CATEGORY OF VILENKIN–KUZNETSOV–SMORODINSKY–SMIRNOV TREES

S.S. MOSKALIUK,[1] N.M. MOSKALIUK[2]

[1]**Bogolyubov Institute for Theoretical Physics**
(14-b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: mss@bitp.kiev.ua)

[2]**Faculty of Mathematics, Uzhgorod National University**
(46, Pidhirna Str., Uzhgorod 88000, Ukraine)

PACS 02.10.De; 02.20.Bb; 02.90.+p

First, we briefly review the definitions and the basic properties of operads and trees. There are many useful types of operads, and each type is determined by the choice of two categories: basic symmetric monoidal category $(\mathcal{C}, \boxtimes)$, which supports the classical linear operads, and a category of graphs Γ reflecting the combinatorics of operadic data and axioms [1–6]. From this viewpoint, the specific operad is a functor $\Gamma \to \mathcal{C}$. Second, our aim is the construction of the category of Vilenkin–Kuznetsov–Smorodinsky–Smirnov (VKSS) trees, which contains VKSS-trees as objects and morphisms generated by a rotation of the n-dimensional space and transforming functions of VKSS-trees.

1. Introduction

The method of trees was invented to elucidate some problems of Lie group representations, and it was developed in the 1960-1970s by Vilenkin, Kuznetsov, and Smorodinsky (VKS) [7–11]. The further development of the Vilenkin–Kuznetsov–Smorodinsky method to Vilenkin–Kuznetsov–Smorodinsky–Smirnov (VKSS) trees was given by Yu. F. Smirnov [12, 13] on the base of the conception of complementarity of Lie groups (in the sense by Moshinsky [14]), which exists between groups $O(n)$ and $Sp(2, R)$. In particular, the problem of calculation of operator matrix elements in the method of K-harmonics is solved, which determined, in the long run, the practical significance of these results.

2. Classical Operads

Let us recall (see [15]) that the *symmetric monoidal category* $(\mathcal{C}, \boxtimes)$ is a category endowed with the bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ together with the involutive commutativity constraint and the associativity constraint. Taken together, they define a family of compatible and functorial isomorphisms $s_* : X_1 \boxtimes \cdots \boxtimes X_n \tilde{\to} X_{s^{-1}(1)} \boxtimes \cdots \boxtimes X_{s^{-1}(n)}$, for any of objects $X_1, \ldots, X_n$ of $\mathcal{C}$ and all $s \in \mathbf{S}_n$. The symmetric group $\mathbf{S}_n$ is defined as a group of bijections $\underline{n} \to \underline{n}$, where $\underline{n} = \{1, \ldots, n\}$.

Most of our monoidal categories will have an identity object $\mathbf{1}_{\mathcal{C}} = \mathbf{1}$. The functors $\mathbf{1}\boxtimes$ and $\boxtimes\mathbf{1} : \mathcal{C} \to \mathcal{C}$ are canonically isomorphic to the identity functor. We assume that $\mathcal{C}$ has small colimits preserved by any functor $X\boxtimes$. In particular, $\mathcal{C}$ must have an initial object 0.

Classical linear operads arise in the same way, when we start with a linear space V endowed with a family $\mathcal{P}$ of *poly*linear operators $V^{\otimes m} \to V, m = 1, 2, 3, \ldots$ (e.g., an associative algebra is such a space endowed with a multiplication map $V^{\otimes 2} \to V$). Closing $\mathcal{P}$ with respect to compositions (of functions with many variables) and linear combinations, we get a (specific) classical operad together with its linear representation in V. Axiomatizing the universal properties of such an object, we arrive at the following notion.

Definition 2.1 *A* ***classical operad*** $\mathcal{P}$ *consists of data a) – d) satisfying axioms A) – C) below.*
a) A family of linear spaces $\mathcal{P}(l)$*, for all* $l \geq 1$.
b) A left/right linear action of $\mathbf{S}_l$ *on* $\mathcal{P}(l)$*, for all* $l \geq 1$ *:* $s \in \mathbf{S}_l$ *maps* $f \in \mathcal{P}(l)$ *to* $fs = s^{-1}f$.
c) A family of composition maps $\gamma(k_1, \ldots, k_l)$*, for all* $l \geq 1, k_a \geq 1$*:*

$$\gamma(k_1, \ldots, k_l) : \quad \mathcal{P}(l) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_l) \to \mathcal{P}(k_1 + \cdots + k_l) . \tag{1}$$

d) (Optional). An identity element $I \in \mathcal{P}(1)$.

