

<https://doi.org/10.15407/ujpe71.5.449>

O.O. VAKHNENKO,¹ V.O. VAKHNENKO²

¹ Department for Theory of Nonlinear Processes in Condensed Matter,
Bogolyubov Institute for Theoretical Physics, The National Academy of Sciences of Ukraine
(14-B Metrologichna Street, Kyiv 03143, Ukraine;
e-mail: vakhnenko@bitp.kyiv.ua; <https://orcid.org/0000-0001-8371-9499>)

² Department of Dynamics of Deformable Solids, Subbotin Institute of Geophysics,
The National Academy of Sciences of Ukraine
(63-B, Bohdan Khmel'nyts'kyi Str., Kyiv 01054, Ukraine;
<https://orcid.org/0000-0002-1250-9563>)

INTEGRABLE PARAMETRICALLY DRIVEN NONLINEAR DYNAMICAL SYSTEM OF PSEUDO-EXCITATIONS ON A TWO-LEG LADDER LATTICE. EXPLICITLY MANAGEABLE FORMULATION

The basic features of the integrable parametrically driven nonlinear dynamical system of pseudo-excitations on a two-leg ladder lattice are formulated in clear and understandable terms. The explicit but rather general realization of parametric drive suitable for the strict development of system's Darboux–Bäcklund integration technique is proposed and described in details.

Keywords: nonlinear dynamics, integrable system, two-leg ladder lattice, parametric drive, Hamiltonian dynamics.

1. Introduction

In the year of 2002 we paid attention on the very interesting and prospective property of semi-discrete (differential-difference) nonlinear integrable systems overlooked or deliberately ignored at the time in scientific literature. It concerns the incorporation of arbitrary parametric drive into certain semi-discrete nonlinear integrable systems without the loss of their integrability [1]. Any of such a parametrically driven system preserves its Hamiltonian formulations but excludes the conservation of total energy from the hierarchy of conservation laws. In so doing, the specifics of parametric drive consists in multiplication of conserving quantities on some time-dependent parameters to form the proper Hamiltonian function.

Frankly speaking our observation about strict Hamiltonian formulations of parametrically driven

systems is in lines with the key statements formulated for time-dependent Hamiltonian systems [2–5].

Recently we have developed the integrable parametrically driven nonlinear dynamical system of pseudo-excitations on a two-leg ladder lattice where the parametric drive is incorporated both through the multiplication coefficients and via the spatially independent field functions combined [6]. In general, such a parametric drive is not explicitly manageable and up to now we specified only the drive oscillating on a constant temporal background [6].

In the present paper we formulate the advanced approach permitting to treat explicitly a wide class of arbitrary time-varying parametric drives as applied to the integrable parametrically driven nonlinear dynamical system of pseudo-excitations on a two-leg ladder lattice [6].

2. Semi-Discrete Parametrically Driven Nonlinear Integrable System in Concise Hamiltonian Form

The concise Hamiltonian form of our parametrically driven nonlinear integrable system reads as follows [6]

$$\frac{d}{d\tau} g_{21}(n) = -\frac{\partial H}{\partial g_{12}(n)} \quad (2.1)$$

Citation: Vakhnenko O.O., Vakhnenko V.O. Integrable parametrically driven nonlinear dynamical system of pseudo-excitations on a two-leg ladder lattice. Explicitly manageable formulation. *Ukr. J. Phys.* **71**, No. 5, 449 (2026). <https://doi.org/10.15407/ujpe71.5.449>.

© Publisher PH “Akademperiodyka” of the NAS of Ukraine, 2026. This is an open access article under the CC BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/4.0/>)

$$\frac{d}{d\tau}g_{12}(n) = +\frac{\partial H}{\partial g_{21}(n)} \quad (2.2)$$

$$\frac{d}{d\tau}g_{23}(n) = -\frac{\partial H}{\partial g_{32}(n)} \quad (2.3)$$

$$\frac{d}{d\tau}g_{32}(n) = +\frac{\partial H}{\partial g_{23}(n)} \quad (2.4)$$

