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FACTORIZATION OF THE LORENTZ TRANSFORMATIONS

The article shows how the factorization of an arbitrary Lorentz transformation is performed. That is, the representation of an arbitrary Lorentz transformation as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Relations are obtained that determine the required boosts and turns.

Keywords: hypercomplex numbers, Lorentz group.

1. Introduction

In quantum electrodynamics, the most convenient and natural form of Lorentz transformations is the hypercomplex form based on 16 Dirac matrices [1]. In the hypercomplex representation, a scalar is associated with the matrix $a\hat{1}$, a pseudoscalar is associated with the matrix $a\hat{\iota}$, a 4-vector is associated with the matrix $a_\alpha\gamma^\alpha$, a 4-pseudovector is associated with the matrix $a_\alpha\pi^\alpha$, and an antisymmetric 4-tensor of the second rank is associated with the matrix $a_{\alpha\beta}\sigma^{\alpha\beta}$. Here, $\tilde{1}$ is the identity matrix 4×4,

$$\gamma^\alpha\gamma^\beta+\gamma^\beta\gamma^\alpha=2g^{\alpha\beta}, \tag{1}$$

$$\hat{\iota}=\gamma^0\gamma^1\gamma^2\gamma^3,\ \pi^\alpha=\gamma^\alpha\hat{\iota},\ 2\sigma^{\alpha\beta}=\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha. \tag{2}$$

As always, Greek indices take values 0, 1, 2, 3, Latin ones take values 1, 2, 3.

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Let us name the numbers

$$a\hat{1}+b\hat{\iota}+c_\alpha\gamma^\alpha+d_\alpha\pi^\alpha+\frac{1}{2}f_{\alpha\beta}\sigma^{\alpha\beta}. \tag{3}$$

Dirac numbers [2]. The hypercomplex system of Dirac numbers contains a subsystem based on 8 matrices $\hat{1}$, $\hat{\iota}$, $\sigma^{\alpha\beta}$. The numbers of this subsystem have the form

$$a\hat{1}+b\hat{\iota}+\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}=a\hat{1}+b\hat{\iota}+L_{01}\sigma^{01}+$$
$$+L_{02}\sigma^{02}+L_{03}\sigma^{03}+L_{23}\sigma^{23}+L_{31}\sigma^{31}+L_{12}\sigma^{12}. \tag{4}$$

We will call these numbers the Lorentz numbers, since it is with their help that Lorentz transformations are carried out. More precisely, with the help of the matrix exponent

$$e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}=\hat{1}+\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}+\frac{1}{2!}\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}\times$$
$$\times\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}+...=\hat{1}+\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}-\frac{L^2}{2!}-\frac{L^2}{3!}\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}+$$
$$+\frac{L^4}{4!}+...=\cos L+\frac{L_{\alpha\beta}\sigma^{\alpha\beta}}{2L}\sin L. \tag{5}$$

Here, we have used the equality

$$\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = \\ = -\frac{1}{2}(L^{\alpha\beta}L_{\alpha\beta} - \hat{\imath}L^{\alpha\beta}L^{\diamond}_{\alpha\beta}) = -L^2 \quad (6)$$

and the designation

$$L^{\diamond}_{\alpha\beta} = \frac{1}{2}\,\varepsilon_{\alpha\beta\mu\nu}L^{\mu\nu}, \quad \varepsilon^{0123} = 1, \quad \varepsilon_{0123} = -1. \quad (7)$$

The tensor $L^{\diamond}_{\alpha\beta}$ dual to $L_{\alpha\beta}$ has components

$$L^{\diamond}_{01} = -L_{23}, L^{\diamond}_{02} = -L_{31}, L^{\diamond}_{03} = -L_{12}, \\ L^{\diamond}_{23} = L_{01}, L^{\diamond}_{31} = L_{02}, L^{\diamond}_{12} = L_{03}. \quad (8)$$

The Lorentz transformation of scalars, pseudo-scalars, vectors, pseudo-vectors, and second-rank antisymmetric tensors is performed by the operations

$$a\hat{1} = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}a\hat{1}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ b\hat{\imath} = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}b\hat{\imath}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ c'_{\alpha}\gamma^{\alpha} = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}c_{\alpha}\gamma^{\alpha}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \quad (9) \\ d'_{\alpha}\pi^{\alpha} = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}d_{\alpha}\pi^{\alpha}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}, \\ f'_{\alpha\beta}\sigma^{\alpha\beta} = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}f_{\alpha\beta}\sigma^{\alpha\beta}e^{-\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}.$$

