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EXACT SOLUTIONS FOR THE KEMMER OSCILLATOR IN 1 + 1 RINDLER COORDINATES

This work presents exact solutions of the Kemmer equation for spin-1 particles in $(1+1)$-dimensional Rindler spacetime, motivated by the need to understand vector bosons under uniform acceleration, including non-inertial effects and the Unruh temperature, which distinguish them from spin-0 and spin-1/2 systems. Starting from the free Kemmer field in an accelerated reference frame, we establish eigenvalue equations resembling those of the Klein–Gordon equation in Rindler coordinates. By introducing the Dirac oscillator interaction through a momentum substitution, we derive an exact closed-form spectrum for the Kemmer oscillator, revealing how the acceleration parameter modifies the characteristic length, shifts the discrete energy spectrum, and lifts degeneracies. In the Minkowski limit $a \to 0$, the standard Kemmer oscillator spectrum is recovered, ensuring consistency with flat-spacetime results. These findings provide a tractable framework for analyzing acceleration-induced effects, with implications for quantum field theory in curved spacetime, quantum gravity, and analogue gravity platforms.

Keywords: Kemmer oscillator, spin-1 particles, vector bosons, Dirac oscillator, Unruh effect, temperature.

1. Introduction

The primary objective of theoretical physics is to develop a coherent conceptual framework that elucidates natural phenomena, enables precise predictions validated through observation, and provides a mathematical foundation for understanding reality. Quantum gravity constitutes a distinct and critical field within physics, aiming to reconcile relativistic quantum mechanics with general relativity in the context of quantum fields. Achieving this unification necessitates integrating all fundamental interactions, relying on mathematical advancements that hinge on solutions to this challenge. A breakthrough in quantum gravity would represent a significant milestone in this endeavor. Quantum gravity remains one of the most profound challenges in physics, primarily due to uncertainties surrounding the system to be quantized–potentially the spacetime metric itself. Both general relativity and quantum field theory rely on the concept of curved spacetime, which is central to the description of gravity. In general relativity, gravity is conceptualized as the curvature of spacetime, mathematically expressed through the Riemann tensor. This geometric framework underpins general relativity, which generalizes Minkowski spacetime to curved spacetime as a foundational principle [1–5].

The Dirac oscillator was first investigated by Itô *et al.* [6], who modified the Dirac equation by replacing the momentum p with $p - im\omega r$, where r is the position vector, m is the particle's mass, and ω is the oscillator frequency. Moshinsky and Szczepaniak [7] revitalized interest in this system by terming it the Dirac oscillator (DO), noting that in the nonrelativistic limit, it reduces to a harmonic oscillator with a strong spin-orbit coupling term. Furthermore, the DO interaction can be interpreted as the interaction of an anomalous magnetic moment with a linear electric field [8, 9]. Benítez *et al.* [10] identified the electromagnetic potential associated with the DO. In recent years, the DO has been explored from various perspectives as a problem in relativistic quantum mechanics, garnering attention for its numerous physical applications and its status as one of the few exactly solvable cases of the Dirac equation [11–31].

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This study employs Rindler spacetime coordinates, a subset of Minkowski spacetime tailored to observers undergoing constant proper acceleration [32]. Rindler spacetime is a coordinate patch of Minkowski spacetime adapted to uniformly accelerated observers, providing a natural framework for analyzing relativistic phenomena in non-inertial frames. In $(1+1)$ dimensions, the line element is given by

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2), \tag{1}$$

where η denotes the Rindler time coordinate, ξ is the spatial coordinate, and a is a constant with dimensions of acceleration. Observers at fixed ξ follow hyperbolic worldlines with proper acceleration $ae^{-a\xi}$; equivalently, along such worldlines, the proper time satisfies $d\tau{=}e^{a\xi}d\eta$. This metric arises from a coordinate transformation of Minkowski coordinates (tx) to $(\eta\xi)$, in which the accelerated trajectories become stationary. Rindler geometry plays a central role in theoretical physics because it captures near-horizon kinematics. The Rindler horizon at $\xi \to -\infty$ mirrors key features of black-hole event horizons, making the spacetime an effective laboratory for quantum field theory in curved backgrounds.

A hallmark result is the Unruh effect: a uniformly accelerated observer perceives the Minkowski vacuum as a thermal state with temperature $T = a/(2\pi)$ (in natural units $\hbar = c = k_{\mathrm{B}}1$). This connection underpins parallels with Hawking radiation, as the near-horizon region of a Schwarzschild black hole is locally Rindler. Analyzing quantum systems such as the Kemmer oscillator within the Rindler patch elucidates how uniform acceleration modifies particle dynamics, spectra, and state structure. Because the Rindler setting admits exact treatments, it serves as a tractable bridge between special relativity and curved-spacetime effects, yielding insights relevant to foundational questions in quantum gravity as well as to relativistic quantum information. In addition, uniform acceleration probes aspects of vector bosons that have no analogue for scalars or spin-$1/2$ fields. In the Kemmer spin-1 sector, equivalent to Proca in appropriate limits, acceleration couples differently to longitudinal and transverse polarizations, modifies spectral degeneracies, and reshapes normalizability domains. Establishing exact solutions in the Rindler patch supplies a controlled baseline for these effects, against which approximate or numerical treatments can be tested. This is relevant both conceptually (Unruh-type responses of vector fields, near-horizon kinematics) and pragmatically, since Rindler-like backgrounds can be engineered in analogue settings. Our aim is therefore to quantify, in closed form, how constant proper acceleration deforms spin-1 spectra and to delineate the parameter regime that yields real, physically admissible energies.

The research focuses on the Kemmer oscillator in a $(1+1)$-dimensional context to investigate physical phenomena influenced by acceleration [33–43]. Introduced in 1939, the Kemmer equation [44] is analogous to the Dirac equation and describes spin-1 particles. It models a system comprising two spin-$\frac{1}{2}$ particles, resembling a two-body Dirac equation, rather than a single spin-1 particle. In addition, the spin-1 sector of the DKP equation was first derived by Barut [45] through the quantization of a classical model of the zitterbewegung. Additionally, Unal [46–48] employed Schrödinger's second quantization procedure to eliminate the spin-0 sector, thereby isolating the spin-1 sector of the DKP equation in both $(3+1)$- and $(1+1)$-dimensional spacetimes. In this approach, the wave function is expressed as a symmetric rank-2 spinor, representing a spin-1 particle as a system of two identical spin-$\frac{1}{2}$ particles, with the wave function constructed as the direct product of two Dirac spinors. In the $(1+1)$-dimensional case, Unal demonstrated the equivalence of the DKP equation to the complex Proca equation, showing that in the massless limit, the Proca equation reduces to the Maxwell equations. Furthermore, Unal established that in $(1+1)$ dimensions, the DKP equation simplifies to a four-component wave equation [46–48]. This spin-1 sector was derived by adapting Barut's classical zitterbewegung model from $(3+1)$ to $(1+1)$ dimensions (for more details see Ref. [34]).