with the expression

$$\begin{aligned} H = & \sum_{m=-\infty}^{\infty} [g_{21}(m)a_{11}g_{12}(m) - g_{21}(m+1)f_{11}g_{12}(m)] + \\ & + \sum_{m=-\infty}^{\infty} [g_{21}(m)a_{13}g_{32}(m) - g_{21}(m+1)f_{13}g_{32}(m)] + \\ & + \sum_{m=-\infty}^{\infty} [g_{23}(m)a_{33}g_{32}(m) - g_{23}(m+1)f_{33}g_{32}(m)] + \\ & + \sum_{m=-\infty}^{\infty} [g_{23}(m)a_{31}g_{12}(m) - g_{23}(m+1)f_{31}g_{12}(m)] + \\ & + \sum_{m=-\infty}^{\infty} \frac{[g_{21}(m)g_{12}(m) + g_{23}(m)g_{32}(m)]^2}{2\sqrt{f_{11}f_{33} - f_{13}f_{31}}} \quad (2.5) \end{aligned}$$

standing for the Hamiltonian function. Here the symbol n denotes the discrete longitudinal spatial coordinate variable running from minus infinity to plus infinity, while the symbol τ stands for the continuous time variable. The field functions $g_{21}(n)$ and $g_{12}(n)$ are treated as the canonically conjugated dynamical field amplitudes settled on the one leg of a ladder lattice, while the field functions $g_{23}(n)$ and $g_{32}(n)$ as the canonically conjugated dynamical field amplitudes settled on the another leg of a ladder lattice.

The system's parametric drive is determined by the time dependencies of quantities a_{jk} as well as by the time dependencies of spatially independent functions f_{jk} . Here the functions f_{jk} are governed by the set of linear ordinary differential equations

$$\frac{d}{d\tau}f_{11} = a_{13}f_{31} - f_{13}a_{31} \quad (2.6)$$

$$\frac{d}{d\tau}f_{13} = a_{11}f_{13} + a_{13}f_{33} - f_{11}a_{13} - f_{13}a_{33} \quad (2.7)$$

$$\frac{d}{d\tau}f_{33} = a_{31}f_{13} - f_{31}a_{13} \quad (2.8)$$

$$\frac{d}{d\tau}f_{31} = a_{33}f_{31} + a_{31}f_{11} - f_{33}a_{31} - f_{31}a_{11} \quad (2.9)$$

with the time-dependent coefficients a_{jk} . In general, these coefficients a_{jk} can be arbitrary functions of

time τ . However, such tough driving conditions are proved to be relaxed by the easily verifying properties

$$\frac{d}{d\tau}(f_{11} + f_{33}) = 0 \quad (2.10)$$

$$\frac{d}{d\tau}(f_{11}f_{33} - f_{13}f_{31}) = 0 \quad (2.11)$$

imposing two fundamental mutual constraints upon the driving functions f_{jk} .

In addition to the parametric drive the functions f_{11} and f_{33} are seen to establish the one-sided coupling between the neighboring sites along the chains, while the functions f_{13} and f_{31} as well as parameters a_{13} and a_{31} establish the coupling between the neighboring sites across the chains.

The one-sided (asymmetric) type of inter-site linear coupling along a particular chain is in evident contrast with the two-sided (symmetric) inter-site linear coupling along a particular chain typical of the conventional molecular excitations [7–9]. It is the reason, why the intra-site excitations of our semi-discrete nonlinear integrable system (2.1)–(2.5) should be referred to as the pseudo-excitonic ones.

3. Parental Form of Semi-Discrete Parametrically Driven Nonlinear Integrable System and Its Zero-Curvature Representation

The semi-discrete parametrically driven nonlinear integrable Hamiltonian system of our interest (2.1)–(2.5) is originated from its parental form [6]

$$\begin{aligned} \frac{d}{d\tau}f_{21}(n) = & f_{21}(n+1)f_{11} + f_{23}(n+1)f_{31} - \\ & - f_{21}(n)a_{11} - f_{22}(n)f_{21}(n) - f_{23}(n)a_{31} \quad (3.1) \end{aligned}$$

$$\begin{aligned} \frac{d}{d\tau}f_{12}(n) = & a_{11}f_{12}(n) + f_{12}(n)f_{22}(n) + a_{13}f_{32}(n) - \\ & - f_{11}f_{12}(n-1) - f_{13}f_{32}(n-1) \quad (3.2) \end{aligned}$$