The Lorentz transformation of Dirac spinors is performed by the operation

$$\psi' = e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}}\psi. \quad (10)$$

If the exponents in (9)–(10) have the form $\frac{1}{2}L_{kl}\sigma^{kl}$, then these formations are spatial rotations; but, if they have the form $L_{0k}\sigma^{0k}$, then these transformations are boosts. In the general case, transformations are neither spatial rotations nor boosts. However, any Lorentz transformation can always be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Below, we obtain relations that allow us to do this in an arbitrary case.

2. Biquaternion Representation of the Lorentz Transformations

Hypercomplex Lorentz numbers are isomorphic with biquaternions. This makes it possible to use the well-known quaternion algebra to simplify manipulations with Lorentz transformations. To verify this isomorphism, we firstly note that

$$\sigma^{01} = -\hat{\imath}\sigma^{23}, \quad \sigma^{02} = -\hat{\imath}\sigma^{31}, \quad \sigma^{03} = -\hat{\imath}\sigma^{12}. \quad (11)$$

Therefore, (4) can be written as

$$a + \hat{\imath}b + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = a + \hat{\imath}b - \hat{\imath}\frac{1}{2}L^{\diamond}_{kl}\sigma^{kl} + \frac{1}{2}L_{kl}\sigma^{kl} = \\ = a + \hat{\imath}b + \frac{1}{2}(L_{kl} - \hat{\imath}L^{\diamond}_{kl})\sigma^{kl} = a + \hat{\imath}b + \frac{1}{2}l_{kl}\sigma^{kl}. \quad (12)$$

Here

$$l_{kl} = L_{kl} - \hat{\imath}L^{\diamond}_{kl}. \quad (13)$$

The algebra of matrices $\hat{1}$, σ^{23}, σ^{31}, σ^{12} is isomorphic with the algebra of quaternions:

$\times$	σ^{23}	σ^{31}	σ^{12}
σ^{23}	$-\hat{1}$	σ^{12}	$-\sigma^{31}$
σ^{31}	$-\sigma^{12}$	$-\hat{1}$	σ^{23}
σ^{12}	σ^{31}	$-\sigma^{23}$	$-\hat{1}$

$\times$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	-1	$\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}$	$-\boldsymbol{k}$	-1	$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}$	-1

Therefore the numbers

$$a\hat{1} + \frac{1}{2}L_{kl}\sigma^{kl} \quad (14)$$

can be thought of as quaternions, and the numbers

$$a + \hat{\imath}b + \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = \left(a\hat{1} + \frac{1}{2}L_{kl}\sigma^{kl}\right) + \\ + \hat{\imath}\left(b - \frac{1}{2}L^{\diamond}_{kl}\sigma^{kl}\right) \quad (15)$$

like biquaternions [3]. That is, as a system of quaternions, expanded by introducing an additional unit $\hat{\imath}$.

Mathematicians consider three possible options for introducing an additional unit: when $\hat{\imath}\cdot\hat{\imath} = -1$, when $\hat{\imath}\cdot\hat{\imath} = 1$, and when $\hat{\imath}\cdot\hat{\imath} = 0$. In the first case, the resulting numbers are called elliptic (ordinary) biquaternions, in the second, hyperbolic biquaternions, and, in the third, parabolic ones. Since $\hat{\imath}\cdot\hat{\imath} = -1$, we are dealing with elliptic (ordinary) biquaternions.

