ALEXANDER POLYNOMIAL INVARIANTS OF TORUS KNOTS $T(n, 3)$ AND CHEBYSHEV POLYNOMIALS

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The explicit formula, which expresses the Alexander polynomials $\Delta_{n,3}(t)$ of torus knots $T(n, 3)$ as a sum of the Alexander polynomials $\Delta_{k,2}(t)$ of torus knots $T(k, 2)$, is found. Using this result and those from our previous papers, we express the Alexander polynomials $\Delta_{n,3}(t)$ through Chebyshev polynomials. The latter result is extended to general torus knots $T(n, l)$ with n and l coprime.

1. Introduction

The interplay between knot theory and physics manifests itself in various ways [1–4]. Yet in 1975, L.D. Faddeev proposed that knot-like solitons classified by the integer-valued Hopf charge could be constructed in a modified sigma model in the three-dimensional space [5]. But only after the paper by Faddeev and Niemi [6] in 1997, the true hunting for knot configurations in theoretical physics began [7–9]. Basing on the heavy usage of computer facilities, the conjecture that the solitons with minimal energy might take the form of knots was confirmed. The further increase of the computer power demonstrated that a number of linked and knotted configurations, which are local or global solutions with minimal energy, do exist in the well-known Faddeev–Skyrme field-theory model. This issue was thoroughly elaborated in subsequent papers (e.g., in [10, 11]) and is still developed. Let us recall that the concept and the role of knots is of basic importance for the non-perturbative sector of non-Abelian gauge theory (see [12] and references therein).

As for the direct connection of knots with particle physics, one should mention the old paper by H. Jehle with the heuristic assigning of knot structures to real particles such as hadrons, neutrinos, and electrons [13]. More recently and on different grounds (using quantum groups or algebras) the connection "knots↔ particles" was exploited in [14, 15]. In particular, some set of torus knots appears in the context of meson phenomenology, namely in connection with quarkonia mass relations [14].

It is worth to underline the importance (and, thus, a wide usage) of applying different polynomial invariants of knots to modern physics problems. E. Witten, for example, has shown that the Jones polynomial invariants can be realized by the tools of topological quantum field theory in $2+1$ dimensions [16]. In addition, the Alexander polynomials and their well-known generalization, the HOMFLY polynomials, play a great role as well.

In our preceding paper [17], we studied the Alexander polynomial invariants of torus knots $T(n, 2)$ from the viewpoint of their connection with Chebyshev polynomials, using the so-called q-numbers. Therein, we explored a direct implication of the properties of q-numbers and their generalization, p, q-numbers, along with Chebyshev polynomials, for reproducing (from recursion relations) the famous skein relations that determine the Alexander polynomials and also HOMFLY polynomials. We note that the relation of some knots to the Chebyshev polynomials (in the context different from ours) was studied in [18]. It should be emphasized that our whole treatment in paper [17] concerned with the simplest, though nontrivial, set of torus knots – the series $T(n, 2)$, n being an odd integer. Since the Alexander (and HOMFLY) polynomials provide the most important and widely used characteristics of knots and links, their thorough investigation is believed to be helpful for the further understanding of the properties of knot-like structures from the first principles and for clarifying the physical interpretation of these knot invariants.

In the present paper, we first concentrate on the study of the Alexander polynomial invariants $\Delta_{n,3}(t)$ for the set of torus knots $T(n,3)$, with n and 3 coprime, and on their close connection with relevant Chebyshev polynomials. Then, we extend our treatment to the general class of torus knots $T(n,l)$, $l \geq 2$, with n and l coprime. Like in our preceding paper, the concept of q-numbers turns out to be very helpful.