$$\begin{aligned} \frac{d}{d\tau}f_{23}(n) = & f_{21}(n+1)f_{13} + f_{23}(n+1)f_{33} - \\ & - f_{21}(n)a_{13} - f_{22}(n)f_{23}(n) - f_{23}(n)a_{33} \quad (3.3) \end{aligned}$$

$$\begin{aligned} \frac{d}{d\tau}f_{32}(n) = & a_{31}f_{12}(n) + f_{32}(n)f_{22}(n) + a_{33}f_{32}(n) - \\ & - f_{31}f_{12}(n-1) - f_{33}f_{32}(n-1) \quad (3.4) \end{aligned}$$

$$\begin{aligned} f_{22}(n) = & \frac{f_{21}(n)f_{33}f_{12}(n) + f_{23}(n)f_{11}f_{32}(n)}{f_{11}f_{33} - f_{13}f_{31}} - \\ & - \frac{f_{21}(n)f_{13}f_{32}(n) + f_{23}(n)f_{31}f_{12}(n)}{f_{11}f_{33} - f_{13}f_{31}} \quad (3.5) \end{aligned}$$

admitting the semi-discrete zero-curvature formulation [10–13]

$$\frac{d}{d\tau}L(n|\lambda) = A(n+1|\lambda)L(n|\lambda) - L(n|\lambda)A(n|\lambda) \quad (3.6)$$

with the following auxiliary spectral and evolutionary matrices [6]

$$L(n|\lambda) = \begin{pmatrix} f_{11} & f_{12}(n) & f_{13} \\ f_{21}(n) & f_{22}(n) + \lambda & f_{23}(n) \\ f_{31} & f_{32}(n) & f_{33} \end{pmatrix} \quad (3.7)$$

and

$$A(n|\lambda) = \begin{pmatrix} a_{11} & f_{12}(n-1) & a_{13} \\ f_{21}(n) & \lambda & f_{23}(n) \\ a_{31} & f_{32}(n-1) & a_{33} \end{pmatrix}. \quad (3.8)$$

Here λ stands for the time-independent spectral parameter.

According to the general rules [14–17] the parental semi-discrete nonlinear system (3.1)–(3.5) is proved to be integrable in the Lax sense.

4. Relationship between the Parental and Hamiltonian Semi-Discrete Parametrically Driven Nonlinear Integrable Systems

The relationship between the parental field functions $f_{21}(n)$, $f_{12}(n)$, $f_{23}(n)$, $f_{32}(n)$ and the Hamiltonian ones $g_{21}(n)$, $g_{12}(n)$, $g_{23}(n)$, $g_{32}(n)$ is based on the transformation formulas [6]

$$g_{21}(n) = f_{21}(n)e_{11} + f_{23}(n)e_{31} \quad (4.1)$$

$$g_{12}(n) = e_{11}f_{12}(n) + e_{13}f_{32}(n) \quad (4.2)$$

$$g_{23}(n) = f_{21}(n)e_{13} + f_{23}(n)e_{33} \quad (4.3)$$

$$g_{32}(n) = e_{31}f_{12}(n) + e_{33}f_{32}(n) \quad (4.4)$$

subjected to the physically motivated condition

$$\begin{aligned} g_{21}(n)g_{12}(n) + g_{23}(n)g_{32}(n) &= \\ &= f_{22}(n)\sqrt{f_{11}f_{33} - f_{13}f_{31}}. \end{aligned} \quad (4.5)$$

The adopted condition (4.5) substantiates the time-dependent coefficients e_{jk} to be given by formulas

$$e_{11} = \frac{1}{2e} + \frac{f_{33}}{2e\sqrt{f_{11}f_{33} - f_{13}f_{31}}} \quad (4.6)$$

$$e_{13} = -\frac{f_{13}}{2e\sqrt{f_{11}f_{33} - f_{13}f_{31}}} \quad (4.7)$$

$$e_{33} = \frac{1}{2e} + \frac{f_{11}}{2e\sqrt{f_{11}f_{33} - f_{13}f_{31}}} \quad (4.8)$$

$$e_{31} = -\frac{f_{31}}{2e\sqrt{f_{11}f_{33} - f_{13}f_{31}}}, \quad (4.9)$$

where

$$e^2 = \frac{1}{2} + \frac{f_{11} + f_{33}}{4\sqrt{f_{11}f_{33} - f_{13}f_{31}}} \quad (4.10)$$

and the identity

$$e_{11}e_{33} - e_{13}e_{31} \equiv 1 \quad (4.11)$$

is taken place.