This paper investigates the application of the Kemmer equation in $(1+1)$-dimensional Rindler spacetime, an area previously unexplored. The study is structured as follows: Section II presents solutions derived using an initial approximation; Section III explores solutions via a second approximation; and Section IV provides interpretations and conclusions. Natural units ($\hbar = c = 1$) are employed throughout this study to simplify calculations and emphasize dimensionless quantities.

By analyzing the Kemmer equation in Rindler spacetime, this research contributes to the understanding of quantum systems in accelerated reference frames and their potential implications for quantum gravity.

2. The Kemmer Equation in a Flat Spacetime Framework

The relativistic Kemmer equation, which provides a Dirac-like formalism for spin-1 particles, is expressed as [44]

$$(\beta^\mu p_\mu - Mc)\,\Psi_K = 0, \tag{2}$$

where $M = 2m$ represents the total mass of two identical spin-$1/2$ particles. The 16×16 Kemmer matrices β^μ $(\mu = 0, 1, 2, 3)$ satisfy the following relation

$$\beta^\mu\beta^\nu\beta^\lambda+\beta^\lambda\beta^\nu\beta^\mu=g^{\mu\nu}\beta^\lambda+g^{\lambda\nu}\beta^\mu, \tag{3}$$

with the explicit representation

$$\beta^\mu = \gamma^\mu \otimes I + I \otimes \gamma^\mu. \tag{4}$$

Here, I is the 4×4 identity matrix, γ^μ denotes the Dirac gamma matrices, and $\otimes$ indicates the direct product. The relativistic covariance of Eq. (3) has been thoroughly analyzed by Kemmer [44].

Now, when the Dirac oscillator interaction is introduced, the momentum operator $\mathbf{p}$ in the free Kemmer equation is replaced according to

$$\mathbf{p} \to \mathbf{p} - iM\omega B\mathbf{r},$$

where the additional term is linear in the spatial coordinate $\mathbf{r}$. Consequently, the Kemmer equation in flat spacetime, incorporating the Dirac oscillator potential, takes the form

$$\Big[\left(\gamma^0 \otimes I + I \otimes \gamma^0\right) E - c\left(\gamma^0 \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \gamma^0\right) \times \\ \times (\mathbf{p} - iM\omega B\mathbf{r}) - Mc^2\gamma^0 \otimes \gamma^0\Big]\Psi_K = 0, \tag{5}$$

where ω is the oscillator frequency. Following Ref. [49], the operator is chosen as $B = \gamma^0 \otimes \gamma^0$ (with $B^2 = 1$), instead of the operator η^0 used in Refs. [42, 43].

The stationary state Ψ_K of Eq. (5) is a sixteen-component wave function of the Kemmer equation and may be expressed as

$$\Psi_K = \Psi_D \otimes \Psi_D, \tag{6}$$

where

$$\Psi_D = (\psi_1, \psi_2, \psi_3, \psi_4)^T \tag{7}$$

denotes the solution of the Dirac equation. In the $(1 + 1)$-dimensional case: (i) the stationary state Ψ_K becomes a four-component wave function, and (ii) the matrices $\boldsymbol{\alpha}$ reduce to the Pauli matrices:

$$\alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \alpha_y = \sigma_y = \\ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \alpha_z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. The Kemmer Equation in a Curved Spacetime Framework

The Kemmer equation in curved spacetime is expressed as follows [34, 35, 40]:

$$\{i\beta^\mu\nabla_\mu - M\}\,\psi_K = 0. \tag{8}$$

Here, M represents the mass of the bosons, and β^μ are the Kemmer matrices. These matrices satisfy the algebraic relation [44]:

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\lambda\nu}\beta^\mu. \tag{9}$$

The Kemmer matrices, in curved spacetime, are defined as:

$$\beta^\mu = \gamma^\mu(x) \otimes I + I \otimes \gamma^\mu(x), \tag{10}$$

where $\gamma^\mu(x)$ are the Dirac matrices, and the β-matrices, referred to as Kemmer matrices, are extensions of these.

In curved spacetime, they relate to their flat Minkowski spacetime counterparts as:

$$\beta^\mu(x) = e^\mu_i(x)\beta^i, \tag{11}$$

where $e^\mu_i(x)$ are the tetrads. The stationary state ψ_K of Eq. (8), in $(1 + 1)$ dimensions, is a four-component wave function for the Kemmer equation, written as:

$$\psi_K = \psi_D \otimes \psi_D = (\psi_1\psi_2\psi_3\psi_4)^T. \tag{12}$$

The covariant derivative is given by:

$$\nabla_\mu=\partial_\mu - \Sigma_\mu. \tag{13}$$

For spin-$\frac{1}{2}$ particles, the spinorial connections are expressed as:

$$\Sigma_\mu = \Gamma_\mu \otimes I + I \otimes \Gamma_\mu, \tag{14}$$

where $\Gamma_\mu(x)$, the spinorial connections for spin-$\frac{1}{2}$ particles, are defined as:

$$\Gamma_\mu(x) = \frac{1}{8}\omega_{\mu ab}\left[\gamma^a, \gamma^b\right]. \tag{15}$$

The spin connection $\omega_{\mu ab}$ is given by:

$$\omega_{\mu ab} = e^\nu_a \Gamma^b_{\sigma\nu} e^\sigma_b + e_{a\nu}\partial_\mu e^\nu_b. \tag{16}$$

Here, $\Gamma^b_{\sigma\nu}$ are the Christoffel symbols, expressed as:

$$\Gamma^b_{\sigma\nu} = \frac{1}{2}g^{b\rho}[\partial_\sigma g_{\rho\nu} + \partial_\nu g_{\rho\sigma} - \partial_\rho g_{\sigma\nu}]. \tag{17}$$

Finally, in one-dimensional spacetime, the gamma matrices are typically represented as [46–48]:

$$\gamma^\mu = (\sigma^3 i\sigma^1). \tag{18}$$

This formulation provides a framework for analyzing bosonic particles in curved spacetime using the Kemmer equation and its associated mathematical structures.