2. Chebyshev Polynomials

Since we will exploit the Chebyshev polynomials below, let us give some sketch of them. Chebyshev polynomials of the first kind $T_n(x)$ can be defined as

$$T_n(x) = 2\cos(n\theta), \qquad 2\cos\theta = x. \tag{1}$$

This normalization means that the Chebyshev polynomials are monic (with unit coefficient at x^n). These Chebyshev polynomials, with account of the formula

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

are seen to satisfy the relation

$$T_{n+1} = xT_n - T_{n-1}, \qquad T_0 = 2, \quad T_1 = x, \tag{2}$$

which is nothing but the recurrence relation. The latter can be used as an alternative definition of $T_n(x)$. Some few low-degree cases of $T_n(x)$ read

$$T_0 = 2, \quad T_1 = x, \quad T_2 = x^2 - 2, \quad T_3 = x^3 - 3x,$$
$$T_4 = x^4 - 4x^2 + 2, \quad T_5 = x^5 - 5x^3 + 5x.$$

Chebyshev polynomials of the second kind can be defined as

$$V_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad 2\cos\theta = x \tag{3}$$

or through the recurrence relation

$$V_{n+1} = xV_n - V_{n-1}, \qquad V_0 = 1, \quad V_1 = x. \tag{4}$$

First few low-order Chebyshev polynomials of the second kind are as follows:

$$V_0 = 1, \quad V_1 = x, \quad V_2 = x^2 - 1, \quad V_3 = x^3 - 2x,$$
$$V_4 = x^4 - 3x^2 + 1, \quad V_5 = x^5 - 4x^3 + 3x.$$

There exists a connection between $T_n(x)$ and $V_n(x)$, namely

$$T_n(x) = V_n(x) - V_{n-2}(x), \qquad n \geq 1, \tag{5}$$

where, and throughout the text, we omit polynomials with negative indices (at $n = 1$ we omit $V_{-1}(x)$).

Note that Chebyshev polynomials of the first kind can be presented in the form

$$T_n(x) = t^n + t^{-n}, \qquad t + t^{-1} = x, \tag{6}$$

while Chebyshev polynomials of the second kind as

$$V_n(x) = \frac{t^{n+1} - t^{-n-1}}{t - t^{-1}}, \qquad t + t^{-1} = x. \tag{7}$$

This is obvious if we put $t = e^{i\theta}$.

3. Alexander Polynomials for Torus Knots

The Alexander polynomials $\Delta(t)$ for knots and links can be defined by the skein relation [19]

$$\Delta_+(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_O(t) + \Delta_-(t) \tag{8}$$

along with the normalization condition (for an unknot)

$$\Delta_{\text{unknot}} = 1. \tag{9}$$

Using (8)-(9), one can find the Alexander polynomial for any knot or link by applying, in a standard way, the operations of "switching" and "elimination".

We focus on the torus knots $T(n,l)$, where n and l are coprime positive integers (the torus knots $T(n,l)$ and $T(l,n)$ are equivalent). As known [19], the Alexander polynomial invariant for $T(n,l)$ is given by the formula

$$\widetilde{\Delta}_{n,l}(t) = \frac{(t^{nl} - 1)(t - 1)}{(t^n - 1)(t^l - 1)}. \tag{10}$$

For $l = 2$ this yields

$$\widetilde{\Delta}_{n,2}(t) = \frac{t^n + 1}{t + 1}, \qquad n = 1, 3, 5, 7, \ldots. \tag{11}$$

Let us list few examples of polynomial (11):

$$\widetilde{\Delta}_{1,2}(t) = 1, \quad \widetilde{\Delta}_{3,2}(t) = t^2 - t + 1,$$

$$\widetilde{\Delta}_{5,2}(t) = t^4 - t^3 + t^2 - t + 1.$$

For $l = 3$, relation (10) yields

$$\widetilde{\Delta}_{n,3}(t) = \frac{t^{2n} + t^n + 1}{t^2 + t + 1}, \qquad n = 1, 2, 4, 5, 7, 8, \ldots. \tag{12}$$

The first entries of (12) are

$$\widetilde{\Delta}_{1,3}(t) = 1, \quad \widetilde{\Delta}_{2,3}(t) = t^2 - t + 1,$$

$$\widetilde{\Delta}_{4,3}(t) = t^6 - t^5 + t^3 - t + 1\,.$$

Note that $\widetilde{\Delta}_{1,2}(t)$, $\widetilde{\Delta}_{1,3}(t)$, and $\widetilde{\Delta}_{1,l}(t)$ all agree with (9). The same concerns $\widetilde{\Delta}_{n,1}(t)$, see (10).