The derivatives of time-dependent coefficients e_{jk} are given by formulas

$$\frac{d}{d\tau}e_{11} = a_{13}e_{31} - e_{13}a_{31} \quad (4.12)$$

$$\frac{d}{d\tau}e_{13} = a_{11}e_{13} - e_{11}a_{13} + a_{13}e_{33} - e_{13}a_{33} \quad (4.13)$$

$$\frac{d}{d\tau}e_{33} = a_{31}e_{13} - e_{31}a_{13} \quad (4.14)$$

$$\frac{d}{d\tau}e_{31} = a_{33}e_{31} - e_{33}a_{31} + a_{31}e_{11} - e_{31}a_{11} \quad (4.15)$$

following from the the evolutionary equations for the driving functions (2.6)–(2.9).

An elementary but somewhat tedious manipulations with the transformation formulas (4.1)–(4.11) and the evolutionary equations for the parental system (3.1)–(3.5) supplemented by the expressions (4.12)–(4.15) for the time derivatives $de_{jk}/d\tau$ convert the parental semi-discrete nonlinear integrable system (3.1)–(3.5) into the Hamiltonian one (2.1)–(2.5).

The established relationship between the Hamiltonian (2.1)–(2.5) and parental (3.1)–(3.5) forms of our parametrically driven semi-discrete nonlinear integrable system is proved to be very useful for the practical application of future Darboux–Bäcklund integration technique based on the matrix-valued auxiliary linear problem

$$X(n+1|\lambda) = L(n|\lambda)X(n|\lambda) \quad (4.16)$$

$$\frac{d}{d\tau}X(n|\lambda) = A(n|\lambda)X(n|\lambda) \quad (4.17)$$

with the spectral $L(n|\lambda)$ and evolutionary $A(n|\lambda)$ matrices (3.7) and (3.8) generic to the parental system (3.1)–(3.5).

5. Inappropriate Explicit Version of Accompanying Parametric Drive

Before the preparation of our previous paper [6] we have tried to construct the Darboux–Bäcklund dressing technique for the explicit solutions of suggested parametrically driven semi-discrete nonlinear integrable system (2.1)–(2.5) by relying upon the substitution [6]

$$f_{jk} = u_{jk} + v_{jk} \tag{5.1}$$

$$a_{jk} = u_{jk} - v_{jk}, \tag{5.2}$$

where the summands u_{jk} are assumed to be time-independent

$$\frac{d}{d\tau} u_{jk} = 0. \tag{5.3}$$

As a result we have obtained harmonically oscillating expressions for the time-dependent summands v_{jk} [6].

Unfortunately, any attempts to incorporate this allegedly promising result into the Darboux–Bäcklund method have been failed inasmuch as they have not led to the explicit temporal dependencies for the seed field solutions.

As to the approximate recommendations given by the theory of linear ordinary differential equations with the variable coefficients [2, 3] they are basically unsuitable for the construction of explicit solutions claimed by the exactly integrable systems.

6. Appropriate Explicit Version of Accompanying Parametric Drive

To overcome the stumbling blocks of the inappropriate explicit version of accompanying parametric drive outlined in the previous Section let us postulate the following substitutions

$$a_{11} = f_{11} + v_{11} \tag{6.1}$$

$$a_{13} = f_{13} \tag{6.2}$$

$$a_{33} = f_{33} + v_{33} \tag{6.3}$$

$$a_{31} = f_{31}. \tag{6.4}$$

Then the set of ordinary differential equations (2.6)–(2.9) acquires very simple form

$$\frac{d}{d\tau} f_{11} = 0 \tag{6.5}$$

$$\frac{d}{d\tau} f_{13} = v_{11} f_{13} - f_{13} v_{33} \tag{6.6}$$

$$\frac{d}{d\tau} f_{33} = 0 \tag{6.7}$$

$$\frac{d}{d\tau} f_{31} = v_{33} f_{31} - f_{31} v_{11} \tag{6.8}$$

explicitly integrable for any arbitrary temporal dependencies of driving coefficients v_{11} and v_{33} .