4. Eigen Solutions of the Free Kemmer Equation in One-Dimensional Rindler Spacetime

The Rindler metric describes an accelerated reference frame within Minkowski spacetime, where the line element is expressed as [32]:

$$ds^2 = e^{\sigma(\xi)}\left(d\eta^2 - d\xi^2\right), \tag{19}$$

where $\sigma(\xi)$ is a static dilaton function given by:

$$\sigma(\xi) = 2a\xi. \tag{20}$$

The corresponding tetrad and its inverse are derived from this metric.

To compute the Christoffel symbols $\Gamma^\nu_{\sigma\mu}$, we use the standard formula:

$$\Gamma^\nu_{\sigma\mu} = \frac{1}{2}g^{\nu\beta}\left[\partial_\sigma g_{\beta\mu} + \partial_\mu g_{\beta\sigma} - \partial_\beta g_{\sigma\mu}\right]. \tag{21}$$

By solving for the spinorial connection, we obtain:

$$\Gamma_0 = -\frac{1}{2}a\gamma^2. \tag{22}$$

The spinorial operator Σ_μ is given by:

$$\Sigma_0 = \Gamma_0 \otimes I + I \otimes \Gamma_0, \tag{23}$$

which simplifies to:

$$\Sigma_0 = -\frac{1}{2}a\begin{pmatrix} 0 & i & i & 0 \\ i & 0 & 0 & i \\ i & 0 & 0 & i \\ 0 & i & i & 0 \end{pmatrix}. \tag{24}$$

The Kemmer matrices β^μ are defined as:

$$\beta^0 = \gamma^0(x) \otimes I + I \otimes \gamma^0(x), \tag{25}$$

which evaluates to:

$$\beta^0 = e^{-\frac{\sigma}{2}}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}. \tag{26}$$

Similarly, the matrixβ^1 is given by:

$$\beta^1 = \gamma^1(x) \otimes I + I \otimes \gamma^1(x), \tag{27}$$

which simplifies to:

$$\beta^1 = e^{-\frac{\sigma}{2}}\begin{pmatrix} 0 & i & i & 0 \\ i & 0 & 0 & i \\ i & 0 & 0 & i \\ 0 & i & i & 0 \end{pmatrix}. \tag{28}$$

The Kemmer equation in Rindler spacetime can be expressed as:

$$\left\{i\beta^0\left(\partial_0 - \Sigma_0\right) + i\beta^1\left(\partial_\xi\right) - M\right\}\psi_k = 0. \tag{29}$$

To solve equation (30), we assume that the wave function takes the form:

$$\psi_k(\eta\xi) = e^{iE\eta}\psi(\xi). \tag{30}$$

Through matrix algebra, we derive the following system of equations:

$$2E\begin{pmatrix} \psi_1 \\ 0 \\ 0 \\ -\psi_4 \end{pmatrix} + \frac{i}{2}\frac{\partial\sigma}{\partial\xi}\begin{pmatrix} \psi_2+\psi_3 \\ 0 \\ 0 \\ \psi_2+\psi_3 \end{pmatrix} + \\ + \partial_\xi\begin{pmatrix} -\psi_2-\psi_3 \\ \psi_1+\psi_4 \\ \psi_1+\psi_4 \\ \psi_2+\psi_3 \end{pmatrix} - z\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = 0, \tag{31}$$

where $z{=}Me^{a\xi}$.

To simplify the system, we apply the following transformation [50, 51] :

$$U(\xi) = e^{\frac{1}{4}\sigma}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

This leads to the transformed wave function:

$$\psi = U\psi = U \begin{pmatrix} \acute{\psi}_1 \\ \acute{\psi}_2 \\ \acute{\psi}_3 \\ \acute{\psi}_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \quad (33)$$

Moayedi and Darabi [50], as well as Santos [51], in their analysis, employ a transformation, designated as U, which they term unitary. It is pertinent from a technical standpoint to clarify that these transformations are not unitary in the strict Hilbert space sense, where an operator U must satisfy the condition $UU^\dagger = U^\dagger U = 1$.

Now, substituting equation (33) into equation (31), we obtain the following system of four linear algebraic equations:

$$(2E + z)\psi_1 - \partial_\xi\psi_2 - \partial_\xi\psi_3 = 0, \quad (34)$$

$$\partial_\xi\psi_1 - z\psi_2 + \partial_\xi\psi_4 = 0, \quad (35)$$

$$\partial_\xi\psi_1 - z\psi_3 + \partial_\xi\psi_4 = 0, \quad (36)$$

$$(2E - z)\psi_4 + \partial_\xi\psi_2 + \partial_\xi\psi_3 = 0. \quad (37)$$

From the system of equations, we find the following relationships:

$$\psi_2 = \psi_3, \quad (38)$$

$$\psi_1 = \frac{2}{2E + z}\partial_\xi\psi_2, \quad (39)$$

$$\psi_4 = \frac{-2}{2E - z}\partial_\xi\psi_2. \quad (40)$$

Substituting the expressions for ψ_1, ψ_3 and ψ_4 in terms of ψ_2, we obtain the final differential equation:

$$\left\{\partial_\xi^2 + E^2 - \frac{z^2}{4}\right\}\psi_2(\xi) = 0. \quad (41)$$

To solve Eq. (42), we consider small values of the acceleration a, allowing us to expand z as:

$$z^2 = \frac{M^2}{4}(1 + 2a\xi). \quad (42)$$

Thus, the equation transforms into:

$$\left\{\partial_\xi^2 - \frac{1}{2}M^2 a\xi + E^2 - \frac{M^2}{4}\right\}\psi(\xi) = 0. \quad (43)$$

Rewriting in a more compact form:

$$\left\{\frac{\partial^2}{\partial\xi^2} - \frac{1}{2}M^2 a\xi + \varsigma\right\}\psi(\xi) = 0, \quad (44)$$

which can also be expressed as:

$$[\ddot{\psi}(\xi) - (\alpha\xi - \varsigma)\psi(\xi) = 0. \quad (45)$$

Applying the boundary conditions [52, 53]:

$$\psi(0) = \psi(\infty) \to 0 \quad (46)$$

we define:

$$\alpha = \frac{1}{2}M^2 a = \frac{1}{l^3}, \quad (47)$$

where the characteristic length l is:

$$l = \frac{1}{\left(\frac{M^2 a}{2}\right)^{1/3}} \quad (48)$$

and the parameter ς is given by:

$$\varsigma = E^2 - \frac{M^2}{4} = \frac{\lambda}{l^2}. \quad (49)$$

Rewriting in terms of a dimensionless variable:

$$\eta = \frac{\xi}{l} - \lambda, \quad (50)$$

where l is the characteristic length, and the boundary conditions are given by

$$\psi(-\eta) = 0; \quad \psi(\infty) \to 0. \quad (51)$$

Within the two turning points, the classically allowed motion occurs in the interval:

$$\eta = -\lambda \text{ to } \eta = 0, \quad (52)$$

which is entirely within the negative values of η.