The Alexander polynomials (10) are reduced to the standard polynomial form and have the degree $nl - n - l + 1 = (n-1)(l-1)$. However, the Alexander polynomials $\widetilde{\Delta}_{n,l}(t)$ can also be presented in the form of the Laurent polynomials $\Delta_{n,l}(t)$ possessing the explicit $t \leftrightarrow t^{-1}$ symmetry. This (Laurent type) form, with account of the multiplier $t^{-\frac{nl-n-l+1}{2}}$, satisfies the relation (note the absence of a tilde on the l.h.s.)

$$\Delta_{n,l}(t) = \widetilde{\Delta}_{n,l}(t)\cdot t^{-\frac{nl-n-l+1}{2}} = \frac{(t^{\frac{nl}{2}} - t^{-\frac{nl}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(t^{\frac{l}{2}} - t^{-\frac{l}{2}})}\,. \tag{13}$$

By m, we denote the highest positive degree of the Alexander polynomial $\Delta_{n,l}(t)$ given as a Laurent polynomial in (13):

$$m = \frac{1}{2}(n-1)(l-1)\,. \tag{14}$$

From now on, we consider two cases:

$$l = 2\,, \quad n = 2m+1\,; \qquad l = 3\,, \quad n = m+1\,. \tag{15}$$

For $l = 2$, Eq. (13) gives

$$\Delta_{n,2}(t) = \frac{t^{\frac{n}{2}} + t^{-\frac{n}{2}}}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}} \tag{16}$$

of which the first several examples (n=1, 3, 5) are

$$\Delta_{1,2}(t) = 1\,, \qquad \Delta_{3,2}(t) = t - 1 + t^{-1}\,,$$

$$\Delta_{5,2}(t) = t^2 - t + 1 - t^{-1} + t^{-2}\,.$$

If $l = 3$, (13) yields

$$\Delta_{n,3}(t) = \frac{t^n + 1 + t^{-n}}{t + 1 + t^{-1}}\,. \tag{17}$$

A few examples for n=1, 2, 4 are

$$\Delta_{1,3}(t) = 1\,, \qquad \Delta_{2,3}(t) = t - 1 + t^{-1}\,,$$

$$\Delta_{4,3}(t) = t^3 - t^2 + 1 - t^{-2} + t^{-3}$$

(note the coincidence of $\Delta_{2,3}(t)$ with $\Delta_{3,2}(t)$ above).

Below, we find a connection between the Alexander polynomials for $T(n,3)$ and $T(k,2)$ torus knots.

4. Presenting $\Delta_{n,3}(t)$ in Terms of $\Delta_{k,2}(t)$

The goal of this section is to express any Alexander polynomial $\Delta_{n,3}(t)$ as an algebraic sum of Alexander polynomials for $T(k,2)$ torus knots.

Proposition 1. *The (Laurent form) Alexander polynomials for the torus knots $T(n,3)$ and the torus knots $T(k,2)$ are connected through the relation*

$$\Delta_{n,3}(t) - \Delta_{n-3,3}(t) = \Delta_{2n-1,2}(t) - \Delta_{2n-5,2}(t)\,. \tag{18}$$

The proof goes by straightforward checking with the use of formulas (16) and (17).

Proposition 2. *The Alexander polynomial of torus knot $T(n,3)$ is expressible as a sum of the Alexander polynomials of torus knots $T(k,2)$ by the formula*[1]

$$\Delta_{n,3}(t) = \sum_{j=0}^{d}\bigl(\Delta_{2n-1-6j,\,2}(t) - \Delta_{2n-5-6j,\,2}(t)\bigr)\,, \tag{19}$$

where d is the integer part of $\frac{2n-1}{6}$: $d = \left[\frac{2n-1}{6}\right]$.