We will state the axioms for these data in two forms: directly in terms of γ and in functional notation. For the latter, put $\underline{\mathcal{P}} = \oplus_{k=1}^{\infty}\mathcal{P}(k)$ and notice that (1) allows us to consider each $f \in \mathcal{P}(l)$ as a polylinear function $\underline{\mathcal{P}}^l \to \underline{\mathcal{P}}$:

$$f(g_1,\dots,g_l) := \gamma(f \otimes g_1 \otimes \dots \otimes g_l)\ , \tag{2}$$

where $\gamma = \gamma((k_1,\dots,k_l)$ if $g_a \in \mathcal{P}(k_a)$. We will often write simply γ for such multigraded components of the operadic composition.

A) The symmetric group $\mathbf{S}_l$ acts on the functions (represented by) $\mathcal{P}(l)$ by a permutation of arguments:

$$(fs)(g_1,\dots,g_l) = f(s(g_1,\dots,g_l))\ . \tag{3}$$

In γ-notation:

$$\gamma(fs \otimes g_1 \otimes \dots \otimes g_l) = \gamma(f \otimes s(g_1 \otimes \dots \otimes g_l))\ . \tag{4}$$

In addition, for $s_1 \in S_{k_1},\dots,s_l \in S_{k_l}$, we denote, by $s_1 \times \dots \times s_l \in S_{k_1+\dots+k_l}$, the image of $(s_1,\dots,s_l)$ acting blockwise upon

$$(1,\dots,k_1|k_1+1,\dots,k_1+k_2|\dots|k_1+\dots+k_{l-1}+1,\dots,\\ \dots,k_1+\dots+k_l)\ .$$

Then

$$f(g_1s_1,\dots,g_ls_l) = (f(g_1,\dots,g_l))(s_1 \times \dots \times s_l)\ . \tag{5}$$

In γ-notation:

$$\gamma(f\otimes g_1s_1\otimes\dots\otimes g_ls_l) = (\gamma(f\otimes g_1\otimes\dots\otimes g_l))(s_1\times\dots\times s_l)\ . \tag{6}$$

B) The composition maps are associative with respect to the substitution (in functional notation). That is, for any $f \in \mathcal{P}(l)$, $g_a \in \mathcal{P}(k_a), a = 1,\dots,l$, and $h_{a,b} \in \mathcal{P}(l_{a,b}), b = 1,\dots,k_a$, we have

$$[f(g_1,\dots,g_l)](h_{1,1},\dots,h_{1,k_1};\dots;h_{l,1},\dots,h_{l,k_l}) = \\ = f(g_1(h_{1,1},\dots,h_{1,k_1}),\dots,g_l(h_{l,1},\dots,h_{l,k_l}))\ . \tag{7}$$

In γ-notation:

$$\gamma(\gamma(f \otimes g_1 \otimes \dots \otimes g_l) \otimes h_{1,1} \otimes \dots \otimes h_{l,k_l}) = \\ = \gamma(f \otimes \gamma(g_1 \otimes h_{1,1} \otimes \dots \otimes h_{1,k_1}) \otimes \cdots \\ \dots \otimes \gamma(g_l \otimes h_{l,2} \otimes \dots \otimes h_{l,k_l}))\ . \tag{8}$$

C) (Optional). If $\mathcal{P}$ is endowed with identity $I \in \mathcal{P}(1)$, then I and $I^{\otimes n}$ become, respectively, left and right identical functions:

$$I(g) = g; \quad f(I,\dots,I) = f\ , \tag{9}$$

$$\gamma(I \otimes g) = g; \quad \gamma(f \otimes I \otimes \dots \otimes I) = f\ . \tag{10}$$

