LOCAL CONSERVATION LAWS IN A NONLINEAR ELECTRODYNAMICS

O.I. BATSULA

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: `batsula@bitp.kiev.ua`*)*

PACS 11.25Uv, 04.70Bw, 03.50De, 04.40Nr

By using a generalization of the Lie–Poisson brackets for the dual Maxwell and Born–Infeld field strength tensors, we construct the gauge invariant axial-vector conserved currents for Born–Infeld and Heisenberg–Euler nonlinear electrodynamics in the 4-dimensional Minkowski space-time. The infinite hierarchies of the currents given by Lie brackets for generally covariant conserved vector and axial vector currents are established. These currents are conserved upon action of the gravitational fields, but the conservation is broken in the Einstein–Cartan theory (over a Riemann–Cartan space-time). The axial-vector currents are conserved only in the $(3+1)$-dimensional space-time.

1. Introduction

The Born–Infeld and Heisenberg–Euler Lagrangians provide particular examples of theories of nonlinear electrodynamics in the 4-dimensional Minkowski space-time. It has its origin in searching the classical singularity-free theory of an electron by Born and Infeld [1]. Later on, it was realized that the creation of virtual electron-positron pairs induces a self-coupling of the electromagnetic field. For a slowly varying, but arbitrarily strong electromagnetic field, the self-interaction energy was computed by Heisenberg and Euler [2].

The propagation of a photon in an external electromagnetic field can be described efficiently by the Heisenberg–Euler Lagrangian. Moreover, the transition amplitude for the photon splitting in quantum electrodynamics is nonvanishing in this case. In principle, this might lead to observational effects, e.g., on the electromagnetic radiation coming from neutron stars which are known to have strong magnetic fields [4, 5]. In particular, certain features in spectra of pulsars can be explained by the photon splitting [6]. For an inflationary universe where electrodynamics is assumed to be nonlinear, the creation of large-scale magnetic fields was studied in [7]. In [8], the method of measurement of the birefringence induced in vacuum by a magnetic field was described: this effect was evaluated with the use of the Euler–Heisenberg–Weisskopf Lagrangian [2, 3]. Finally, the Born–Infeld type action also appears as the low-energy effective action of open strings or the M-theory [9, 10]. It was also shown [11] that the low-energy dynamics of D-branes is described by the Dirac–Born–Infeld action.

2. Symmetries and the Conservation Laws for an Electromagnetic Field

It is well known that, from the electromagnetic tensor $\mathcal{F}_{\alpha\beta}$ and its dual

$$ {}^*\mathcal{F}^{\gamma\delta} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta}, \quad \eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}\varepsilon^{\alpha\beta\gamma\delta}, \tag{1} $$

one can construct two invariants:

$$ Q = \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}, \quad R = \frac{1}{4}\,{}^*\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}. \tag{2} $$

In a pseudo-Cartesian frame at some point, we have

$$ Q = \frac{1}{2}(\mathbf{H}^2 - \mathbf{E}^2), \quad R = \mathbf{H}\cdot\mathbf{E}, $$

where $\mathbf{E}$ and $\mathbf{H}$ are the ordinary 3-vectors of the electric and magnetic fields.

Let us introduce the 4-potential A_α through

$$ \mathcal{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{3} $$

and the Lagrangian $\mathcal{L}(\mathcal{Q},\mathcal{R})$ as an arbitrary function of invariants (2). Then the variational principle applied to

$$ \int \sqrt{-g}\mathcal{L}d^4x \tag{4} $$

yields the following Euler equations [12]

$$ \nabla_\alpha P^{\alpha\beta} = 0, \quad P^{\alpha\beta} = \mathcal{L}_Q\mathcal{F}^{\alpha\beta} + \mathcal{L}_R\,{}^*\mathcal{F}^{\alpha\beta}, $$

$$ \mathcal{L}_Q = \frac{\partial\mathcal{L}}{\partial Q}, \quad \mathcal{L}_R = \frac{\partial\mathcal{L}}{\partial R}, \tag{5} $$