To prove (19), we rewrite Eq. (18) first as

$$\Delta_{n,3}(t) = \Delta_{2n-1,2}(t) - \Delta_{2n-5,2}(t) + \Delta_{n-3,3}(t) \tag{20}$$

and, shifting n to $n-3$, we deduce the relation

$$\Delta_{n-3,3}(t) = \Delta_{2n-7,2}(t) - \Delta_{2n-11,2}(t) + \Delta_{n-6,3}(t) \tag{21}$$

from (20). Substituting (21) in (20) yields

$$\Delta_{n,3}(t) = \Delta_{2n-1,2}(t) - \Delta_{2n-5,2}(t) + \Delta_{2n-7,2}(t) -$$

$$-\Delta_{2n-11,2}(t) + \Delta_{n-6,3}(t)\,. \tag{22}$$

Similarly to (21), we further expand $\Delta_{n-6,3}(t)$ and put the result into (22), and so on. At the last stage of the process for $\Delta_{n,3}(t)$, we come to $\Delta_{2,3}(t)$ and/or $\Delta_{1,3}(t)$. Since $\Delta_{2,3}(t) = \Delta_{3,2}(t)$ and $\Delta_{1,3}(t) = 1 = \Delta_{1,2}(t)$, we obtain the expression for the Alexander polynomial of torus knot $T(n,3)$ given through a sum of the Alexander polynomials for torus knots $T(n,2)$ only.

It is worth to give, from (19), a few examples of expressing the Alexander polynomials $\Delta_{n,3}(t)$ by a sum of the Alexander polynomials $\Delta_{n,2}(t)$:

$$\Delta_{1,3}(t) = \Delta_{1,2}(t)\,,$$

$$\Delta_{2,3}(t) = \Delta_{3,2}(t)$$

[1] Note that the remark similar to the one after (5) is applied here too.

$$\Delta_{4,3}(t)=\Delta_{7,2}(t)-\Delta_{3,2}(t)+\Delta_{1,2}(t)\,,$$

$$\Delta_{5,3}(t)=\Delta_{9,2}(t)-\Delta_{5,2}(t)+\Delta_{3,2}(t)\,,$$

$$\Delta_{7,3}(t)=\Delta_{13,2}(t)-\Delta_{9,2}(t)+\Delta_{7,2}(t)-$$

$$-\Delta_{3,2}(t)+\Delta_{1,2}(t)\,,$$

$$\Delta_{8,3}(t)=\Delta_{15,2}(t)-\Delta_{11,2}(t)+\Delta_{9,2}(t)-$$

$$-\Delta_{5,2}(t)+\Delta_{3,2}(t)\,,$$

$$\Delta_{10,3}(t)=\Delta_{19,2}(t)-\Delta_{15,2}(t)+\Delta_{13,2}(t)-$$

$$-\Delta_{9,2}(t)+\Delta_{7,2}(t)-\Delta_{3,2}(t)+\Delta_{1,2}(t)\,. \qquad (23)$$

Thus, we have obtained the formula expressing (each from) the set $\Delta_{n,3}(t)$ through the sum of a definite number of the Alexander polynomials $\Delta_{k,2}(t)$ with proper signs.

It would be of interest to generalize this result to any torus knot $\Delta_{n,l}(t)$.

5. Expressing the Alexander Polynomials by Chebyshev Polynomials

In this section, we present the Alexander polynomials $\Delta_{n,2}(t)$ and $\Delta_{n,3}(t)$ in terms of the Chebyshev polynomials $T_n(x)$ and $V_n(x)$, $x=t+t^{-1}$.

Proposition 3. *There exists the following relation between the Alexander polynomials $\Delta_{n,2}(t)$ for torus knots $T(n,2)$ and Chebyshev polynomials of the second kind:*

$$\Delta_{n,2}(t)\equiv\Delta_{2m+1,2}(t)=$$

$$=V_m(x)-V_{m-1}(x)\,,\quad x=t+t^{-1}\,. \qquad (24)$$

To prove (24), we use formula (16), along with $V_{m-1}(t)$ and $V_m(t)$ taken from (7). Note that an analog of (24) was first found in [14] on the base of using the q-numbers.

Equation (24) can also be written as

$$\Delta_{n,2}(t)=V_{\frac{n-1}{2}}(x)-V_{\frac{n-3}{2}}(x)\,,\quad x=t+t^{-1}\,. \qquad (25)$$

From (24) and (5), we find ($m\geq 1$):

$$T_m(x)=\Delta_{2m+1,2}(t)+\Delta_{2m-1,2}(t)\,,\quad t+t^{-1}=x\,. \qquad (26)$$