*An operad endowed with identity which is considered as a part of its structure will be called a **unital operad**.*

We can now define *a classical operad* $\mathcal{P}$ in $\mathcal{C}$ by closely following Definition 2.1. Components $\mathcal{P}(n)$ will be objects of $\mathcal{C}$ endowed with the action of $\mathbf{S}_n, \otimes$ will be replaced by $\boxtimes$, and operadic multiplications γ will be morphisms in $\mathcal{C}$. Axioms $A) - C)$ must be written down as commutative diagrams involving, in particular, permutation isomorphisms of tensor products in $\mathcal{C}$.

REMARK 2.1 *Let us consider the main classes of monoidal categories. Sets with direct product and linear spaces with tensor product form two archetypal classes of symmetric monoidal categories. Variations include imposing an additional structure on the objects. Sets more often appear endowed with a topology or a manifold structure (in the smooth or analytic category). Linear spaces come equipped with grading and/or differential. In this way, we get classical topological operads, classical operads in the category of complexes, and so on. Monoidal functors between symmetric monoidal categories extend to the respective categories of operads.*

3. Combinatorial Trees

We give here a graph-theoretic definition of (finite, rooted, planar) tree. The main subtlety is that the trees we use are not quite finite graphs in the usual sense: some of the edges have a vertex at only one of their ends. This suggests the following definitions.

DEFINITION 3.1

*A (planar) **input − output graph**(Fig. 1,a) consists of*

- *a finite set V (the **vertices**)*
- *a finite set E (the **edges**), a subset $I \subseteq E$ (the **input edges**), and an element $o \in E$ (the **output edge**)*

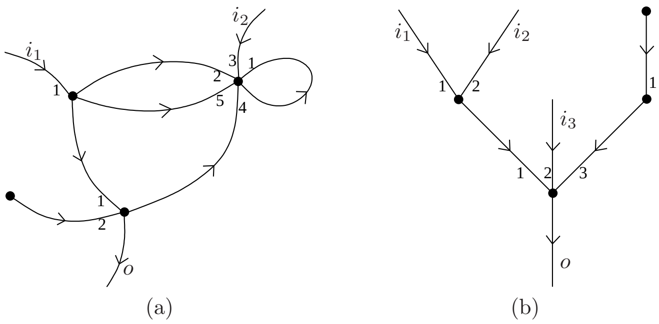


Fig. 1. (*a*) Input-output graph with 4 vertices and 2 input edges i_1, i_2, (*b*) combinatorial tree with 4 vertices and 3 input edges i_1, i_2, i_3. In both, the numbers indicate the order on the edges arriving at each vertex

- *a function* $s : E\backslash I \longrightarrow V$ *(**source**) and a function* $t : E\backslash\{o\} \longrightarrow V$ *(**target**)*
- *for each* $v \in V$*, a total order* $\leq$ *one on* $t^{-1}\{v\}$*.*

We write $v \xrightarrow{e}$ to mean that e is a non-input edge with $s(e) = v$ and, similarly, $\xrightarrow{e} v'$ to mean that e is a non-output edge with $t(e) = v'$. Of course, $v \xrightarrow{e} v'$ means that e is a non-input, non-output edge with $s(e) = v$ and $t(e) = v'$. A tree is, roughly speaking, a connected, simply connected graph, and the following notion of path allows us to express this.

Definition 3.2 *A **path**from a vertex v to an edge e in an input-output graph is a diagram*

$$v = v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{l-1}} v_l \xrightarrow{e_l = e}$$

in the graph. That is, a path from v to e consists of

- *an integer* $l \geq 1$,
- *a sequence* $(v_1, v_2, \ldots, v_l)$ *of vertices with* $v_1 = v$,
- *a sequence* $(e_1, \ldots, e_{l-1}, e_l)$ *of edges with* $e_l = e$

such that

$$v_1 = s(e_1),\ t(e_1) = v_2 = s(e_2),\ \ldots,\ t(e_{l-1}) = v_l = s(e_l),$$

and all of these sources and targets are defined.