which, together with

$$\nabla_\alpha \, {}^*\mathcal{F}^{\alpha\beta} = 0, \tag{6}$$

are locally satisfied by (3), and the Einstein equations [1]

$$G_{\alpha\beta} = \chi T_{\alpha\beta},$$

where

$$T^{\alpha\beta} = \mathcal{L} g^{\alpha\beta} - P^{\alpha\delta}\mathcal{F}^{\beta}_{\delta} = \mathcal{L}_Q \tau^{\alpha\beta} + (\mathcal{L} - Q\mathcal{L}_Q - R\mathcal{L}_R) g^{\alpha\beta}, \tag{7}$$

which form the system of field equations. We recall that

$$\tau^{\alpha\beta} = Q g^{\alpha\beta} - \mathcal{F}^{\alpha\delta}\mathcal{F}^{\beta}_{\delta} \tag{8}$$

is the Maxwellian energy tensor.

In 1921, Bessel-Hagen [13] applied the Noether theorems [14] to the calculus of variations of electromagnetic fields in vacuum and showed that the invariance of the Maxwell equations under the fifteen-parameter conformal group implies the existence of fifteen divergenceless expressions. These mathematical results were assumed to represent fifteen conservation laws for electromagnetic fields. A review of symmetries and conservation laws of the Maxwell equations is given in [15]. A complete explicit classification of all independent conservation laws of the Maxwell equations in the four-dimensional Minkowski space was given in [16].

The most significant nonlinear theory of electrodynamics is the Born–Infeld theory [1]. Among its many special properties, we mention the exact $SO(2)$ electric-magnetic duality invariance [17]. The Lagrangian density describing of Born–Infeld theory in four dimensions is as follows:

$$\mathcal{L}_{\mathrm{BI}} = \frac{4\pi}{b^2}\{1 - \sqrt{1 + \frac{1}{2} b^2 \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{16} b^4 \left(\mathcal{F}_{\alpha\beta} \, {}^*\mathcal{F}^{\alpha\beta}\right)^2}, \tag{9}$$

which coincides with the usual Maxwell Lagrangian in the weak field limit with $b = 2\pi\alpha'$ (α' is inverse of the string tension).

It is useful to define the second-rank tensor

$$P^{\alpha\beta} = -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\alpha\beta}} =$$

$$= \frac{\mathcal{F}^{\alpha\beta} - \frac{1}{4} b^2 (\mathcal{F}_{\mu\nu} \, {}^*\mathcal{F}^{\mu\nu})^2 \, {}^*\mathcal{F}_{\alpha\beta}}{\sqrt{1 + \frac{1}{2} b^2 \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{16} b^4 \left(\mathcal{F}_{\alpha\beta} \, {}^*\mathcal{F}^{\alpha\beta}\right)^2}} \tag{10}$$

(so that $P^{\alpha\beta} \approx \mathcal{F}^{\alpha\beta}$ for weak fields) that satisfies the electromagnetic motion equations

$$\partial_\mu P^{\mu\nu} = 0, \quad \partial_\mu \, {}^*\mathcal{F}^{\mu\nu} = 0, \tag{11}$$

which are highly nonlinear in $\mathcal{F}_{\mu\nu}$.

The Lagrangian [2, 3, 7]

$$\mathcal{L} = Q + K_0 Q^2 + K_1 R^2 \tag{12}$$

describes the Heisenberg–Euler theory with the constants

$$K_0 = \frac{8\alpha^2}{45 m_e^4}, \quad K_1 = \frac{14\alpha^2}{45 m_e^4},$$

where α is the fine structure constant, m_e is the electron mass, and Q, R are the invariants of the electromagnetic field (2).

A model of nonlinear electrodynamics which is described by the Lagrangian

$$\mathcal{L} = -\left[\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right)^2 / \Lambda^8\right]^{\frac{\delta-1}{2}} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{13}$$

where Λ and δ are the parameters, gives an example of a theory, where the nonlinearities act in a way to sufficiently amplify the initial magnetic field [7]. Originally, the non-Abelian version of this model had been proposed to describe the low-energy QCD [21].