The numbers $a\hat{1} + b\hat{\imath}$ are isomorphic with complex numbers and commute with $\sigma^{\alpha\beta}$. We will call such numbers $\hat{\imath}$-complex numbers. Accordingly, biquaternions (15) can be considered as quaternions with $\hat{\imath}$-complex coefficients. In particular, exponent (5) can be written as

$$e^{\frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta}} = e^{\boldsymbol{l\varsigma}} = 1 + \boldsymbol{l\varsigma} + \frac{1}{2!}\boldsymbol{l\varsigma}\cdot\boldsymbol{l\varsigma} + \\ + \frac{1}{3!}\boldsymbol{l\varsigma}\cdot\boldsymbol{l\varsigma}\cdot\boldsymbol{l\varsigma} + ... = \cos l + \frac{\boldsymbol{l\varsigma}}{l}\sin l. \quad (16)$$

Here, we use the notation

$$\boldsymbol{l}\boldsymbol{\zeta} \equiv \frac{1}{2}L_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{2}l_{kl}\sigma^{kl} = \frac{1}{2}(L_{kl} - \hat{\imath}L^{\diamond}_{kl})\sigma^{kl} =$$
$$= (L_{23} - \hat{\imath}L^{\diamond}_{23})\sigma^{23} + (L_{31} - \hat{\imath}L^{\diamond}_{31})\sigma^{31} + (L_{12} - \hat{\imath}L^{\diamond}_{12})\sigma^{12} =$$
$$= (\boldsymbol{r} + \imath\hat{\boldsymbol{b}})\boldsymbol{\varsigma} = \tag{17}$$

$$(r_x + \hat{\imath}b_x)\sigma^{23} + (r_y + \hat{\imath}b_y)\sigma^{31} + (r_z + \hat{\imath}b_z)\sigma^{12}, \tag{18}$$

$$l^2 = -\boldsymbol{l\varsigma}\cdot\boldsymbol{l\varsigma} = \boldsymbol{l}\cdot\boldsymbol{l} = (\boldsymbol{r}+\hat{\imath}\boldsymbol{b})^2 = \boldsymbol{r}^2 - \boldsymbol{b}^2 + \hat{\imath}2\boldsymbol{b}\cdot\boldsymbol{r}, \tag{19}$$

$$l = \sqrt{-\boldsymbol{l\varsigma}\cdot\boldsymbol{l\varsigma}} = \sqrt{\boldsymbol{l}\cdot\boldsymbol{l}} = \sqrt{\boldsymbol{r}^2 - \boldsymbol{b}^2 + \hat{\imath}2\boldsymbol{b}\cdot\boldsymbol{r}}. \tag{20}$$

If $\boldsymbol{l} = \boldsymbol{r}$, $\boldsymbol{b} = 0$, then transformations (9)–(10) describe the space direct rotation of the reference frame around the $\boldsymbol{r}$-axis by an angle $2r$. But if $\boldsymbol{l} = \hat{\imath}\boldsymbol{b}$, $\boldsymbol{r} = 0$, then transformations (9)–(10) describe the boost. Namely, if $\frac{\boldsymbol{b}}{b} = \frac{\boldsymbol{v}}{v}$, $\tanh 2b = \frac{v}{c}$, then they will describe the transition to the reference frame that moves relative to the original system with a speed $\boldsymbol{v}$. In the general case, when $\boldsymbol{l} = \boldsymbol{r} + \hat{\imath}\boldsymbol{b}$, transformations (9)–(10) are neither spatial rotations nor boosts. However, as we will see below, they can always be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation.

Operations with biquaternion exponents (16) are much more convenient to perform, if, instead of $\hat{\imath}$-complex vectors $\boldsymbol{l}$, $\hat{\imath}$-complex vectors $\boldsymbol{\lambda}$ are used as parameters of the Lorentz transformations

$$\boldsymbol{\lambda}(\boldsymbol{l}) = \frac{\boldsymbol{l}}{l}\tan l. \tag{21}$$

Obviously, in the case of a spatial rotation, when $\boldsymbol{l}$ is a $\hat{\imath}$-real vector $\boldsymbol{l} = \boldsymbol{r}$, the parameter $\boldsymbol{\lambda}$ is also a $\hat{\imath}$-real vector

$$\boldsymbol{\rho}(\boldsymbol{r}) = \frac{\boldsymbol{r}}{r}\tan r. \tag{22}$$

In the boost case, when $\boldsymbol{l}$ is the $\hat{\imath}$-imaginary vector $\boldsymbol{l} = \hat{\imath}\boldsymbol{b}$, the parameter $\boldsymbol{\lambda}$ is also the $\hat{\imath}$-imaginary vector

$$\boldsymbol{\beta}(\hat{\imath}\boldsymbol{b}) = \frac{\hat{\imath}\boldsymbol{b}}{b}\tanh b. \tag{23}$$