Definition 3.3 *A **combinatorial tree**is an input-output graph such that, for every vertex v, there is precisely one path from v to the output edge.*

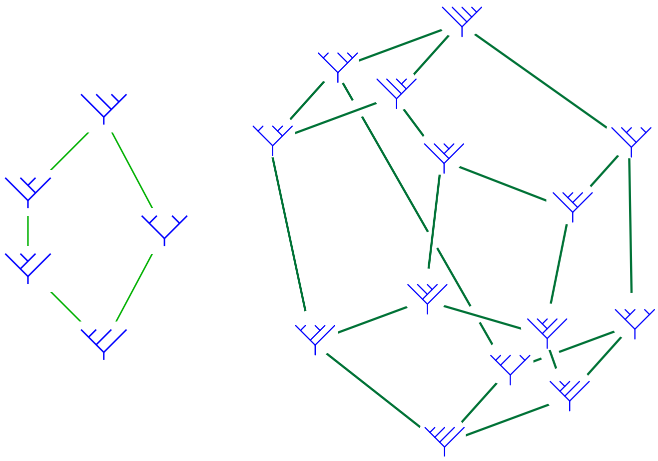
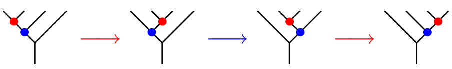


Fig. 2. Tamari order on T_3 and T_4

Figure 1,*b* shows a combinatorial tree. The ordering of the edges arriving at each vertex encodes the planar embedding. "Tree" is an abbreviation for "finite, rooted,planar tree". If we were doing symmetric operads, then we would use non-planar trees; if we were doing cyclic operads, then we would use non-rooted trees; and so on.

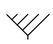
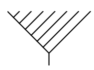
4. Geometric Interpretation of Trees

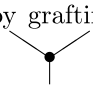
Let T_n be the set of rooted planar binary trees with n interior nodes (and thus $n+1$ leaves). The *Tamari order* (see [5]) on T_n is the partial order, whose cover relations are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus,



is an increasing chain in T_3 (the moving vertices are marked with dots). Only basic properties of the Tamari order are needed in this subsection; their proofs will be provided. For more properties, see [5]. Figure 2 shows the Tamari order on T_3 and T_4.

Let 1_n be the minimum tree in T_n. It is called a *right comb*, as all of its leaves are right pointing:

1_4 =  , 1_7 =  .

Given trees $s \in T_p$ and $t \in T_q$, the tree $s \vee t \in T_{p+q+1}$ is obtained by grafting the root of s onto the left leaf of the tree $\left\{ \right.$  $\left. \right\}$ and the root of t onto its right

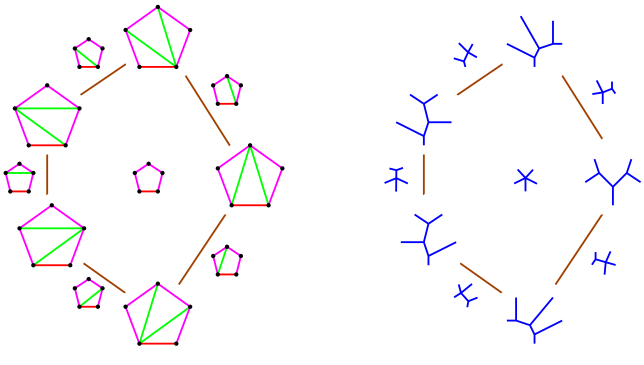
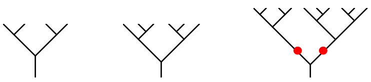
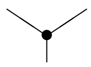


Fig. 3. Two views of the associahedron $\mathcal{A}_{\ni}$

leaf. Below, we display trees s, t, and $s \vee t$, indicating the position of the grafts with dots.



For $n > 0$, every tree $t \in T_n$ has the unique decomposition $t = t_l \vee t_r$ with $t_l \in T_p$, $t_r \in T_q$, and $n = p+q+1$. Thus, T_n is in bijection with $\bigsqcup_{p+q-1} T_p \times T_q$. Since $T_0 = \{|\}$ and $T_1 = \Big\{$  $\Big\}$, T_n contains the Catalan number $\frac{(2n)!}{n!(n+1)!}$ of trees [7].