In general, the Bessel function of order $\frac{1}{3}$ provides solutions to the differential equation satisfying the boundary condition $\psi(\infty) = 0$. The solution that meets this condition is the Airy function

$$\psi(\eta) = CAi(\eta). \quad (53)$$

At this stage, two cases need to be considered.

For positive values of η, the Airy function can be expressed as [16, 53]:

$$Ai(\eta) = \frac{1}{\pi}\sqrt{\frac{\eta}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right) \text{ for } \eta > 0, \quad (54)$$

where $K_{\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right)$ is the modified Hankel function.

For negative values of η, the Airy function is described using Bessel functions [16, 52, 53]:

$$Ai(-\eta) = \frac{1}{3}\sqrt{\eta}\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right)\right\}. \quad (55)$$

Thus, the wave function takes the form:

$$\psi(\eta) = \frac{C}{3}\sqrt{\eta}\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right)\right\}. \quad (56)$$

To satisfy the boundary condition at $\eta = -\lambda$, the Airy function must vanish:

$$Ai(-\lambda) = 0, \quad (57)$$

which implies:

$$J_{\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\eta^{\frac{3}{2}}\right) = 0. \quad (58)$$

These conditions determine the allowed energy levels of the system.

For higher energy levels, where $\lambda \gg 1$, we use the asymptotic forms of the Bessel functions [53]:

$$J_{\frac{1}{3}}(\eta) \to \sqrt{\frac{2}{\pi\eta}}\cos\left(\eta - \frac{5\pi}{12}\right) \quad (59)$$

and

$$J_{-\frac{1}{3}}(\eta) \to \sqrt{\frac{2}{\pi\eta}}\cos\left(\eta - \frac{\pi}{12}\right). \quad (60)$$

From these expressions, we find:

$$\psi(\eta) \to \frac{C}{3}\sqrt{|\eta|}\cos\left(\frac{2}{3}|\eta|^{\frac{3}{2}} - \frac{\pi}{4}\right). \quad (61)$$

For large values of η, the eigenvalue condition is given by:

$$\frac{2}{3}\lambda_n^{\frac{3}{2}} = \left(2n - \frac{1}{2}\right)\frac{\pi}{2}. \quad (62)$$

Solving for λ_n:

$$\lambda_n = \left\{\frac{3\pi}{4}\left(2n - \frac{1}{2}\right)\right\}^{\frac{2}{3}}. \quad (63)$$

This leads to the energy relation:

$$E^2 = \frac{M^2}{4} + \frac{1}{l^2}\left\{\frac{3\pi}{4}\left(2n - \frac{1}{2}\right)\right\}^{\frac{2}{3}}. \quad (64)$$

Thus, the energy levels take the form:

$$E_n = \pm M\sqrt{\frac{1}{4} + \frac{1}{(Ml)^2}\left\{\frac{3\pi}{4}\left(2n - \frac{1}{2}\right)\right\}^{\frac{2}{3}}}. \quad (65)$$

The characteristic length l depends on the acceleration parameter a and is given by:

$$l = \frac{1}{\left(\frac{M^2 a}{2}\right)^{1/3}}. \quad (66)$$

This parameter enters the spectrum as the characteristic length scale associated with the oscillator. It arises naturally from the separation of variables and quantization conditions, and it essentially sets the quantum of spatial extension of the bound state. Its presence indicates that the eigenvalues depend not only on the quantum numbers but also on the confinement scale fixed by the dynamics of the oscillator.

Equation (65) describes the discrete spectrum of the free Kemmer field in (1+1)-dimensional Rindler spacetime. The parameter l, defined in Eq. (66), represents the characteristic length associated with the system. It emerges naturally from the interplay between the particle's mass M and the acceleration parameter a, and it sets both the localization scale of the Airy-type wave functions and the spacing of the allowed energy levels. As a increases, l decreases, which enhances the level separation, whereas in the limit $a \to 0$, one finds $l \to \infty$, recovering the continuous Minkowski-space spectrum. Thus, the parameter l encapsulates the influence of uniform acceleration on the quantization of the Kemmer oscillator and provides a direct measure of how acceleration modifies the relativistic energy structure. Finally, energy values can be computed for different values of a, demonstrating the dependence of the energy spectrum on the acceleration parameter.

5. Eigen Solutions of the One-Dimensional Kemmer Oscillator

In the presence of a Dirac oscillator potential, we introduce the following modification:

$$\partial_1 \to \partial_1 + M\omega B\xi \quad (67)$$

with $B = \gamma^0 \otimes \gamma^0$. Thus, the Kemmer equation with Dirac oscillator interaction takes the form:

$$\left\{i\beta^0(\partial_0 - \Sigma_0) + i\beta^1(\partial_1 + M\omega B\xi) - M\right\}\psi_k = 0. \quad (68)$$

To solve equation (68), we assume that the wave function is given by:

$$\psi_k(\eta\xi) = e^{iE\eta}\psi(\xi). \tag{69}$$

The solution of the Kemmer equation is then expressed as:

$$2E\begin{pmatrix}\psi_1\\0\\0\\-\psi_4\end{pmatrix}+\frac{i}{2}\frac{\partial\sigma}{\partial\xi}\begin{pmatrix}\psi_2+\psi_3\\0\\0\\\psi_2+\psi_3\end{pmatrix}+\partial_\xi\begin{pmatrix}-\psi_2-\psi_3\\\psi_1+\psi_4\\\psi_1+\psi_4\\\psi_2+\psi_3\end{pmatrix}+$$
$$+M\omega\xi\begin{pmatrix}\psi_2+\psi_3\\\psi_1+\psi_4\\\psi_1+\psi_4\\-\psi_2-\psi_3\end{pmatrix}-z\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}=0, \tag{70}$$

where $z = Me^{a\xi}$.