Hence, Eq. (26) expresses the Alexander polynomials in terms of Chebyshev polynomials of the first kind. Here are some examples:

$$\Delta_{3,2}(t)=T_1(x)-\Delta_{1,2}(t)=T_1(x)-1\,,$$

$$\Delta_{5,2}(t)=T_2(x)-T_1(x)+1\,,$$

$$\Delta_{7,2}(t)=T_3(x)-T_2(x)+T_1(x)-1\,. \qquad (27)$$

The general formula covering (27) is

$$\Delta_{2m+1,2}(t)=T_m(x)-T_{m-1}(x)+T_{m-2}(x)-\cdots+$$

$$+(-1)^{m-1}T_1(x)+(-1)^m=\sum_{k=0}^{m-1}(-1)^kT_{m-k}(x)+(-1)^m\,. \qquad (28)$$

Proposition 4. *The Alexander polynomial for torus knot $T(n,3)$ is expressed as the sum of Chebyshev polynomials of the second kind by the relation*

$$\Delta_{n,3}(t)=V_{n-1}(x)+$$

$$+\sum_{k=0}^{d}\big(-V_{n-2-3k}(x)-V_{n-3-3k}(x)+2V_{n-4-3k}(x)\big), \qquad (29)$$

where $d=\left[\frac{n-2}{3}\right]$, $x=t+t^{-1}$.

This immediately follows from (19) and (24).

From formula (29), we have

$$\Delta_{n,3}(t)=V_{n-1}(x)-V_{n-2}(x)-V_{n-3}(x)+$$

$$+2V_{n-4}(x)-V_{n-5}(x)-V_{n-6}(x)+$$

$$+2V_{n-7}(x)-V_{n-8}(x)-\cdots\,,\quad x=t+t^{-1}\,. \qquad (30)$$

Some first examples of (30) are

$$\Delta_{1,3}(t)=V_0(x),\quad \Delta_{2,3}(t)=V_1(x)-V_0(x),$$

$$\Delta_{4,3}(t)=V_3(x)-V_2(x)-V_1(x)+2V_0(x),$$

$$\Delta_{5,3}(t)=V_4(x)-V_3(x)-V_2(x)+2V_1(x)-V_0(x),$$

$$\Delta_{7,3}(t)=V_6(x)-V_5(x)-V_4(x)+2V_3(x)-V_2(x)-$$

$-V_1(x) + 2V_0(x)$.

Proposition 5. *The Alexander polynomial for the torus knot $T(n,3)$ is expressed through Chebyshev polynomials of the first kind by the formula*

$$\Delta_{n,3}(t) = \sum_{k=0}^{d} \big(T_{n-1-3k}(x) - T_{n-2-3k}(x)\big) + (-1)^{n-d}, \quad (31)$$

where $d = \left[\frac{n-1}{3}\right]$, $x = t + t^{-1}$.

The proof follows from the joint use of (19) and (26).

Here are some special cases of (31):

$$\Delta_{1,3}(t) = T_0(x) - 1, \quad \Delta_{2,3}(t) = T_1(x) - T_0(x) + 1,$$

$$\Delta_{4,3}(t) = T_3(x) - T_2(x) + T_0(x) - 1,$$

$$\Delta_{5,3}(t) = T_4(x) - T_3(x) + T_1(x) - T_0(x) + 1,$$

$$\Delta_{7,3}(t) = T_6(x) - T_5(x) + T_3(x) - T_2(x) +$$

$$+T_0(x) - 1.$$

6. Some Useful Formulas for the Alexander Polynomials

Putting definition (3) of the Chebyshev polynomials in (24), we obtain the following formula for the Alexander polynomials $\Delta_{2m+1,2}(t)$:

$$\Delta_{2m+1,2}(t) = \frac{\cos(m+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} \qquad 2\cos\theta = t + t^{-1}\,, \quad (32)$$

or, setting $m = \frac{1}{2}(n-1)$,

$$\Delta_{n,2}(t) = \frac{\cos\frac{n\theta}{2}}{\cos\frac{\theta}{2}}\,, \qquad 2\cos\theta = t + t^{-1}\,, \quad (33)$$

where $n = 1, 3, 5, 7, \cdots$. From the latter, remembering that $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$, we have

$$\Big(\Delta_{n,2}(t)\Big)^2 = \frac{T_n(x)+2}{x+2}\,, \qquad x = t + t^{-1}\,. \quad (34)$$