The Hasse diagram of T_n is isomorphic to the 1-skeleton of the *associahedron* $\mathcal{A}_n$, an $(n-1)$-dimensional polytope. (See [16] and [17].) The faces of $\mathcal{A}_n$ are in one-to-one correspondence with collections of non-intersecting diagonals of a polygon with $n+2$ sides (an $(n+2)$-gon). Equivalently, the faces of $\mathcal{A}_n$ correspond to polygonal subdivisions of an $n+2$-gon with facets corresponding to diagonals and vertices to triangulations. The dual graph of a polygonal subdivision is a planar tree, and the dual graph of a triangulation is a planar binary tree. If we distinguish one edge to be the root edge, the trees are rooted, and this furnishes a bijection between the vertices of $\mathcal{A}_n$ and T_n. Figure 3 shows two views of the associahedron $\mathcal{A}_3$; the first as polygonal subdivisions of the pentagon, and the second as the corresponding dual graphs (planar trees). The root is at the bottom.

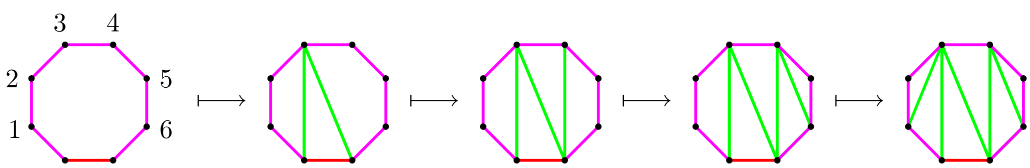
Let $\mathfrak{S}_n$ be the group of permutations of $[n]$, which denotes the set $\{1, 2, \ldots, n\}$. We describe the map $\lambda\colon \mathfrak{S}_n \to T_n$ in terms of triangulations of the $(n+2)$-gon, where we label the vertices with $0, 1, \ldots, n, n+1$ beginning with the left vertex of the root edge and proceeding clockwise. Let $\sigma \in \mathfrak{S}_n$, and let $w_i := \sigma^{-1}(n+1-i)$, for $i = 1, \ldots, n$. This records the positions of the values of σ taken in decreasing order. We inductively construct the triangulation, beginning with the empty triangulation consisting of the root edge. After i steps, we have a triangulation τ_i of the polygon

$$P_i \ := \ \mathrm{Conv}\{0, n+1, w_1, \ldots, w_i\}\,.$$

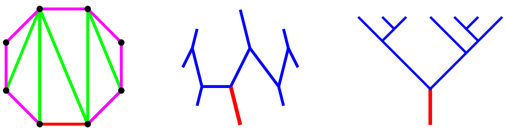
Some edges of P_i will be edges of the original $(n+2)$-gon, and others will be diagonals. Each diagonal cuts the $(n+2)$-gon into two pieces, one containing P_i and the other containing a polygon, which is not yet triangulated and whose root edge we take to be that diagonal. Subsequent steps add to the triangulation τ_i and its support P_i.

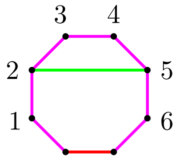
First set $\tau_1 := \mathrm{Conv}\{0, n+1, w_1\}$, the triangle with base the root edge and apex the vertex $w_1 = \sigma^{-1}(n)$. Set $P_1 := \tau_1$ and continue. After i steps, we have constructed τ_i and P_i in such a way that the vertex w_{i+1} is not in P_i. Hence, it must lie in some untriangulated polygon consisting of some consecutive edges of the $(n+2)$-gon and a diagonal that is an edge of P_i. Add the join of the vertex w_{i+1} and the diagonal to the triangulation to obtain a triangulation τ_{i+1} of the polygon P_{i+1}. The process terminates when $i = n$.

