Uw

For the one-parameter Alexander (Jones) skein relation we introduce the Alexander (Jones) "bosonic" q-numbers, and for the two-parameter HOMFLY skein relation we propose the HOMFLY "bosonic" (q, p)-numbers ("bosonic" numbers connected with deformed bosonic oscillators). With the help of these deformed "bosonic" numbers, the corresponding skein relations can be reproduced. Analyzing the introduced "bosonic" numbers, we point out two ways of obtaining the two-parameter HOMFLY skein relation ("bosonic" (q, p)-numbers) from the one-parameter Alexander and Jones skein relations (from the corresponding "bosonic" q-numbers). These two ways of obtaining the HOMFLY skein relation are equivalent.

K e y w o r d s: polynomial invariant; knot; link; Alexander, Jones, and HOMFLY skein relations; "bosonic" q-numbers; "bosonic" (q, p)-numbers.

1. Introduction

The knot theory is substantially based on the axioms of skein relation and normalization [1] allowing one to describe every knot and link by a definite polynomial. These polynomials form the set of polynomial invariants. The goal of this paper is to show that every of the three polynomial invariants (Alexander, Jones, HOMFLY) can be put into correspondence to the definite "bosonic" q-numbers/(q, p)-numbers ($[n]^A$, $[n]^V$, $[n]^H$), which allow one to reproduce the corresponding skein relation. Comparing these deformed numbers, we find the rule of obtaining the two-parameter HOMFLY polynomial invariants of knots and links from one-parameter (Alexander, Jones) polynomial invariants.

The so-called (q, p)-numbers, which are an important ingredient of (q, p)-calculus, generalizing the well-known q-calculus [2], appear in the connection with (q, p)-deformed bosonic oscillators [3]. Recently, with the help of these (q, p)-numbers, we have shown for the important case of torus knots that the generalized (two-variable) Alexander polynomials can be obtained [4, 5], as well as the generalized (two-variable) Jones polynomials [6, 7].

2. Skein Relations

The Alexander polynomials $\Delta(t)$ for knots and links are defined by the Alexander skein relation [8] together with the normalization condition for the unknot:

$$\Delta_+(t) - \Delta_-(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_O(t), \Delta_{\text{unknot}} = 1. \quad (1)$$

The Jones polynomials $V(t)$ are described by the following skein relation and normalization condition [9]:

$$t^{-1}V_+(t) - tV_-(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_O(t), V_{\text{unknot}} = 1. \quad (2)$$

The HOMFLY polynomials $H(a, z)$ are introduced in the following way [10]:

$$a^{-1}H_+(a, z) - aH_-(a, z) = zH_O(a, z), \ H_{\text{unknot}} = 1.$$

For our goal, it is necessary to make a change of the variable $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. Thus, the HOMFLY skein relation can be rewritten in the form

$$a^{-1}H_+(a, t) - aH_-(a, t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})H_O(a, t). \quad (3)$$

Let us write the skein relation in the general form

$$P_{L_+}(t) = l_1 P_{L_O}(t) + l_2 P_{L_-}(t), \quad (4)$$

where l_1 and l_2 are coefficients. The capital letter "L" stands for "Link" and denotes one of the two: knot or link (unknot belongs to knots). Here, three polynomials $P_{L_+}(t)$, $P_{L_O}(t)$, $P_{L_-}(t)$ correspond to the overcrossing Link L_+ ("overcrossing" refers to a chosen

crossing of the Link), zero crossing Link L_O, and undercrossing Link L_-. Thus, applying the surgery operation of elimination of a crossing to an initial Link L_+, one obtains a simpler Link L_O. The Link L_- is obtained from the same initial Link L_+ by the another surgery operation of switching of the crossing.

Consider the simplest torus knots $T(2m+1,2)$ and torus links $L(2m,2)$, where $m = 0,1,2,3,\ldots$. The common notation for these torus knots and links $L_{n,2}$ corresponds to torus knots, if n is odd, and to torus links for even n. The surgery operation of elimination turns $L_{n,2}$ into $L_{n-1,2}$, and the switching operation turns it into $L_{n-2,2}$. Because of it, the very important property follows from (4), namely, the series of polynomials $P_{L_{n,2}}(t)$ is characterized by the recurrence relation

$$P_{L_{n+1,2}}(t) = l_1 P_{L_{n,2}}(t) + l_2 P_{L_{n-1,2}}(t), \qquad (5)$$

which repeats itself in the skein relation (4). We now rewrite formula (5) in a simpler notation

$$P_{n+1,2}(t) = l_1 P_{n,2}(t) + l_2 P_{n-1,2}(t). \qquad (6)$$

For the polynomials $P_{L_{n,2}}(t)$ with odd n, relation (6) yields the recurrence relation referred only to the torus knots $T(2m+1,2)$

$$P_{n+2,2}(t) = k_1 P_{n,2}(t) + k_2 P_{n-2,2}(t), \qquad (7)$$

where the coefficients k_1 and k_2 are expressed through l_1 and l_2 [4] as

$$k_1 = l_1^2 + 2l_2, \quad k_2 = -l_2^2. \qquad (8)$$

From (7) and the normalization condition

$$P_{1,2} = 1 \qquad (9)$$

we obtain for the trefoil

$$P_{3,2} = k_1 + k_2. \qquad (10)$$

3. "Bosonic" (q,p)-numbers

The one-parameter "bosonic" q-number (structural function) characteristic of a Biedenharn–Macfarlane deformed bosonic oscillator corresponding to an integer n is defined as [11, 12]