If the parameter $\boldsymbol{\lambda}(\boldsymbol{l})$ is known, then the parameter $\boldsymbol{l}$ is determined by the relation

$$\boldsymbol{l} = \frac{\boldsymbol{\lambda}}{\sqrt{\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}}}\arctan\sqrt{\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}} = \frac{\boldsymbol{\lambda}}{\lambda}\arctan\lambda,$$
$$l = \arctan\sqrt{\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}} = \arctan\lambda. \tag{24}$$

When using the parameter $\boldsymbol{\lambda}(\boldsymbol{l})$, the exponent $e^{\boldsymbol{l\varsigma}}$ takes the form

$$e^{\boldsymbol{l\varsigma}} = \cos l + \frac{\boldsymbol{l\varsigma}}{l}\sin l = \cos l\left(1 + \frac{\boldsymbol{l\varsigma}}{l}\tan l\right) =$$
$$= \frac{1}{\sqrt{1+\tan^2 l}}\left(1 + \frac{\boldsymbol{l\varsigma}}{l}\tan l\right) = \frac{1+\boldsymbol{\lambda\varsigma}}{\sqrt{1+\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}}}, \tag{25}$$

and the product of two exponents $e^{\boldsymbol{l}_2\boldsymbol{\varsigma}}e^{\boldsymbol{l}_1\boldsymbol{\varsigma}}$ is of the form

$$e^{\boldsymbol{l}_2\boldsymbol{\varsigma}}e^{\boldsymbol{l}_1\boldsymbol{\varsigma}} = \frac{(1+\boldsymbol{\lambda}_2\boldsymbol{\varsigma})(1+\boldsymbol{\lambda}_1\boldsymbol{\varsigma})}{\sqrt{(1+\boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}_2)(1+\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_1)}} =$$
$$= \frac{1-\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2}{\sqrt{(1+\boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}_2)(1+\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_1)}}\times$$
$$\times\left(1 + \frac{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_2\times\boldsymbol{\lambda}_1}{1-\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2}\boldsymbol{\varsigma}\right). \tag{26}$$

Here, $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}(\boldsymbol{l}_1)$, $\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}(\boldsymbol{l}_2)$. Putting

$$\boldsymbol{\lambda}(\boldsymbol{l}) = \frac{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_2\times\boldsymbol{\lambda}_1}{1-\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2}, \tag{27}$$

we get

$$1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}(\boldsymbol{l}) = \frac{(1+\boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}_2)(1+\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_1)}{(1-\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2)^2}. \tag{28}$$

The product of the exponents takes the form of exponent (16) with the exponent $\boldsymbol{l\zeta}$:

$$e^{\boldsymbol{l}_2\boldsymbol{\varsigma}}e^{\boldsymbol{l}_1\boldsymbol{\varsigma}} = \frac{1+\boldsymbol{\lambda\varsigma}}{\sqrt{1+\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}}} = e^{\boldsymbol{l\varsigma}}. \tag{29}$$

Relation (26) determines the rule for the composition of the parameters $\boldsymbol{\lambda}(\boldsymbol{l}_1)$ and $\boldsymbol{\lambda}(\boldsymbol{l}_2)$, when multiplying the exponents $e^{\boldsymbol{l}_2\boldsymbol{\varsigma}}e^{\boldsymbol{l}_1\boldsymbol{\varsigma}}$. Note that, in [4], the same relation was obtained for another parameter not related to biquaternions.

3. Factorization of the Lorentz Transformations

Any Lorentz transformation can be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Accordingly, the exponent $e^{\boldsymbol{l\varsigma}}$ can be represented as a product

$$e^{\boldsymbol{l\varsigma}} = e^{\hat{\imath}\boldsymbol{b\varsigma}}e^{\boldsymbol{r\varsigma}}, \tag{30}$$

or

$$e^{\boldsymbol{l\varsigma}} = e^{\boldsymbol{r}'\boldsymbol{\varsigma}}e^{\hat{\imath}\boldsymbol{b}'\boldsymbol{\varsigma}}. \tag{31}$$

Let us take a few steps to find the parameters of boosts $\boldsymbol{b}$, $\boldsymbol{b}'$ and rotations $\boldsymbol{r}$, $\boldsymbol{r}'$.