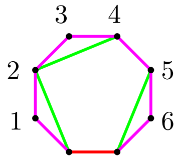
For example, we display this process for the permutation $\sigma = 316524$, where we label the vertices of the first octagon:



The last two steps are suppressed, as they add no new diagonals. The dual graph to the triangulation τ_n is the planar binary tree $\lambda(\sigma)$. Here is the triangulation, its dual graph, and a "straightened" version, which we recognize as the tree $\lambda(316524)$.



$\{1,2,5,6\} \longleftrightarrow$ 

$\{2,4,5\} \longleftrightarrow$ 

A subset S of $[n]$ determines a face Φ_{S} of the associahedron $\mathcal{A}_n$ as follows. Suppose that we label the vertices of the $(n{+}2)$-gon as above. Then the vertices labeled $0, n{+}1$ and those labeled by S form a $(\#\mathsf{S}{+}2)$-gon, whose edges include a set E of non-crossing diagonals of the original $(n{+}2)$-gon. These diagonals determine the face Φ_{S} of $\mathcal{A}_n$ corresponding to S. We give two examples of this association when $n = 6$ below.

We determine the image of f_ζ using the above description of the map $\lambda\colon \mathfrak{S}_n \to T_n$. We say that a face of $\mathcal{A}_{\sqrt{}+\mathrm{II}}$ of the form Φ_{S} with $\#\mathsf{S} = q$ has *type* (p,q). If a face has a type, this type is unique. A permutation $\zeta \in \mathfrak{S}^{(p,q)}$ is uniquely determined by the set $\zeta\{p{+}1,\dots,p{+}q\}$. Therefore, a face of type (p,q) is the image of f_ζ for a unique permutation $\zeta \in \mathfrak{S}^{(p,q)}$. This allows us to speak of the vertex of the face corresponding to a pair $(s,t) \in T_p \times T_q$ (under f_ζ).

5. Classical Operads as Functors

By $\mathtt{Tree}_{clas}$, we denote the category, whose objects are finite rooted trees with the following properties: a) the multiplicity of each vertex is at least two; b) at each vertex, either all incoming flags are halves of edges, or all incoming flags are tails. Morphisms are generated by the following two classes of maps:

a) Isomorphisms compatible with orientation.

b) Contraction of all edges having a common vertex with some outgoing flag and keeping orientation.

More formally, a morphism $\varphi : \sigma \to \tau$ consists of two maps $\varphi_V : V_\sigma \to V_\tau$ and $\varphi^F : F_\tau \to F_\sigma$ compatible with boundaries and involutions and such that φ^F sends tails to tails. Composition of the morphisms corresponds to the composition of the induced maps on vertices and flags. A morphism contracts an edge e if φ_V glues its vertices, and both flags of this edge do not belong to the image of φ^F.

Contractions of different edges commute in an evident sense.

Let v be a vertex of a rooted tree T. Its *star* T_v is a one-vertex tree with vertex v, tails $F_T(v)$, and the outcoming flag as a root.

PROPOSITION 5.1 *The category of classical operads (without identity) in a symmetric monoidal category* $(\mathcal{C}, \boxtimes)$ *is equivalent to the category of functors* $\mathcal{P}$: $\mathtt{Tree}_{class} \to \mathcal{C}$ *isomorphic to a functor satisfying the following condition:*

$$\mathcal{P}(T) = \boxtimes_{v \in V_T} \mathcal{P}(T_v)\ . \tag{11}$$

Sketch of Proof see in [6].

DEFINITION 5.1 *From the graph-theoretic viewpoint, it would be more natural to allow all rooted trees with* $|v| \geq 2$ *as objects, and contractions of any subset of edges as morphisms. The functors from this category* $\mathtt{Tree}_M$ *to* $\mathcal{C}$ *satisfying (11) (up to functor isomorphism) are called* ***Markl's operads***.

REMARK 5.1 *Consider now the category* $\mathtt{Tree}_{cyc}$ *of finite non-rooted trees with* $|v| \geq 2$, *with morphisms generated by the contraction of edges and isomorphisms. Neither root nor orientation is a part of the structure. Functors* $\mathtt{Tree}_{cyc} \to \mathcal{C}$ *satisfying (11) are essentially* cyclic operads *in the sense of [18]. The most essential new feature of cyclic operads is the action of* $\mathbf{S}_{l+1}$ *upon* $\mathcal{P}(l)$.