From (33) and (3), we find

$$\Delta_{n,2}(t) = \frac{V_{2n-1}(y)}{yV_{n-1}(y)}\,, \qquad y^2 = t + t^{-1} + 2\,. \quad (35)$$

Rewriting (6) as

$$T_n(y) = t^{\frac{n}{2}} + t^{-\frac{n}{2}}\,, \qquad y = t^{\frac{1}{2}} + t^{-\frac{1}{2}} \quad (36)$$

and putting it in (16), we have

$$\Delta_{n,2}(t) = \frac{T_n(y)}{y}\,. \quad (37)$$

Using (17) and (36), we obtain the formula for $\Delta_{n,3}(t)$ in terms of $T_n(y)$:

$$\Delta_{n,3}(t) = \frac{T_n^2(y) - 1}{y^2 - 1}\,,$$

where $n = 1, 2, 4, 5, 7, \ldots$. Using (6), Eq. (17) can also be written in the form

$$\Delta_{n,3}(t) = \frac{T_n(x)+1}{x+1}\,, \qquad x = t + t^{-1}\,.$$

7. Alexander Polynomials of $T(n,l)$ in Terms of q-Numbers

In this section, we extend the formula (24) valid for $l = 2$ to the general case of arbitrary l. To this end, like in [17], we use the q-numbers [20–23]. Recall that the q-number corresponding to an integer n is defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,, \quad (38)$$

with q being a parameter. If $q \to 1$, we recover n: $[n]_q \xrightarrow{q\to 1} n$. If q is viewed as a variable, we arrive at q-polynomials.

With account of (38), Eq. (13) can be immediately rewritten in terms of q-numbers to yield ($q \equiv t$):

$$\Delta_{n,l}(t) = \frac{[nl]_{t^{\frac{1}{2}}}}{[n]_{t^{\frac{1}{2}}}[l]_{t^{\frac{1}{2}}}} = \frac{[n]_{t^{\frac{l}{2}}}}{[n]_{t^{\frac{1}{2}}}} = \frac{[l]_{t^{\frac{n}{2}}}}{[l]_{t^{\frac{1}{2}}}}\,. \quad (39)$$

From (7) and (38), we have

$$V_n(x) = [n+1]_q\,, \qquad x = q + q^{-1}\,. \quad (40)$$

Formula (24) in terms of q-numbers reads

$$\Delta_{2m+1,2}(t) = [m+1]_t - [m]_t\,, \qquad t \equiv q\,, \quad (41)$$

or, since $n = 2m+1$,

$$\Delta_{n,2}(t) = \left[\frac{n+1}{2}\right]_t - \left[\frac{n-1}{2}\right]_t\,. \quad (42)$$

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It is easy to verify that (42) is reduced to (16). If $l=3$, relation (17) yields the formula somewhat similar to (42):

$$\Delta_{n,3}(t)=\Big[\frac{2n+1}{3}\Big]_{t^{\frac{3}{2}}}-\Big[\frac{2n-1}{3}\Big]_{t^{\frac{3}{2}}}+\Big[\frac{1}{3}\Big]_{t^{\frac{3}{2}}}. \qquad (43)$$

Each of the three terms can be rewritten, because

$$\Big[\frac{X}{3}\Big]_{t^{\frac{3}{2}}}=\frac{[X]_{t^{\frac{1}{2}}}}{[3]_{t^{\frac{1}{2}}}}.$$

As a result, Eq. (43) becomes

$$\Delta_{n,3}(t)=\frac{1}{[3]_{t^{\frac{1}{2}}}}\Big([2n+1]_{t^{\frac{1}{2}}}-[2n-1]_{t^{\frac{1}{2}}}+1\Big). \qquad (44)$$

In the case of arbitrary l, the generalization of (42) and (43) can be obtained from (13), namely

$$\Delta_{n,l}(t)=\Big[\tfrac{n(l-1)+1}{l}\Big]_{t^{\frac{l}{2}}}-\Big[\tfrac{n(l-1)-1}{l}\Big]_{t^{\frac{l}{2}}}+\frac{[l-2]_{t^{\frac{n}{2}}}}{[l]_{t^{\frac{1}{2}}}}. \qquad (45)$$