At the first step, we multiply the left and right parts of equalities (29)–(30) by the left and right parts of their Hermitian conjugate equalities

$$(e^{\boldsymbol{l}\varsigma})^{\dagger} = e^{-\boldsymbol{l}^{\star}\varsigma} = e^{-\boldsymbol{r}\varsigma}e^{\hat{\imath}\boldsymbol{b}\varsigma}. \quad (32)$$

or

$$(e^{\boldsymbol{l}\varsigma})^{\dagger} = e^{-\boldsymbol{l}^{\star}\varsigma} = e^{\hat{\imath}\boldsymbol{b}'\varsigma}e^{-\boldsymbol{r}'\varsigma}. \quad (33)$$

Here, * denotes the $\hat{\imath}$-complex conjugate: $\boldsymbol{l} = \boldsymbol{r} + \hat{\imath}\boldsymbol{b}$, $\boldsymbol{l}^{\star} = \boldsymbol{r} - \hat{\imath}\boldsymbol{b}$.

We multiply equality (29) by (31) from the right

$$e^{\boldsymbol{l}\varsigma}e^{-\boldsymbol{l}^{\star}\varsigma} = e^{\hat{\imath}\boldsymbol{b}\varsigma}e^{\boldsymbol{r}\varsigma}e^{-\boldsymbol{r}\varsigma}e^{\hat{\imath}\boldsymbol{b}\varsigma} = e^{\hat{\imath}2\boldsymbol{b}\varsigma}, \quad (34)$$

and equality (30) is multiplied by (32) from the left

$$e^{-\boldsymbol{l}^{\star}\varsigma}e^{\boldsymbol{l}\varsigma} = e^{\hat{\imath}\boldsymbol{b}'\varsigma}e^{\boldsymbol{r}'\varsigma}e^{-\boldsymbol{r}'\varsigma}e^{\hat{\imath}\boldsymbol{b}'\varsigma} = e^{\hat{\imath}\boldsymbol{b}'\varsigma}e^{\hat{\imath}\boldsymbol{b}'\varsigma} = e^{\hat{\imath}2\boldsymbol{b}'\varsigma}. \quad (35)$$

Using (26), we find the parameters $\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b})$ and $\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}')$ corresponding to the parameters $\boldsymbol{l} = \hat{\imath}2\boldsymbol{b}$ and $\boldsymbol{l}' = \hat{\imath}2\boldsymbol{b}'$:

$$\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}) = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}^{\star}(-\boldsymbol{l}) + \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}^{\star}(-\boldsymbol{l})}{1 - \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^{\star}(-\boldsymbol{l})} = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}^{\star}(\boldsymbol{l}) - \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}^{\star}(\boldsymbol{l})}{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^{\star}(\boldsymbol{l})}, \quad (36)$$

$$\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}') = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}^{\star}(-\boldsymbol{l}) + \boldsymbol{\lambda}^{\star}(-\boldsymbol{l}) \times \boldsymbol{\lambda}(\boldsymbol{l})}{1 - \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^{\star}(-\boldsymbol{l})} = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}^{\star}(\boldsymbol{l}) - \boldsymbol{\lambda}^{\star}(\boldsymbol{l}) \times \boldsymbol{\lambda}(\boldsymbol{l})}{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^{\star}(\boldsymbol{l})}. \quad (37)$$

At the second step, we will find the parameters $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})$ and $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')$ we need. They differ from $\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b})$ and $\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}')$ by factors

$$\frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}\cdot\hat{\imath}\boldsymbol{b}}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}\cdot\hat{\imath}2\boldsymbol{b}}} \quad \text{and} \quad \frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}'\cdot\hat{\imath}\boldsymbol{b}'}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}'\cdot\hat{\imath}2\boldsymbol{b}'}}. \quad (38)$$