6. Classifying Space of the Category of Stable Trees

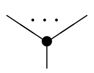
Let us consider a graphical definition of a category of trees. By Definition 3.3, $\mathbf{tr}$ is the free plane operad on the terminal object of $\mathbf{Set}^{\mathbb{N}}$, and an n-leafed tree is an element of $\mathbf{tr}_n$. As we saw, the sets $\mathbf{tr}_n$ also admit the following recursive description:

- $| \in \mathbf{tr}_1$
- if $n, k_1, \ldots, k_n \in \mathbb{N}$ and $\tau_1 \in \mathbf{tr}_{k_1}, \ldots, \tau_n \in \mathbf{tr}_{k_n}$ then $(\tau_1, \ldots, \tau_n) \in \mathbf{tr}_{k_1+\cdots+k_n}$.

A category of trees $\mathtt{Tree}$ is the disjoint union $\coprod_{n\in\mathbb{N}} \mathtt{Tree}_n$. An object of $\mathtt{Tree}_n$ is an n-leafed tree. The set of maps in $\mathtt{Tree}_n$ is

$$(T_2^2 1)(n) = (T_2(\mathbf{tr}))(n),$$

that is, a map is an n-leafed tree τ, in which each k-ary vertex v has assigned to it a k-leafed tree σ_v; the domain of the map is the tree obtained by gluing the σ_v's together in the way dictated by the shape of τ, and the codomain is τ itself. Put another way, what a map does is to take a tree σ (the domain), partition it into a finite number of (possibly trivial) subtrees, and replace each of these subtrees by the corolla



with the same number of leaves, to give the codomain τ. Figure 4 depicts a certain map $\sigma \longrightarrow \tau$ in $\mathtt{Tree}_4$ in three different ways: in (a) as a 4-leafed tree τ with a k-leafed tree σ_v assigned to each k-ary vertex v, in (b) as a 4-leafed tree σ partitioned into subtrees σ_v, and in (c) as something looking more like a function. We will return to the third point of view later; for now, just observe that there is an induced function from the vertices of σ to the vertices of τ, in which the inverse image of a vertex v of τ is the set of vertices of σ_v. In some texts, a map of trees is described as something that "contracts some internal edges". (Here, an *internal edge* is an edge that is not the root or a leaf; maps of trees keep the root and leaves fixed. To "contract"an internal edge

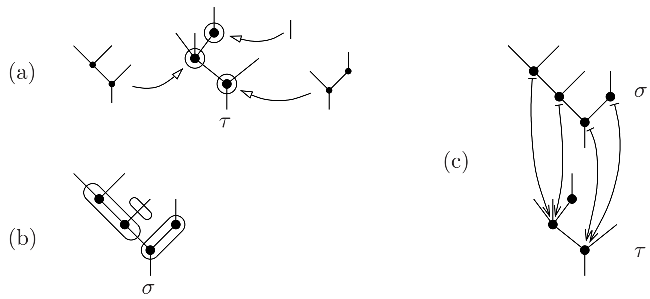


Fig. 4. Three pictures of a map in $\mathtt{Tree}_4$

means to shrink it down to a vertex.) With one important caveat, this is what our maps of trees do: for in a map $\sigma \longrightarrow \tau$, the replacement of each partitioning subtree σ_v by the corolla with the same number of leaves amounts to the contraction of all the internal edges of σ_v. For example, Fig. 5,a shows a tree σ with some of its edges marked for contraction, and Figs. 5,b and 5,c show the corresponding maps $\sigma \longrightarrow \tau$ in two different styles (as in Figs. 4,b and c); so τ is the tree obtained by contracting the marked edges of σ. The caveat is that some of σ_v's may be the trivial tree, and these are replaced by the 1-leafed corolla. This does *not* amount to the contraction of internal edges: it is rather the addition of a vertex to the middle of a (possibly external) edge. Any map of trees can be viewed as a combination of contractions of internal edges and additions of vertices to existing edges. For example, the map illustrated in Fig. 4 contracts two internal edges and adds a vertex to one edge.