A legitimate question to ask is whether there exists an extension of the Maxwell action for an arbitrary dimension that possesses the conformal invariance. The answer is positive, and the conformally invariant Maxwell action is given as [22]

$$S_{\mathrm{M}} = -\alpha \int d^d x \sqrt{-g} \left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right)^{\frac{d}{4}}. \tag{14}$$

It is not hard to see that, under the conformal transformation which acts on the fields as $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $A_\mu \to A_\mu$, action (14) is not changed. Note that the conformal action (14) for $d = 4$ is reduced to the Maxwell action, as it should be.

From action (14), we obtain the energy-momentum tensor [22]

$$T_{\mu\nu} = 4\alpha \left(\frac{d}{4}\mathcal{F}_{\mu\rho}\mathcal{F}^{\rho}_{\nu}\mathcal{F}^{\frac{d}{4}-1} - \frac{1}{4} g_{\mu\nu}\mathcal{F}^{\frac{d}{4}}\right), \tag{15}$$

where $\mathcal{F} = \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ is the Maxwell invariant, and the conformal invariance of the action is encoded in the traceless condition $T^{\mu}_{\mu} = 0$.

We say that an arbitrary bilinear function $j^\mu = j^\mu\left(P^{\alpha\beta}(\mathcal{F}),\ {}^*\mathcal{F}^{\alpha\beta}, \partial^\gamma P^{\alpha\beta}, \partial^\gamma\ {}^*\mathcal{F}^{\alpha\beta}\right)$ is a conserved current if it satisfies the continuity equation

$$\partial_\mu j^\mu = 0, \quad \mu = 0,1,2,3. \tag{16}$$

Here, $\mathcal{F}^{\mu\nu}$ is the tensor of an electromagnetic field, and $P^{\alpha\beta}(\mathcal{F})$ is given by (5).

According to the Ostrogradskii–Gauss theorem, it follows from (16) that the following quantity is conserved in time:

$$\mathcal{J}_0 = \int d^3x j_0. \tag{17}$$

A generalization of the Lie–Poisson brackets [18] for $P^{\mu\nu}$ and ${}^*\mathcal{F}^{\mu\nu}$ is given by

$$j^\alpha_{\rm BI} = \{P,\ {}^*\mathcal{F}\} = P_{\mu\nu}\partial^\nu\ {}^*\mathcal{F}^{\mu\alpha} -\ {}^*\mathcal{F}_{\mu\nu}\partial^\nu P^{\mu\alpha}. \tag{18}$$

Notice that

$$\partial_\alpha j^\alpha_{\rm BI} = j^\alpha_{{\rm BI},\alpha} = P_{\mu\nu,\alpha}\ {}^*\mathcal{F}^{\mu\alpha,\nu} + P_{\mu\nu}\ {}^*\mathcal{F}^{\mu\alpha,\nu}{}_\alpha -$$

$$-\ {}^*\mathcal{F}_{\mu\nu,\alpha}P^{\mu\alpha,\nu} -\ {}^*\mathcal{F}_{\mu\nu}P^{\mu\alpha,\nu}{}_\alpha = P_{\mu\nu,\alpha}\ {}^*\mathcal{F}^{\mu\alpha,\nu} -$$

$$-\ {}^*\mathcal{F}^{\mu\nu,\alpha}P_{\mu\alpha,\nu} + P_{\mu\nu}\ {}^*\mathcal{F}^{\mu\alpha,\nu}{}_\alpha -\ {}^*\mathcal{F}_{\mu\nu}P^{\mu\alpha,\nu}{}_\alpha = 0, \tag{19}$$

where we used the motion equation (11) and changed the index summation $\alpha \leftrightarrow \nu$.