To find these factors, we use the trigonometric equality

$$\frac{\tan z}{\tan 2z} = \frac{1}{1+\sqrt{1+\tan^2 2z}}. \quad (39)$$

Thus, we get

$$\frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}\cdot\hat{\imath}\boldsymbol{b}}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}\cdot\hat{\imath}2\boldsymbol{b}}} = \frac{1}{1+\sqrt{1+\tan^2\sqrt{\hat{\imath}2\boldsymbol{b}\cdot\hat{\imath}2\boldsymbol{b}}}} = \frac{1}{1+\sqrt{1+\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b})\cdot\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b})}}, \quad (40)$$

$$\frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}'\cdot\hat{\imath}\boldsymbol{b}'}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}'\cdot\hat{\imath}2\boldsymbol{b}'}} = \frac{1}{1+\sqrt{1+\tan^2\sqrt{\hat{\imath}2\boldsymbol{b}'\cdot\hat{\imath}2\boldsymbol{b}'}}} = \frac{1}{1+\sqrt{1+\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}')\cdot\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}')}}. \quad (41)$$

Let us calculate

$$\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b})\cdot\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}) = \boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}')\cdot\boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}') = \frac{\left[\begin{array}{c}\boldsymbol{\lambda}^2(\boldsymbol{l}) + \boldsymbol{\lambda}^{\star 2}(\boldsymbol{l}) - 2\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^{\star}(\boldsymbol{l}) + \\ + \boldsymbol{\lambda}^2(\boldsymbol{l})\boldsymbol{\lambda}^{\star 2}(\boldsymbol{l}) - [\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^{\star}(\boldsymbol{l})]^2\end{array}\right]}{[1+\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^{\star}(\boldsymbol{l})]^2} = \frac{[1+\boldsymbol{\lambda}^2(\boldsymbol{l})][1+\boldsymbol{\lambda}^{\star 2}(\boldsymbol{l})] - [1+\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^{\star}(\boldsymbol{l})]^2}{[1+\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^{\star}(\boldsymbol{l})]^2} = \frac{[1+\boldsymbol{\lambda}^2(\boldsymbol{l})][1+\boldsymbol{\lambda}^{\star 2}(\boldsymbol{l})]}{[1+\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^{\star}(\boldsymbol{l})]^2} - 1. \quad (42)$$

Respectively,

$$\frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}\cdot\hat{\imath}\boldsymbol{b}}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}\cdot\hat{\imath}2\boldsymbol{b}}} = \frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}'\cdot\hat{\imath}\boldsymbol{b}'}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}'\cdot\hat{\imath}2\boldsymbol{b}'}} = \frac{1+\boldsymbol{\lambda}(\boldsymbol{l})\,\boldsymbol{\lambda}^{\star}(\boldsymbol{l})}{1+\boldsymbol{\lambda}(\boldsymbol{l})\,\boldsymbol{\lambda}^{\star}(\boldsymbol{l})+\sqrt{[1+\boldsymbol{\lambda}^2(\boldsymbol{l})][1+\boldsymbol{\lambda}^2(\boldsymbol{l})]^{\star}}}. \quad (43)$$

Thus, the parameters $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})$ and $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')$ corresponding to the parameters $\boldsymbol{l} = \hat{\imath}\boldsymbol{b}$ and $\boldsymbol{l}' = \hat{\imath}\boldsymbol{b}'$, have the form

$$\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}) = \boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b})\frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}\cdot\hat{\imath}\boldsymbol{b}}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}\cdot\hat{\imath}2\boldsymbol{b}}} = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}^{\star}(\boldsymbol{l}) - \boldsymbol{\lambda}(\boldsymbol{l})\times\boldsymbol{\lambda}^{\star}(\boldsymbol{l})}{1+\boldsymbol{\lambda}(\boldsymbol{l})\,\boldsymbol{\lambda}^{\star}(\boldsymbol{l})+\sqrt{[1+\boldsymbol{\lambda}^2(\boldsymbol{l})][1+\boldsymbol{\lambda}^2(\boldsymbol{l})]^{\star}}}, \quad (44)$$

$$\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}') = \boldsymbol{\lambda}(\hat{\imath}2\boldsymbol{b}')\frac{\tan\sqrt{\hat{\imath}\boldsymbol{b}'\cdot\hat{\imath}\boldsymbol{b}'}}{\tan\sqrt{\hat{\imath}2\boldsymbol{b}'\cdot\hat{\imath}2\boldsymbol{b}'}} = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}^{\star}(\boldsymbol{l}) + \boldsymbol{\lambda}(\boldsymbol{l})\times\boldsymbol{\lambda}^{\star}(\boldsymbol{l})}{1+\boldsymbol{\lambda}(\boldsymbol{l})\,\boldsymbol{\lambda}^{\star}(\boldsymbol{l})+\sqrt{[1+\boldsymbol{\lambda}^2(\boldsymbol{l})][1+\boldsymbol{\lambda}^2(\boldsymbol{l})]^{\star}}}. \quad (45)$$