From this, we deduce a generalization of Eq. (44):

$$\Delta_{n,l}(t)=\frac{[n(l-1)+1]_{t^{\frac{1}{2}}}}{[l]_{t^{\frac{1}{2}}}}-\frac{[n(l-1)-1]_{t^{\frac{1}{2}}}}{[l]_{t^{\frac{1}{2}}}}+$$

$$+\frac{[n(l-2)]_{t^{\frac{1}{2}}}}{[n]_{t^{\frac{1}{2}}}}\cdot\frac{1}{[l]_{t^{\frac{1}{2}}}}. \qquad (46)$$

8. Alexander Polynomials of $T(n,l)$ and Chebyshev Polynomials

We can express (44) in terms of the Chebyshev polynomials:

$$\Delta_{n,3}(t)=\frac{V_{2n}(y)-V_{2n-2}(y)+1}{V_2(y)}, \qquad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}}. \quad (47)$$

Likewise, as an extension of the latter to the $T(n,l)$ torus knots, relation (46) yields

$$\Delta_{n,l}(t)=\frac{V_{n(l-1)}(y)-V_{n(l-1)-2}(y)}{V_{l-1}(y)}+$$
$$+\frac{V_{n(l-2)-1}(y)}{V_{l-1}(y)\cdot V_{n-1}(y)}, \qquad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}}. \qquad (48)$$

In addition, using Eqs. (39), we obtain another three formulas giving $\Delta_{n,l}(t)$ via the Chebyshev polynomials:

$$\Delta_{n,l}(t)=\frac{V_{nl-1}(y)}{V_{n-1}(y)\cdot V_{l-1}(y)}, \qquad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}} \qquad (49)$$

or

$$\Delta_{n,l}(t)=\frac{V_{n-1}(z_1)}{V_{n-1}(y)}, \quad z_1=t^{\frac{l}{2}}+t^{-\frac{l}{2}}, \quad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}}, \quad (50)$$

or

$$\Delta_{n,l}(t)=\frac{V_{l-1}(z_2)}{V_{l-1}(y)}, \quad z_2=t^{\frac{n}{2}}+t^{-\frac{n}{2}}, \quad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}}. \quad (51)$$

Now let us pay attention to the following very interesting point: it follows from (36) and (50) that

$$\Delta_{n,l}(t)=\frac{V_{n-1}\big(T_l(y)\big)}{V_{n-1}(y)}, \qquad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}}. \qquad (52)$$

Analogously, from (36) and (51), we obtain

$$\Delta_{n,l}(t)=\frac{V_{l-1}\big(T_n(y)\big)}{V_{l-1}(y)}, \qquad y=t^{\frac{1}{2}}+t^{-\frac{1}{2}}. \qquad (53)$$

As is seen, the numerators of both Eqs. (52) and (53) include the expression $V(\ldots)$, i.e. a Chebyshev polynomial of the second kind, in which the role of an argument, in turn, is played by the Chebyshev polynomial of the first kind.

9. Dependence of $\Delta_{n,l}(t)$ on n Through $\Delta_{n,2}(t)$

In this section, we show that the whole dependence of the Alexander polynomial $\Delta_{n,l}(t)$ on the number n can be given solely through some lower Alexander polynomial $\Delta_{n,k}(t)$, $k<l$, k being a fixed positive integer coprime both with l and n. Let us first consider the simplest case $k=2$.

Rewriting (16) in the form

$$\Delta_{n,2}(t)=\frac{Z+Z^{-1}}{t^{\frac{1}{2}}+t^{\frac{-1}{2}}}, \qquad Z\equiv t^{\frac{n}{2}},$$