Furthermore, the current

$$j^\alpha_{\rm M} = \{\mathcal{F},\ {}^*\mathcal{F}\} = \mathcal{F}_{\mu\nu}\partial^\nu\ {}^*\mathcal{F}^{\mu\alpha} -\ {}^*\mathcal{F}_{\mu\nu}\partial^\nu\mathcal{F}^{\mu\alpha} =$$

$$=\ {}^*\mathcal{F}^{\alpha\nu}\partial^\nu\mathcal{F}_{\mu\nu} - \mathcal{F}^{\alpha\mu}\partial^\nu\ {}^*\mathcal{F}_{\mu\nu} \tag{20}$$

satisfies the equation

$$j^\alpha_{\rm M} = 0 \tag{21}$$

with the Maxwell equations for the electromagnetic field.

Current (20) coincides with the vector Lagrangian for an electromagnetic field [20]. Namely, it is the minimum equation (20) from other theories (9), (12)–(14) of nonlinear electrodynamics.

3. General Covariant and Gauge Invariant Infinite Hierarchy of Conservation Currents

The equations that describe the dynamics of electromagnetic fields for the Lagrangians $\mathcal{L}(Q,R)$ (9), (12), (13) and action (14) in a curved space-time are

$$\nabla_\alpha P^{\alpha,\beta} = 0, \quad \text{(field equations)} \tag{22}$$

$$\nabla_\alpha\ {}^*\mathcal{F}^{\alpha\beta} = 0, \quad \text{(Bianchi identities)} \tag{23}$$

where ∇_α denotes the covariant differentiation. We define

$$j^\mu = P^{\mu\alpha}\mathcal{L}_{;\alpha}\,, \tag{24}$$

$${}^*j^\mu =\ {}^*\mathcal{F}^{\mu\nu}\mathcal{L}_{;\alpha}\,, \tag{25}$$

where $;\alpha$ denotes the covariant derivative.

Then the equations of motion (22) and (23) and relation [24] yield

$$\mathcal{L}_{;\alpha\beta} = \mathcal{L}_{;\beta\alpha} \tag{26}$$

and

$$j^\mu_{;\mu} = 0, \quad {}^*j^\mu_{;\mu} = 0 \tag{27}$$

in the 4-dimensional space-time of general relativity (Riemann space-time).

The general covariant conserved currents (24) and (25) satisfy the composition law [23]

$$[\delta j, \delta\ {}^*j] = \delta R, \tag{28}$$

where R is also a general covariant and gauge invariant conserved current [23]:

$$R^\alpha = j^\mu\nabla_\mu\ {}^*j^\alpha -\ {}^*j^\mu\nabla_\mu j^\alpha. \tag{29}$$

These infinite hierarchy of the currents j^μ, ${}^*j^\mu$, and R^μ are conserved upon the action of gravitational fields, but the conservation is broken in the Einstein–Cartan theory. The axial-vector currents (18) and (25) are conserved only in a $(3+1)$-dimensional space-time.

4. Discussion

We note that the currents $\tilde{j}^\mu = j^\mu\mathcal{L}^{-1}$ and ${}^*\tilde{j}^\mu = {}^*j^\mu\mathcal{L}^{-1}$ also conserved, and their physical dimension $[j^\mu] \sim m^3$ coincides with that of the Maxwell current in the interaction term $j^\mu A_\mu$.

We have shown that there is a class of gauge field theories that have an infinite set of conservation laws in the $(3+1)$-dimensional space-time of general relativity. It is a more delicate question to decide when the existence of an infinite number of independent conservation laws implies the complete integrability [25]. This is the open question, as well as the possible physical sense of the existence of currents $\tilde{j}^\mu$ and ${}^*\tilde{j}^\mu$ possessing a physical dimension of the Maxwell current.

I am grateful to V.P. Gusynin and V.N. Shadura for the fruitful discussions. I thank also the Referee for helpful suggestions.