Thus, we obtain

$$f_{11} = u_{11} \tag{6.9}$$

$$f_{13} = u_{13} \exp(+V_{11} - V_{33}) \tag{6.10}$$

$$f_{33} = u_{33} \tag{6.11}$$

$$f_{31} = u_{31} \exp(+V_{33} - V_{11}), \tag{6.12}$$

where

$$\frac{d}{d\tau} V_{11} \equiv v_{11} \tag{6.13}$$

$$\frac{d}{d\tau} V_{33} \equiv v_{33} \tag{6.14}$$

while the parameters u_{jk} are time-independent ones

$$\frac{d}{d\tau} u_{jk} = 0. \tag{6.15}$$

We clearly see that the fundamental properties (2.10)–(2.11) of driving functions f_{jk} are fulfilled.

As the matter of fact the whole parametric drive in the semi-discrete nonlinear integrable system in either parental (3.1)–(3.5) or Hamiltonian (2.1)–(2.5) incarnation is strictly regulated by the temporal behavior of just two functions V_{11} and V_{33} that can be chosen arbitrarily according to the needs of a particular solving problem.

Here we would like to stress that any straightforward integration procedure should be developed exclusively upon the properties of parental system as the generic one. Then, the obtained solutions can be readily reformulated in terms of Hamiltonian system using the results of our present research.

7. Explicit Representation of Seed Fields as One of the Key Points in Darboux–Bäcklund Integration Approach

One of the crucial points in developing an appropriate Darboux–Bäcklund integration technique for semi-discrete nonlinear integrable systems is known

to be the apposite explicit choice of so-called seed solutions [12, 13, 15].

Any seed solution to the parental semi-discrete nonlinear integrable system of our interest (3.1)–(3.5) is substantiated by the explicit version of accompanying parametric drive formulated in Section 6. Presently we announce the simplest productive seed solution given by formulas

$${}^0 f_{12}(n) = u_{12} \exp(+V_{11}) \quad (7.1)$$

$${}^0 f_{21}(n) = 0 \quad (7.2)$$

$${}^0 f_{23}(n) = 0 \quad (7.3)$$

$${}^0 f_{32}(n) = u_{32} \exp(+V_{33}). \quad (7.4)$$

Here the space-independent coefficients u_{12} and u_{32} are set to be time-independent too

$$\frac{d}{d\tau} u_{12} = 0 = \frac{d}{d\tau} u_{32}. \quad (7.5)$$

The main objective of future Darboux–Bäcklund integration technique consists in dressing the above seed solution (7.1)–(7.4) to the so-called crop solution encompassing analytical features of system’s spatio-temporal dynamics.

8. Conclusion

In present paper we made a crucial technical step for the actual analytical integration of parametrically driven nonlinear dynamical system of pseudo-excitations on a two-leg ladder lattice. The nonlinear dynamical system of interest has been suggested in our previous article [6]. However up to now it was unclear how its integrability in the Lax sense could be explored in explicitly manageable terms inasmuch as earlier proposed particular form of parametric drive has not led to the explicit temporal dependencies for the seed field solutions. Now this problem has been resolved by means of rather general form of parametric drive admitting explicit temporal dependencies for the seed field solutions suitable for the development of a proper Darboux–Bäcklund dressing technique of system’s integration.

The extensive study on Darboux–Bäcklund integration technique as well as on the explicit analytical solutions to the parametrically driven nonlinear dynamical system of pseudo-excitations on a two-leg ladder lattice is now in progress.

Oleksiy O. Vakhnenko acknowledges support from the National Academy of Sciences of Ukraine within the Project No 0122U000887. Vyacheslav O. Vakhnenko acknowledges support from the National Academy of Sciences of Ukraine within the Project No 0123U100182. Oleksiy O. Vakhnenko also acknowledges support from the Simons Foundation (USA) under the Grant SFI-PD-Ukraine-00014573.