As in the previous section, in order to simplify the system, we use the same transformation U which leads to the transformation:

$$\acute{\psi} = U\psi = U\begin{pmatrix}\acute{\psi}_1\\\acute{\psi}_2\\\acute{\psi}_3\\\acute{\psi}_4\end{pmatrix}=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}. \tag{71}$$

By substituting Eq. (71) into Eq. (62), we derive the following system of four linear algebraic equations:

$$(2E+z)\psi_1-(\partial_\xi+M\omega\xi)\psi_2-(\partial_\xi+M\omega\xi)\psi_3=0, \tag{72}$$
$$(\partial_\xi+M\omega\xi)\psi_1-z\psi_2+(\partial_\xi-M\omega\xi)\psi_4=0, \tag{73}$$
$$(\partial_\xi+M\omega\xi)\psi_1-z\psi_3+(\partial_\xi-M\omega\xi)\psi_4=0, \tag{74}$$
$$(2E-z)\psi_4+(\partial_\xi-M\omega\xi)\psi_2+(\partial_\xi-M\omega\xi)\psi_3=0. \tag{75}$$

These equations describe the behavior of the system under the influence of the Dirac oscillator potential.

From the system of equations, we find the following relations:

$$\psi_2=\psi_3, \tag{76}$$
$$\psi_1=\frac{2}{2E+z}\left(\partial_\xi-M\omega\xi\right)\psi_2, \tag{77}$$
$$\psi_4=\frac{-2}{2E-z}\left(\partial_\xi-M\omega\xi\right)\psi_2. \tag{78}$$

By substituting ψ_1, ψ_3, and ψ_4 in terms of ψ_2, we obtain the final equation:

$$\left\{\partial_\xi^2+M^2\omega^2\xi^2-M\omega+E^2-\frac{z^2}{4}\right\}\psi_k(\xi)=0, \tag{79}$$

where:

$$z^2=\frac{M^2}{4}\left(1+2a\xi+2a^2\xi^2\right). \tag{80}$$

If we consider small values of the acceleration a, equation (80) simplifies to:

$$\left\{\partial_\xi^2-M^2\omega^2\xi^2-M\omega+E^2-\frac{M^2}{4}-\right.$$
$$\left.-2M^2a\xi-M^2a^2\xi^2\right\}\psi_k(\xi)=0. \tag{81}$$

Rearranging terms, we get:

$$\left\{\partial_\xi^2-M^2\xi^2\omega'\square^2-\frac{1}{2}M^2a\xi+E^2-\right.$$
$$\left.-M\omega-\frac{M^2}{4}\right\}\psi_k(\xi)=0, \tag{82}$$

where:

$$\omega'\square^2=\omega^2-\frac{a^2}{2}. \tag{83}$$

Rewriting in a final form:

$$\left\{\partial_\xi^2-M^2\xi^2\omega'\square^2-\frac{1}{2}M^2a\xi+E^2-\right.$$
$$\left.-M\omega-\frac{M^2}{4}\right\}\psi_k(\xi)=0. \tag{84}$$

After performing the final calculations, we obtain the energy equation:

$$E=\pm\sqrt{2M\omega(n+1)+\frac{M^2}{4}+\frac{M^2a^2}{16\omega'\square^2}}, \tag{85}$$

where:

$$\left(\frac{p_y^2}{2M}-\frac{1}{2}M\omega'\square^2y^2\right)\psi_k(y)=$$
$$=\left(\frac{E^2}{2M}+\frac{1}{2}\omega-\frac{1}{8}M-\frac{Ma^2}{8}\right)\psi_k(y), \tag{86}$$

with:

$$y^2=\left(\xi+\frac{a}{4\omega'\square^2}\right)^2. \tag{87}$$

This leads to the equation of the harmonic oscillator:

$$\left(p_y^2+\frac{1}{2}M\omega'\square^2y^2\right)\psi_k(y)=\acute{E}^2\psi_k(y), \tag{88}$$

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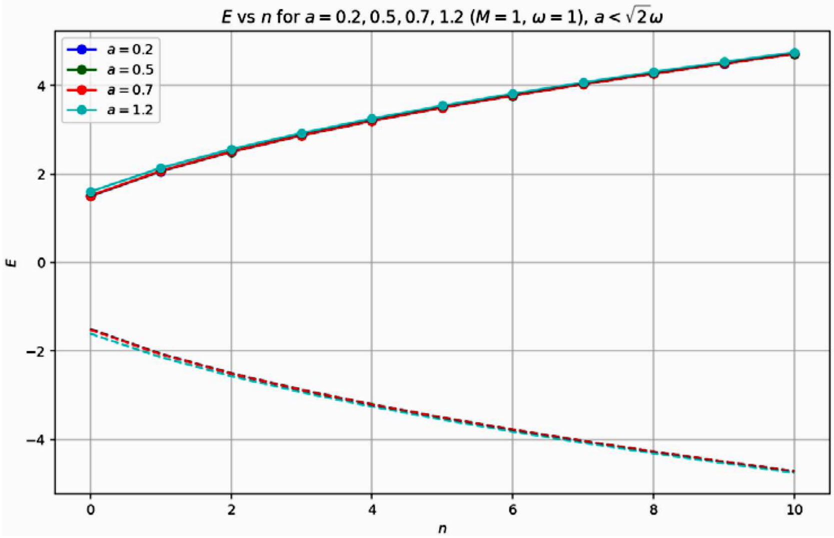

Fig. 1. Energy levels E of the one-dimensional Kemmer oscillator in Rindler spacetime as a function of the quantum number n for various acceleration parameters a

where:

$$\acute{E}^2 = \frac{E^2}{2M} - \frac{1}{2}\omega - \frac{1}{8}M - \frac{Ma^2}{8}. \quad (89)$$

The energy of the harmonic oscillator is:

$$\bar{E} = \omega\left(n + \frac{1}{2}\right). \quad (90)$$

Thus, the final form of the energy levels is:

$$E = \pm\sqrt{2M\omega(n+1) + \frac{M^2}{4} + \frac{M^2a^2}{16\omega^2 - 8a^2}}. \quad (91)$$

Equation (91) gives the energy spectrum of the Kemmer oscillator in $(1+1)$-dimensional Rindler spacetime. The spectrum retains the harmonic-oscillator structure but is modified by two relativistic contributions: the rest-mass term $(M/2)$ and a correction proportional to the acceleration parameter a. The appearance of the term $M^2a^2/\,16(\omega^2 - 8a^2)$ indicates that acceleration effectively renormalizes the oscillator frequency and removes the degeneracy of the energy levels. Physically, the quantum number n labels the oscillator excitations, while the acceleration shifts their energies, with larger a producing stronger deviations from the flat-spacetime spectrum. In the limit $a \to 0$, Eq. (90) reduces to the well-known spectrum of the one-dimensional Kemmer oscillator in Minkowski space, confirming the consistency of the result.