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad (11)$$

with q to be a parameter. Some of the q-numbers are

$$[1]_q = 1, \quad [2]_q = q + q^{-1},$$

$$[3]_q = q^2 + 1 + q^{-2}, \quad [4]_q = q^3 + q + q^{-1} + q^{-3}, \ldots.$$

The recurrence relation for (11) looks as

$$[n+1]_q = (q + q^{-1})[n]_q - [n-1]_q. \qquad (12)$$

The two-parameter "bosonic" (q,p)-number corresponding to the integer number n is defined as [3]

$$[n]_{q,p} = \frac{q^n - p^n}{q - p}, \qquad (13)$$

where q,p are parameters. If $p = q^{-1}$, then $[n]_{q,p} = [n]_q$. Some of the q,p-numbers are given below:

$$[1]_{q,p} = 1, \quad [2]_{q,p} = q + p,$$
$$[3]_{q,p} = q^2 + qp + p^2, \quad [4]_{q,p} = q^3 + q^2 p + qp^2 + p^3, \ldots.$$

The recurrence relation for q,p-numbers is

$$[n+1]_{q,p} = (q+p)[n]_{q,p} - qp[n-1]_{q,p}. \qquad (14)$$

4. Alexander "Bosonic" q-numbers: $[n]^A$

From the Alexander skein relation (1) in the form (4)

$$\Delta_+(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_O(t) + \Delta_-(t), \qquad (15)$$

one has the "Link coefficients"

$$l_1^A = t^{\frac{1}{2}} - t^{-\frac{1}{2}}, \quad l_2^A = 1. \qquad (16)$$

From (15), we have the recurrence relation for Alexander polynomials of torus knots and links $L_{n,2}$ (by analogy to (6))

$$\Delta_{n+1,2}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{n,2}(t) + \Delta_{n-1,2}(t). \qquad (17)$$

Using (8) and (16), we obtain the "knot coefficients"

$$k_1^A = t + t^{-1}, \quad k_2^A = -1. \qquad (18)$$

Therefore, the recurrence relation for Alexander polynomials of torus knots $T(2m+1,2)$ looks as

$$\Delta_{n+2,2}(t) = (t + t^{-1})\Delta_{n,2}(t) - \Delta_{n-2,2}(t). \qquad (19)$$

Comparing (19) and (14) allows us to put (what we call) the Alexander "bosonic" q-numbers $[n]^A$ into correspondence to (19). Indeed, from $q + p = t + t^{-1}$, $qp = 1$, we have $q = t$, $p = t^{-1}$. Therefore, relation (13) yields

$$[n]^A = \frac{t^n - t^{-n}}{t - t^{-1}}, \quad t \equiv q, \qquad (20)$$

which coincides with q-numbers of Biedenharn and Macfarlane.

5. Jones "Bosonic" q-numbers: $[n]^V$

From the Jones skein relation (2) in the form (4)

$$V_+(t) = t(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_O(t) + t^2 V_-(t), \qquad (21)$$

we have the "Link coefficients"

$$l_1^V = t^{\frac{3}{2}} - t^{\frac{1}{2}}, \quad l_2^V = t^2, \qquad (22)$$

and, correspondingly, find the "knot coefficients"

$$k_1^V = t^3 + t, \quad k_2^V = -t^4. \qquad (23)$$

The recurrence relation for the Jones polynomials of torus knots $T(2m+1, 2)$ has the form

$$V_{n+2,2}(t) = (t^3 + t)V_{n,2}(t) - t^4 V_{n-2,2}(t). \qquad (24)$$

Comparing (24) and (14), we obtain what we call the Jones "bosonic" q-numbers

$$[n]^V = \frac{t^{3n} - t^n}{t^3 - t}, \quad t \equiv q, \qquad (25)$$

6. HOMFLY "Bosonic" (q, p)-numbers: $[n]^H$

The HOMFLY skein relation (3) in the form (4)

$$H_+(a,t) = a(t^{\frac{1}{2}} - t^{-\frac{1}{2}})H_O(a,t) + a^2 H_-(a,t) \qquad (26)$$

gives the "Link coefficients"

$$l_1^H = a(t^{\frac{1}{2}} - t^{-\frac{1}{2}}), \quad l_2^H = a^2. \qquad (27)$$

From whence, we find the "knot coefficients"

$$k_1^H = a^2(t + t^{-1}), \quad k_2^H = -a^4, \qquad (28)$$

which are used to introduce the HOMFLY "bosonic" (q, p)-numbers according to the relation

$$[n+1]^H = k_1^H [n]^H + k_2^H [n-1]^H. \qquad (29)$$

Comparing (29) and (14), from $q + p = k_1^H$, $qp = -k_2^H$, we have $q = at$ and $p = at^{-1}$. It follows from (13) what we call the HOMFLY "bosonic" (q, p)-numbers

$$[n]^H = a^{2(n-1)}\frac{t^n - t^{-n}}{t - t^{-1}}, \quad t \equiv q, \ a \equiv p. \qquad (30)$$