1. M. Born and L. Infeld, Proc. Roy. Soc. (London) A **144**, 425 (1934).
2. W. Heisenberg and H. Euler, Z. für Physik **98**, 714 (1936).
3. V.S. Weisskopf and K. Dan, Vidensk. Selsk. Mat. Fys. Medd. **14**, n. 6 (1936).
4. Z. Bialynicka-Birula and J. Bialynicka-Birula, Phys. Rev. D **2**, 2341 (1970).
5. S.L. Adler, J.N. Bahcall, C.G. Callan, and M.N. Rosenbluth, Phys. Rev. Lett. **25**, 1061 (1970); S.L. Adler, Annals Phys. **67**, 599 (1971).
6. M.G. Baring, Phys. Rev. D **62**, 016003 (2000); A.K. Garding and D. Lai, Rept. Prog. Phys. **69**, 2631 (2006).
7. K.E. Kunze, e-print ArXiv: 0710.2435 (2007).
8. E. Jacopini and E. Zavatinni, Phys. Lett. B **85**, 151 (1979).
9. E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B **163**, 123 (1985); A.A. Tseytlin, e-print ArXiv: hep-th/9908105 (1999).
10. G.W. Gibbons and C.A.R. Herdeiro, Phys. Rev. D **63**, 064006 (2001).
11. R.G. Leigh, Mod. Phys. Lett. A **4**, 2767 (1989).
12. G. Boillat, J. Math. Phys. **11**, 941 (1970).
13. E. Bessel-Hagen, Math. Ann. **84**, 258 (1921); B.F. Plybon, Amer. J. Phys. **42**, 998 (1974).
14. E. Noether, Nachr. Ges. Göttingen **2**, 235 (1918).
15. W.I. Fushchych and A.G. Nikitin, *Symmetries of Maxwell Equations* (Reidel, Dordrecht, 1987); W.I. Fushchych and A.G. Nikitin, *Symmetries of the Equations of Quantum Mechanics* (New York, Alberton Press, 1994); W.I. Fushchych and A.G. Nikitin, J. Phys. A **25**, L231 (1992).
16. S.C. Anco and J. Pohjanpelto, Acta Appl. Math. **69**, 285 (2002).
17. G.W. Gibbons and D.A. Rashed, Nucl. Phys. B **454**, 185 (1995).
18. F. Margi, J. Math. Phys. **19**, 1115 (1978); Nuovo Cim. B **34**, 334 (1976).
19. F. Margi, C. Morosi, and O. Ragnico, Commun. Math. Phys. **99**, 115 (1985).
20. T.A. Morgan and D.W. Joseph, Nuovo. Cim. **39**, 494 (1965); W.I. Fushchich, I.Yu. Krivsky, and V.M. Simulik, Nuovo Cim. B **103**, 423 (1989).
21. H. Pagels and E. Tomboulis, Nucl. Phys. B **143**, 485 (1978).
22. V. Hassaine and C. Martinez, Phys. Rev. D **75**, 027502 (2007).
23. E. D'Hoker and R. Jackiw, Phys. Rev. D **26**, 3517 (1982).
24. A.P. Ligtman, W.H. Press, R.H. Price, and S.A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton Univ. Press, Princeton, 1975).
25. E.G.B. Hohler and K. Olaussen, Mod. Phys. Lett. A **8**, 3377 (1993).

Received 10.11.10

ЛОКАЛЬНІ ЗАКОНИ ЗБЕРЕЖЕННЯ В НЕЛІНІЙНІЙ ЕЛЕКТРОДИНАМІЦІ

О.І. Бацула

Р е з ю м е

Використовуючи узагальнення дужок Лі–Пуассона для дуального тензора напруженості Максвелла та тензора напруженості Борна–Інфельда, будуємо калібрувально-інваріантні аксіально-векторні струми, що зберігаються, для нелінійної електродинаміки Борна–Інфельда, а також Гейзенберга–Ейлера в 4-вимірному просторі-часі Мінковського. Встановлено нескінченну ієрархію струмів, що зберігаються, яка задана дужками Лі коваріантних векторного та аксіально-векторного струмів. Ці струми зберігаються в гравітаційному полі, але це збереження порушене в теорії Ейнштейна–Картана (в просторі-часі Рімана–Картана). Аксіально-векторні струми зберігаються тільки в (3+1)-вимірному просторі-часі.