It is important to note, however, that certain parameter ranges can lead to unphysical results. In particular, in Eq. (90) the denominator $16\omega^2 - 8a^2$ may vanish for specific choices of the oscillator frequency ω and acceleration a. This singularity has no physical meaning and instead signals the breakdown of the approximations used in deriving Eq. (90). Similarly, the radicand in Eq. (90) can become negative if the combination of M, ω, and a falls outside the domain ensuring real solutions, leading to imaginary (non-physical) energies. To guarantee well-defined and real spectra, one must impose the restrictions

$$\omega^2 > 1/2a^2E^2 \geq 0, \quad (92)$$

together with the condition that the denominator in Eq. (91) remains positive.

To illustrate and better interpret the final expression of the energy spectrum, three figures have been generated.

In Figure 1, the dependence of the energy levels E of the one-dimensional Kemmer oscillator in Rindler spacetime on the quantum number n is presented for various values of the acceleration parameter a ($a = 0.02, 0.05, 0.7, 1.2$, with $M = 1$ and $\omega = 1$). The results illustrate the modifications to the spectrum arising from uniform acceleration. Owing to the underlying curved Rindler metric $ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2)$, a gravitational-like influence is introduced, altering the energy levels relative to those in flat Minkowski spacetime. As the acceleration parameter increases, the levels are shifted upwards, and the oscillator's characteristic length scale is deformed. These changes may lift degeneracies and modify the level spacing. In

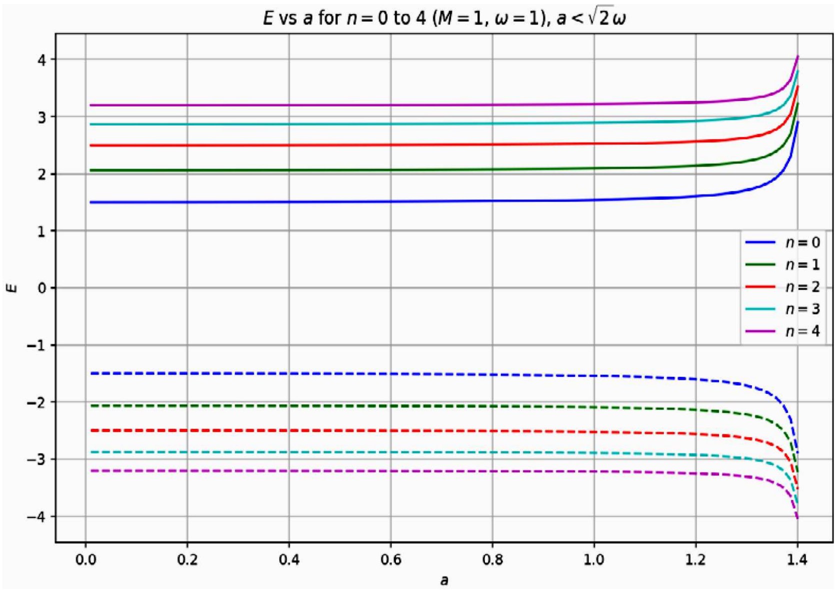

Fig. 2. Energy spectrum E as a function of the acceleration parameter a for the Kemmer oscillator in $(1+1)$-dimensional Rindler spacetime, with quantum numbers $n = 0, 1, 2, 3, 4$ ($a < \sqrt{2}\omega$). Solid lines represent the positive energy branch, while dashed lines indicate the negative branch. The condition $a < \sqrt{2}\omega$ ensures the physical admissibility of the spectrum, as derived from Eq. (91) in Section 5, preventing singularities in the denominator of the energy expression (Eq. (91))

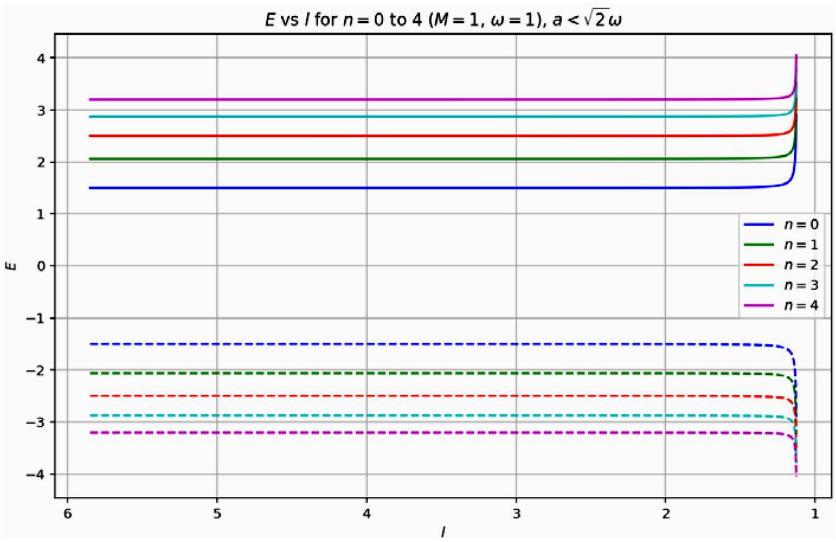

Fig. 3. Energy spectrum E as a function of the characteristic length $l = (2/(M^2 a))^{1/3}$ for the Kemmer oscillator in (1+1)-dimensional Rindler spacetime, with quantum numbers $n =$ $= 0$ to 4 ($a < \sqrt{2}\omega$). Solid lines denote the positive energy branch, and dashed lines denote the negative branch. The inverted x-axis accounts for the inverse relationship between l and a, reflecting the Rindler metric's spatial coordinate transformation (Eq. (19), Section 3). This visualization highlights the acceleration-induced modification of the system's spatial scale, a critical finding for understanding near-horizon kinematics and spectral properties in curved spacetime

the limiting case $a \to 0$, the spectrum smoothly reduces to the standard Kemmer oscillator spectrum in flat spacetime, thus maintaining consistency with established results.