1. O.O. Vakhnenko. Solitons in parametrically driven discrete nonlinear Schrödinger systems with the exploding range of intersite interactions. *J. Math. Phys.* **43** (5), 2587 (2002).
2. V.A. Yakubovich, V.M. Starzhinskii. *Linear Differential Equations with Periodic Coefficients. Vol. 1–2* (John Wiley, New York, 1975).
3. V.A. Yakubovich, V.M. Starzhinskii. *Parametric Resonance in Linear Systems* (Nauka, Moscow, 1987).
4. A. Dewisme, S. Bouquet. First integrals and symmetries of time-dependent Hamiltonian systems. *J. Math. Phys.* **34** (3), 997 (1993).
5. Jü. Struckmeier, C. Riedel. Invariants for time-dependent Hamiltonian systems. *Phys. Rev. E* **64** (2), 026503 (9 pages) (2001).
6. O.O. Vakhnenko, V.O. Vakhnenko. Development and analysis of novel integrable nonlinear dynamical systems on quasi-one-dimensional lattices. Parametrically driven nonlinear system of pseudo-excitations on a two-leg ladder lattice. *Ukr. J. Phys.* **69** (8), 577 (2024).
7. A.S. Davydov. *Theory of Molecular Excitons* (Plenum Press, New York–London, 1971).
8. A.S. Davydov, N.I. Kislukha. Solitary excitons in one-dimensional molecular chains. *Phys. Stat. Solidi B* **59** (2), 465 (1973).
9. A.S. Davydov, N.I. Kislukha. Solitons in one-dimensional molecular chains. *Phys. Stat. Solidi B* **75** (2), 735 (1976).
10. L.D. Faddeev, L.A. Takhtajan. *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 1987).
11. G.-Z. Tu. A trace identity and its applications to the theory of discrete integrable systems. *J. Phys. A: Math. Gen.* **23** (17), 3903 (1990).
12. O.O. Vakhnenko. Semi-discrete integrable nonlinear Schrödinger system with background-controlled inter-site resonant coupling. *J. Nonlin. Math. Phys.* **24** (2), 250 (2017).
13. O.O. Vakhnenko. Nonlinear integrable dynamics of coupled vibrational and intra-site excitations on a regular one-dimensional lattice. *Phys. Lett. A* **405**, 127431 (6 pages) (2021).
14. O.O. Vakhnenko, V.O. Vakhnenko. Development and analysis of novel integrable nonlinear dynamical systems on quasi-one-dimensional lattices. Two-component nonlinear

- system with the on-site and spatially distributed inertial mass parameters. *Ukr. J. Phys.* **69** (3), 168 (2024).
15. O.O. Vakhnenko, A.P. Verchenko. Nonlinear system of \mathcal{PT} -symmetric excitations and Toda vibrations integrable by the Darboux–Bäcklund dressing method. *Proc. R. Soc. A* **477** (2256), 20210562 (18 pages) (2021).
16. O.O. Vakhnenko, V.O. Vakhnenko, A.P. Verchenko. Physical insight into the semi-discrete nonlinear integrable systems with the true and false multicomponentness. *Chaos, Solitons and Fractals* **200** (2), 117043 (24 pages) (2025).
17. O.O. Vakhnenko, V.O. Vakhnenko. Integrable twelve-component nonlinear dynamical system on a quasi-one-dimensional lattice. *SIGMA* **21**, 089 (17 pages) (2025).

Received 25.11.25.
Author's version

O.O. Вахненко, В.О. Вахненко

ІНТЕГРОВНА ПАРАМЕТРИЧНО
УРУХОМЛЮВАНА НЕЛІНІЙНА ДИНАМІЧНА
СИСТЕМА ПСЕВДОЗБУДЖЕНЬ
НА ДВОНІЖКОВІЙ ДРАБИНЧАТІЙ ҐРАТЦІ.
ЯВНО ЗДІЙСНЕННЕ ФОРМУЛЮВАННЯ

Ясно і зрозуміло сформульовано засадничі ознаки інтегрованої параметрично урухомлюваної нелінійної динамічної системи псевдозбуджень на двоніжковій драбинчатій ґратці. Запропоновано та детально описано явну, проте досить загальну реалізацію параметричного урухомлювання, придатного для послідовної побудови техніки інтегрування системи методом Дарбу–Беклунда.

Ключові слова: нелінійна динаміка, інтегровна система, двоніжкова драбинчата ґратка, параметричне урухомлювання, Гамільтонова динаміка.