7. Alexander Skein Relation from Alexander "Bosonic" q-numbers

In Section 4, we obtained the Alexander "bosonic" q-numbers from the Alexander skein relation. In this section, moving in the opposite direction, we obtain the Alexander skein relation (15) from the Alexander "bosonic" q-numbers $[n]^A$ (20). First, comparing (13) and (20), we find $q = t$, $p = t^{-1}$. Putting it into (14), one has the recurrence relation for the Alexander "bosonic" q-numbers:

$$[n+1]^A = (t + t^{-1})[n]^A - [n-1]^A. \qquad (31)$$

From whence, we have "knot coefficients" (18):

$$k_1^A = t + t^{-1}, \quad k_2^A = -1.$$

According to (8), the "Link coefficients" are

$$l_2 = +(-k_2)^{\frac{1}{2}}, \quad l_1 = +(k_1 - 2l_2)^{\frac{1}{2}}. \qquad (32)$$

Thus, we obtain l_2^A and l_1^A, which coincide with (16). Putting them into (4) leads to the Alexander skein relation (15).

In a similar way, the Jones "bosonic" q-numbers $[n]^V$ (25) yield the Jones skein relation (21), and the HOMFLY skein relation (26) follows from the HOMFLY "bosonic" (q, p)-numbers $[n]^H$ (30).

8. HOMFLY invariants from Alexander and Jones invariants

In this section, we consider how to build two-parameter HOMFLY polynomial invariants on the basis of one-parameter Alexander or Jones ones. To formulate proper rule, we compare the HOMFLY "bosonic" (q, p)-numbers $[n]^H$ (30) and the Alexander "bosonic" q-numbers $[n]^A$ (20):

$$[n]^H = a^{2(n-1)}[n]^A. \qquad (33)$$

Then, the first way of obtaining the HOMFLY (q, p)-numbers reduces to introducing the second variable in the form of a multiplier $a^{2(n-1)}$ before $[n]^A$. In the case of the Jones "bosonic" q-numbers $[n]^V$, the multiplier looks as $(aq)^{2(n-1)}$:

$$[n]^H = (aq)^{2(n-1)}[n]^V. \qquad (34)$$

We suggest another way of obtaining the HOMFLY skein relation with the help of the "bosonic" (q, p)-numbers $[n]^{H_1}$. To this end, we make the substitution in (20):

$$t \to q, \quad t^{-1} \to p^{-1}$$

and, thus,

$$[n]^{H_1} = \frac{q^n - p^{-n}}{q - p^{-1}}. \tag{35}$$

One more substitution

$$q^{\frac{1}{4}}p^{-\frac{1}{4}} \to a, \quad q^{\frac{1}{2}}p^{\frac{1}{2}} \to t,$$

in (35) turns $[n]^{H_1}$ into $[n]^H$, which proves their equivalence.

In the case of the Jones "bosonic" q-numbers $[n]^V$, the substitution

$$t^3 \to q^3, \quad t \to p$$

in (20) turns it into

$$[n]^{H_2} = \frac{q^{3n} - p^n}{q^3 - p}. \tag{36}$$

By substituting

$$q^3 p \to a^4, \quad q^{\frac{3}{2}}p^{-\frac{1}{2}} \to t,$$

in (36), we turn $[n]^{H_2}$ into $[n]^H$.

9. Concluding Remarks

The introduced Alexander and Jones "bosonic" q-numbers and the HOMFLY "bosonic" (q, p)-numbers give possibility to operate with these deformed numbers instead of operating with the corresponding skein relations, which is believed to be much easier. We also hope that the dealing with the deformed numbers instead of the skein relations will promote the finding of new polynomial invariants of knots and links. It should be mentioned that the problem of searching for the Reidemeister moves in terms of (q, p)-calculus arises.

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А.М. Павлюк

ПОЛІНОМІАЛЬНІ ІНВАРІАНТИ
ХОМФЛІ ДЛЯ ТОРИЧНИХ ВУЗЛІВ
І БОЗОННЕ (q, p)-ЧИСЛЕННЯ

Р е з ю м е

Для однопараметричного скейн-співвідношення Александера (Джонса) введено "бозонні" q-числа Александера (Джонса), а для двопараметричного скейн-співвідношення Хомфлі – "бозонні" (q, p)-числа Хомфлі ("бозонні" числа пов'язані з деформованими бозонними осциляторами). За допомогою цих деформованих "бозонних" чисел можна відновити відповідні скейн-співвідношення. Аналізуючи введені "бозонні" числа, ми вказуємо на два способи отримання двопараметричного скейн-співвідношення Хомфлі ("бозонних" (q, p)-чисел) із однопараметричних скейн-співвідношень Александера і Джонса (із відповідних "бозонних" q-чисел). Ці два способи отримання скейн-співвідношення Хомфлі еквівалентні.