In Figure 2, the energy spectrum of the free Kemmer field in $(1+1)$-dimensional Rindler spacetime is displayed as a function of the acceleration parameter a for different quantum numbers n ($n = 0, 1, 2, 3, 4$). This figure emphasizes the modifications to the spectrum induced by the non-inertial frame. Uniform acceleration in Rindler coordinates gives rise to an Unruh-like thermal effect with effective temperature $T = a/(2\pi)$, which directly impacts the energy levels. As a increases, the levels experience an upward shift that becomes more pronounced for higher quantum states. This trend reflects the nontrivial coupling of acceleration to the spin-1 components of the Kemmer field.

In Figure 3, the spectrum of the Kemmer oscillator including a Dirac oscillator–type interaction in (1+1)-dimensional Rindler spacetime is shown as a function of the characteristic length l for quantum states

$n = 0, 1, 2, 3, 4$. This analysis reveals the interplay between the oscillator interaction and the geometric effects of Rindler spacetime. The exponential factor of the metric, $e^{2a\xi}$, modifies the effective potential and the spatial confinement properties, thereby deforming the structure of the energy levels. The Dirac oscillator interaction, expressed as a momentum shift $\mathbf{p} \to \mathbf{p} - iM\omega B\mathbf{r}$, further enhances these modifications, producing shifts and broadening of the energy levels with increasing acceleration. Moreover, this interaction affects the normalizability of the wave functions, providing deeper insights into the combined influence of acceleration-induced curvature and oscillator dynamics in non-inertial frames.

6. Conclusion

In conclusion, we have derived exact solutions for the Kemmer oscillator in $(1+1)$-dimensional Rindler spacetime, addressing the unique dynamics of spin-1 bosons under uniform acceleration, including the Unruh effect and their distinction from spin-0 and spin-1/2 systems. The introduction of the Dirac oscillator interaction yields a closed-form spectrum, with the acceleration parameter modifying the characteristic length, shifting energy levels, and lifting degeneracies. Consistency with the Minkowski limit $(a \to 0)$ validates the results against flat-spacetime predictions. The clarified non-unitary transformation and verified sign convention ensure mathematical rigor, while the defined parameter domain $(\omega^2 > \frac{a^2}{2})$ guarantees real energies. These findings establish a tractable framework for exploring acceleration-induced effects, with potential applications in quantum field theory, quantum gravity, and analogue gravity platforms. Future work could extend this analysis to higher dimensions.

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1. S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, 1972).
2. T. Jacobson. Introduction to Quantum Fields in Curved Spacetime and the Hawking Effect. In *Lectures on Quantum Gravity.* Ed. by A. Gomberoff, D. Marolf, Series of the Centro De Estudios Científicos (SCEC) (Springer, 2005), p. 39.
3. K.D. Krori, P. Borgohain, P.K. Kar, D. Das (Kar). Exact scalar and spinor solutions in some rotating universes. *J. Math. Phys.* **29** (7), 1645 (1988).
4. K.D. Krori, P. Borgohain, D. Das (Kar). Exact scalar and spinor solutions in the field of a stationary cosmic string. *J. Math. Phys.* **35** (3), 1032 (1994).
5. K. Bakke, H. Belich. On the influence of a Coulomb-like potential induced by the Lorentz symmetry breaking effects on the harmonic oscillator. *Eur. Phys. J. Plus* **127** (9), 102 (2012).
6. D. Itô, K. Mori, E. Carriere. An example of dynamical systems with linear trajectory. *Il Nuovo Cimento A* **51** (4), 1119 (1967).
7. M. Moshinsky, A. Szczepaniak. The Dirac oscillator. *J. Phys. A: Math. Gen.* **22** (17), L817 (1989).
8. M. Moreno, A. Zentella. Covariance, CPT and the Foldy–Wouthuysen transformation for the Dirac oscillator. *J. Phys. A: Math. Gen.* **22** (17), L821 (1989).
9. R.P. Martínez-y-Romero, A.L. Salas-Brito. Conformal invariance in a Dirac oscillator. *J. Math. Phys.* **33**, 1831 (1992).
10. J. Benítez, R.P. Martínez y Romero, H.N. Núñez-Yépez, A.L. Salas-Brito. Solution and hidden supersymmetry of a Dirac oscillator. *Phys. Rev. Lett.* **64**, 1643 (1990).
11. H. Benzair, M. Merad, T. Boudjedaa, A. Makhlouf. Relativistic oscillators in a noncommutative space: A path integral approach. *Z. Naturforsch. A* **67**, 77 (2012).
12. A. Boumali and H. Hassanabadi. The thermal properties of a two-dimensional dirac oscillator under an external magnetic field. *Eur. Phys. J. Plus* **128**, 124 (2013).
13. A. Boumali. The one-dimensional thermal properties for the relativistic harmonic oscillator. *Electron. J. Theor. Phys.* **12** (32), 121 (2015).
14. A. Boumali, L. Chetouani, H. Hassanabadi. Effect of a minimal length on the thermal properties of a Dirac oscillator. *Acta Phys. Pol. B* **47**, 2067 (2016).
15. A. Boumali, Z. Selama. Two-dimensional Klein–Gordon oscillator in the presence of a minimal length. *Phys. Part. Nucl. Lett.* **15**, 473 (2018).
16. A. Boumali, T.I. Rouabhia. The thermal properties of the one-dimensional Boson particles in rindler spacetime. *Phys. Lett. A* **385**, 126985 (2021).
17. Z. Selama, A. Boumali. Two-dimensional Boson oscillator under a magnetic field in the presence of a minimal length in the non-commutative space. *Rev. Mex. Fis.* **67**, 226 (2021).
18. N. Korichi, A. Boumali, Y. Chargui. Statistical properties of the 1d space fractional Klein–Gordon oscillator. *J. Low Temp. Phys.* **206**, 32 (2022).
19. J.D. Castano-Yepes, I. Lujan-Cabrera, C. Ramirez-Gutierrez. Comments on superstatistical properties of the one-dimensional Dirac oscillator by Abdelmalek Boumali *et al.* *Physica A* **580**, 125206 (2021).
20. A.M. Frassino, D. Marinelli, O. Panella, P. Roy. Thermodynamics of quantum phase transitions of a Dirac oscillator in a homogenous magnetic field. *J. Phys. A: Math. Theor.* **53** (18), 185204 (2020).