we arrive at the equation

$$Z^2-(t^{\frac{1}{2}}+t^{-\frac{1}{2}})\Delta_{n,2}(t)Z+1=0$$

and at the identical one for Z^{-1}, which have the solutions

$$Z=\frac{1}{2}\Big((t^{\frac{1}{2}}+t^{-\frac{1}{2}})\Delta_{n,2}(t)\pm$$

$$\pm\sqrt{(t^{\frac{1}{2}}+t^{-\frac{1}{2}})^2(\Delta_{n,2}(t))^2-4}\Big),$$

$$Z^{-1}=\frac{1}{2}\Big((t^{\frac{1}{2}}+t^{-\frac{1}{2}})\Delta_{n,2}(t)\mp$$

$$\mp\sqrt{(t^{\frac{1}{2}}+t^{-\frac{1}{2}})^2(\Delta_{n,2}(t))^2-4}\,\Big).$$

This allows us to present (13) in the form

$$\Delta_{n,l}(t)=\frac{Z^l-Z^{-l}}{Z^1-Z^{-1}}\frac{t^{\frac{1}{2}}-t^{-\frac{1}{2}}}{t^{\frac{l}{2}}-t^{-\frac{l}{2}}}=$$

$$=\mathcal{F}\big(Z(t,\Delta_{n,2}(t)\ );l,t\big)=\mathcal{F}\big(\Delta_{n,2}(t);l,t\big)\,,$$

which means that all the functional dependence on n in the Alexander polynomial $\Delta_{n,l}(t)$ is contained in $\Delta_{n,2}(t)$ alone, n being coprime with l and 2. For example, the whole dependence on n for $\Delta_{7,5}(t)$ is in $\Delta_{7,2}(t)$, but in $\Delta_{8,3}(t)$ for $\Delta_{8,5}(t)$ (for the latter type connection, see below).

Yet another formula, which means that the explicit dependence on n for the Alexander polynomial $\Delta_{n,l}(t)$, is given by $\Delta_{n,k}(t)$, $k=2$ or 3, with respective n, follows from combining (51), (16), and (17), namely:

$$\Delta_{n,l}(t)=\frac{V_{l-1}(z)}{V_{l-1}(y)},\qquad y{=}t^{\frac{1}{2}}+t^{-\frac{1}{2}},\tag{54}$$

where if $k=2$ and n is coprime with l and 2, we have that

$$z=(t^{\frac{1}{2}}+t^{-\frac{1}{2}})\Delta_{n,2}(t)\,.$$

At last, if $k=3$ and n is coprime with l and 3, then we have, in (54), that

$$z=\Big((t+1+t^{-1})\Delta_{n,3}(t)+1\Big)^{\frac{1}{2}}\,.$$

10. Concluding Remarks

In this paper, we have shown how to express any Alexander polynomial $\Delta_{n,3}(t)$ as a (finite) sum of the summands $\Delta_{k,2}(t)$ with appropriate values of k, see formulas (19) and (29). Note that if one succeeds to solve the similar problem of expressing $\Delta_{n,l}(t)$ by the sum of $\Delta_{k,2}(t)$, it would give us some novel, interesting description for a general torus knot.

We have demonstrated for this, more general case of torus knots than the one treated in [17], the close connection of the Alexander polynomials $\Delta_{n,l}(t)$ with the Chebyshev polynomials. Among the others, we recall the result given by (52) showing that the dependence on n in the $\Delta_{n,l}(t)$ can be incorporated just in the Chebyshev polynomial of the second kind, whereas the dependence on l – in the Chebyshev polynomial of the first kind, the latter being the argument of the former one. We have also shown that the dependence of $\Delta_{n,l}(t)$ on n may be through $\Delta_{n,2}(t)$ $(n,l,2$ are coprime) or through $\Delta_{n,3}(t)$ $(n,l,3$ are coprime).

In the visible future, we hope to find a particular physical realization of the obtained results in the direction of constructing the knot-like configuration(s) within some field theory model.

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ПОЛІНОМІАЛЬНІ ІНВАРІАНТИ АЛЕКСАНДЕРА ТОРИЧНИХ ВУЗЛІВ $T(n,3)$ І ПОЛІНОМИ ЧЕБИШОВА

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Р е з ю м е

Отримана явна формула, яка виражає поліноми Александера $\Delta_{n,3}(t)$ торичних вузлів $T(n,3)$ через суму поліномів Александера $\Delta_{n,2}(t)$ торичних вузлів $T(n,2)$. На основі цього, а також результатів наших попередніх робіт, ми виражаємо поліноми Александера $\Delta_{n,3}(t)$ в термінах поліномів Чебишова. Даний результат поширено на довільні торичні вузли $T(n,l)$, де n та l – взаємно прості цілі числа.