As it should be, the parameters $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})$ and $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')$ are $\hat{\imath}$-imaginary vectors. The direction of these vectors depends on which operation – turn or boost – is performed firstly, and which is secondly. The magnitude of the vectors $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})$ and $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')$ does not depend on this.

At the third step, we find the exponents describing spatial rotations. To do this, we multiply (29) by

$e^{-\hat{\imath}\boldsymbol{b}\varsigma}$ on the left side, and (30) by $e^{-\hat{\imath}\boldsymbol{b}'\varsigma}$ on the right side:

$$e^{-\hat{\imath}\boldsymbol{b}\varsigma}e^{\boldsymbol{l}\varsigma} = e^{-\hat{\imath}\boldsymbol{b}\varsigma}e^{\hat{\imath}\boldsymbol{b}\varsigma}e^{\boldsymbol{r}\varsigma} = e^{\boldsymbol{r}\varsigma}, \quad (46)$$

$$e^{\boldsymbol{l}\varsigma}e^{-\hat{\imath}\boldsymbol{b}'\varsigma} = e^{\boldsymbol{r}'\varsigma}e^{\hat{\imath}\boldsymbol{b}'\varsigma}e^{-\hat{\imath}\boldsymbol{b}'\varsigma} = e^{\boldsymbol{r}'\varsigma}. \quad (47)$$

We use (26) again and, after long, but not complicated transformations, we find the parameters $\boldsymbol{\lambda}(\boldsymbol{r})$ and $\boldsymbol{\lambda}(\boldsymbol{r}')$ corresponding to the parameters $\boldsymbol{r}$ and $\boldsymbol{r}'$:

$$\boldsymbol{\lambda}(\boldsymbol{r}) = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}(-\hat{\imath}\boldsymbol{b}) + \boldsymbol{\lambda}(-\hat{\imath}\boldsymbol{b}) \times \boldsymbol{\lambda}(\boldsymbol{l})}{1 - \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(-\hat{\imath}\boldsymbol{b})} =$$
$$= \frac{\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}) + \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})}{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})} =$$
$$= \Big\{\boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}(\boldsymbol{l})[\boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^\star(\boldsymbol{l})] +$$
$$+ \boldsymbol{\lambda}(\boldsymbol{l})\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} - \boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}^\star(\boldsymbol{l}) +$$
$$+ \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}^\star(\boldsymbol{l}) - \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}^\star(\boldsymbol{l}) -$$
$$- \boldsymbol{\lambda}(\boldsymbol{l})[\boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^\star(\boldsymbol{l})] + \boldsymbol{\lambda}^\star(\boldsymbol{l})[\boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(\boldsymbol{l})]\Big\} \times$$
$$\times \Big\{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star}\Big\}^{-1} \times$$
$$\times \{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})\}^{-1} =$$
$$= \frac{\begin{bmatrix} \boldsymbol{\lambda}(\boldsymbol{l})\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \\ + \boldsymbol{\lambda}^\star(\boldsymbol{l})[1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}(\boldsymbol{l})] \end{bmatrix}}{1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star}} \times$$
$$\times \frac{1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star}}{\begin{bmatrix} 1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \\ + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) \end{bmatrix}} =$$
$$= \frac{\boldsymbol{\lambda}(\boldsymbol{l})\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \boldsymbol{\lambda}^\star(\boldsymbol{l})\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]}}{\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]}}, \quad (48)$$