21. M.H. Pacheco, R.R. Landim, C.A.S. Almeida. One-dimensional Dirac oscillator in a thermal bath. *Phys. Lett. A* **311**, 93 (2003).
22. R. Giachetti, V. Grecchi. PT-symmetric operators and metastable states of the 1d relativistic oscillators. *J. Phys. A: Math. Theor.* **44** (9), 095308 (2011).
23. Y. Chargui, L. Chetouani, A. Trabelsi. Exact solution of the one-dimensional Klein–Gordon equation with scalar and vector linear potentials in the presence of a minimal length. *Chin. Phys. B* **19**, 020305 (2010).
24. Y. Chargui, A. Trabelsi. Path integral treatment of the one-dimensional Klein–Gordon oscillator with minimal length. *Phys. Scr.* **84** (4), 045019 (2011).
25. B. Mirza, M. Mohadesi. The Klein–Gordon and the Dirac oscillators in a noncommutative space. *Commun. Theor. Phys.* **42** (5), 664 (2004).
26. B. Mirza, R. Narimani, S. Zare. Relativistic oscillators in a noncommutative space and in a magnetic field. *Commun. Theor. Phys.* **55** (3), 405 (2011).
27. F. Ahmed. The generalized Klein–Gordon oscillator in the background of cosmic string space-time with a linear potential in the Kaluza–Klein theory. *Eur. Phys. J. C* **80**, 211 (2020).
28. L.C.N. Santos, C.E. Mota, C.C. Barros Jr. Klein–Gordon oscillator in a topologically nontrivial space-time. *Adv. High Energy Phys.* **2019**, 2729352 (2019).
29. N.A. Rao, B.A. Kagali. Energy profile of the one-dimensional Klein–Gordon oscillator. *Phys. Scr.* **77** (1), 015003 (2008).
30. G. Junker. On the supersymmetry of the Klein–Gordon oscillator. *Symmetry* **13** (5), 835 (2021).
31. J. Carvalho, A.M.d.M. Carvalho, E. Cavalcante *et al.* Klein–Gordon oscillator in Kaluza–Klein theory. *Eur. Phys. J. C* **76**, 365 (2016).
32. B.W. Rindler. *Essential Relativity: Special, General, and Cosmological* (Springer, 1977).
33. W.-y. Tsai, A. Yildiz. Motion of charged particles in a homogeneous magnetic field. *Phys. Rev. D* **4**, 3643 (1971).
34. K. Sogut, A. Havare. Spin-1 particle in an electrical field in (1+1)-dimensional Schrödinger spacetime. *Class. Quantum Grav.* **23** (23), 7129 (2006).
35. K. Sogut, A. Havare. Transmission resonances in the Duffin–Kemmer–Petiau Equation in (1+1) dimensions for an asymmetric cusp potential. *Phys. Scr.* **82** (4), 045013 (2010).
36. L.D. Krase, P. Lu, R.H. Good, Jr. Stationary states of a spin-1 particle in a constant magnetic field. *Phys. Rev. D* **3**, 1275 (1971).
37. D. Shay, R.H. Good, Jr. Spin-one particle in an external electromagnetic field. *Phys. Rev.* **179**, 1410 (1969).
38. R.J. Duffin. On the characteristic matrices of covariant systems. *Phys. Rev.* **54**, 1114 (1938).
39. K. Sogut, A. Havare, I. Acikgoz. Energy levels and wave functions of vector Bosons in a homogeneous magnetic field. *J. Math. Phys.* **43**, 3952 (2002).
40. A. Havare, K. Sögüt. Energy eigenvalues of Kemmer equation for a homogeneous magnetic field. arXiv:hep-th/0207088 (2002).
41. Y. Nedjadi, R.C. Barrett. The Duffin–Kemmer–Petiau oscillator. *J. Phys. A: Math. Gen.* **27** (12), 4301 (1994).
42. Y. Nedjadi, R.C. Barrett. Solution of the central field problem for a Duffin–Kemmer–Petiau vector Boson. *J. Math. Phys.* **35**, 4517 (1994).
43. N. Kemmer. The particle aspect of meson theory. *Proc. R. Soc. A* **173** (952), 91 (1939).
44. A.O. Barut. Excited states of zitterbewegung. *Phys. Lett. B* **237**, 436 (1990).
45. N. Ünal. Kernel of the classical zitterbewegung. *Found. Phys.* **27**, 747 (1997).
46. N. Ünal. Path integral quantization of a spinning particle. *Found. Phys.* **28** (5), 755 (1998).
47. N. Ünal. Duffin–Kemmer–Petiau equation, proca equation and maxwell's equation in 1+1 d. *Concepts of Physics* **2**, 273 (2005).
48. A. Boumali. Exact solutions of the Kemmer equation for a Dirac oscillator. *Phys. Lett. A* **346** (4), 261 (2005).
49. S.K. Moayedi, F. Darabi. Exact solutions of Dirac equation on a 2D gravitational background. *Phys. Lett. A* **322** (3), 173 (2004).
50. L.C.N. Santos, C.C. Barros Jr. Fermions in the rindler spacetime. *Int. J. Geom. Methods Mod. Phys.* **16** (09), 1950140 (2019).
51. S. Flügg. *Practical Quantum Mechanics* (Springer Science & Business Media, 2012).
52. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Edited by M. Abramowitz, I.A. Stegun (Dover Publications, 1974).

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ТОЧНІ РОЗВ'ЯЗКИ ДЛЯ ОСЦИЛЯТОРА КЕММЕРА В КООРДИНАТАХ РІНДЛЕРА $(1+1)$

У цій роботі представлено точні розв'язки рівняння Кеммера для частинок зі спіном 1 у $(1+1)$-вимірному просторі-часі Ріндлера. Дослідження зумовлене необхідністю зрозуміти поведінку векторних бозонів в умовах рівномірного прискорення, враховуючи неінерційні ефекти та температуру Унру, що відрізняє їх від систем зі спіном 0 та спіном 1/2. Виходячи з вільного поля Кеммера в прискореній системі відліку, ми встановлюємо рівняння на власні значення, що нагадують рівняння Кляйна-Ґордона в координатах Ріндлера. Шляхом введення взаємодії осцилятора Дірака через підстановку імпульсу, ми отримуємо точний спектр у замкненій формі для осцилятора Кеммера, показуючи, як параметр прискорення змінює характерну довжину, зсуває дискретний енергетичний спектр і знімає виродження. У границі Мінковського $a \to 0$ відновлюється стандартний спектр осцилятора Кеммера, що забезпечує узгодженість із результатами для плоского простору-часу. Ці результати створюють зручну основу для аналізу ефектів, зумовлених прискоренням, що важливо для квантової теорії поля у викривленому просторі-часі, квантової гравітації та платформ аналогової гравітації.

Ключові слова: осцилятор Кеммера, частинки зі спіном 1, векторні бозони, осцилятор Дірака, ефект Унру, температура.