$$\boldsymbol{\lambda}(\boldsymbol{r}') = \frac{\boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}(-\hat{\imath}\boldsymbol{b}') + \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}(-\hat{\imath}\boldsymbol{b}')}{1 - \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(-\hat{\imath}\boldsymbol{b}')} =$$
$$= \frac{\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}') - \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')}{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')} =$$
$$\Big\{\boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}(\boldsymbol{l})[\boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}^\star(\boldsymbol{l})] + \boldsymbol{\lambda}(\boldsymbol{l}) \times$$
$$\times \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} - \boldsymbol{\lambda}(\boldsymbol{l}) + \boldsymbol{\lambda}^\star(\boldsymbol{l}) -$$
$$- \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}^\star(\boldsymbol{l}) + \boldsymbol{\lambda}(\boldsymbol{l}) \times \boldsymbol{\lambda}^\star(\boldsymbol{l}) -$$
$$- \boldsymbol{\lambda}(\boldsymbol{l})[\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l})] + \boldsymbol{\lambda}^\star(\boldsymbol{l})[\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}(\boldsymbol{l})]\Big\} \times$$
$$\times \Big\{1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star}\Big\}^{-1} \times$$
$$\times \{1 + \boldsymbol{\lambda}(\boldsymbol{l}) \cdot \boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')\}^{-1} =$$
$$= \frac{\boldsymbol{\lambda}(\boldsymbol{l})\sqrt{[1+\boldsymbol{\lambda}^2(\boldsymbol{l})][1+\boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \boldsymbol{\lambda}^\star(\boldsymbol{l})[1+\boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}(\boldsymbol{l})]}{1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star}} \times$$
$$\times \frac{1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star}}{\begin{bmatrix} 1 + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})][1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \\ + \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}(\boldsymbol{l}) - \boldsymbol{\lambda}(\boldsymbol{l})\cdot\boldsymbol{\lambda}^\star(\boldsymbol{l}) \end{bmatrix}} =$$
$$= \frac{\boldsymbol{\lambda}(\boldsymbol{l})\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \boldsymbol{\lambda}^\star(\boldsymbol{l})\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]}}{\sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]^\star} + \sqrt{[1 + \boldsymbol{\lambda}^2(\boldsymbol{l})]}}. \quad (49)$$

As expected, the parameters $\boldsymbol{\lambda}(\boldsymbol{r})$ and $\boldsymbol{\lambda}(\boldsymbol{r}')$ are $\hat{\imath}$-real vectors. Neither the magnitude nor the direction of these vectors depend on the sequence in which a turn and a boost are performed.

Having obtained the parameters $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})$, $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')$, $\boldsymbol{\lambda}(\boldsymbol{r})$ and $\boldsymbol{\lambda}(\boldsymbol{r}')$, we can use relations (23) to pass to the parameters $\boldsymbol{l}$, $\boldsymbol{l}'$, $\boldsymbol{r}$, $\boldsymbol{r}'$ and represent an arbitrary Lorentz transformation in the form (29) or (30). It is even simpler to express the exponents in (29), (30) directly in terms of $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b})$, $\boldsymbol{\lambda}(\hat{\imath}\boldsymbol{b}')$, $\boldsymbol{\lambda}(\boldsymbol{r})$, and $\boldsymbol{\lambda}(\boldsymbol{r}')$ using relation (24).

4. Summary

The article shows how any Lorentz transformation can be represented as a sequence of a spatial rotation and a boost or a boost and a spatial rotation. Relations are found that determine the parameters of such turns and boosts. Representing an arbitrary Lorentz transformation in the form of a rotation and a boost or a boost and a rotation makes it possible to give a physical meaning to this transformation and to analyze it.

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ГІПЕРКОМПЛЕКСНЕ ПРЕДСТАВЛЕННЯ ГРУПИ ЛОРЕНЦА

Досліджено гіперкомплексну структуру групи Лоренца, побудовану на матрицях Дірака. Вона подібна до кватерніонної личини групи просторових поворотів. Такий вигляд має низку переваг. По-перше, у ній перетворення різних геометричних об'єктів – векторів, антисиметричних тензорів другого рангу і біспінорів – здійснюється за допомогою тих самих операторів, бо ця личина звідна. По-друге, представлення правила композиції двох довільних перетворень Лоренца має простий вигляд. Ці переваги значно спрощують знаходження багатьох закономірностей, пов'язаних із перетвореннями Лоренца. Зокрема, вони спрощують дослідження зв'язку спіна з псевдовектором Паулі–Любанського та малою групою Вігнера.

Ключові слова: гіперкомплексні числа, група Лоренца.