Some further understanding of the category of trees can be gained by considering just those trees, in which each vertex has at least two branches coming up out of it. We call these "stable trees", following Kontsevich and Manin [19]. Formally, $\mathbf{StTree}_n$ is the full subcategory of $\mathtt{Tree}_n$ with objects defined by the recursive clauses

- $| \in \mathbf{StTree}_1$
- if $n \geq 2$, $k_1, \ldots, k_n \in \mathbb{N}$, and $T_1 \in \mathbf{StTree}_{k_1}, \ldots, T_n \in \mathbf{StTree}_{k_n}$ then $(T_1, \ldots, T_n) \in \mathbf{StTree}_{k_1+\cdots+k_n}$,

and an *n-leafed stable tree*is an object of $\mathbf{StTree}_n$. Since a stable tree can contain no subtree of the form, all maps between stable trees are "surjections", that is, con-

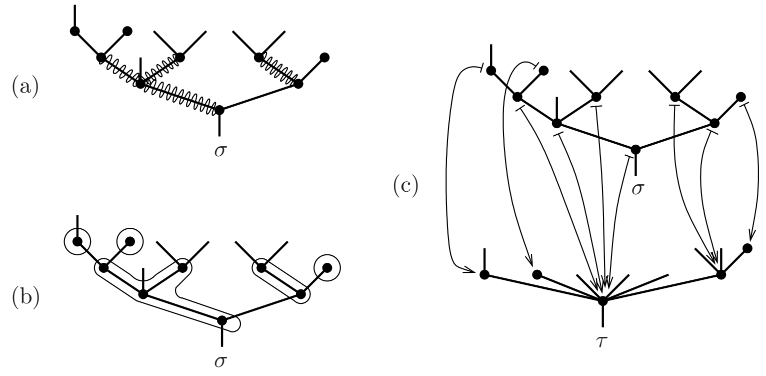


Fig. 5. Three pictures of an epic in $\mathtt{Tree}_6$

Fig. 6. (*a*) The category of 3-leafed stable trees, and (*b*) its classifying space

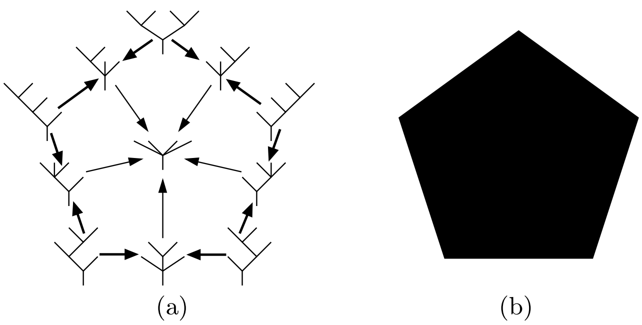


Fig. 7. (*a*) The category of 4-leafed stable trees, and (*b*) its classifying space

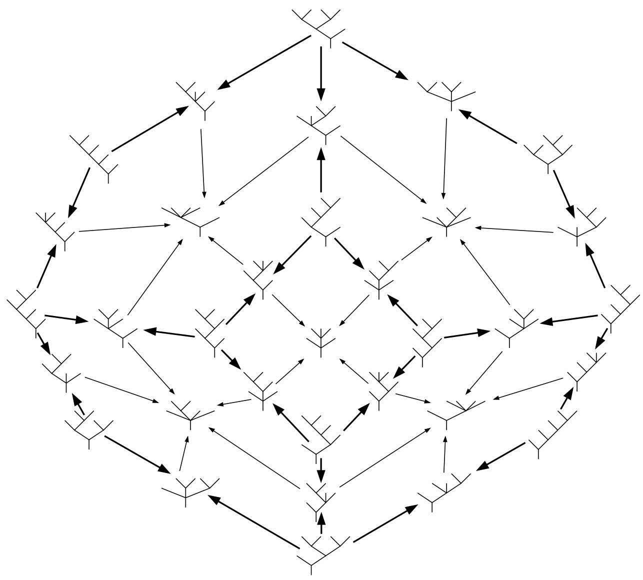


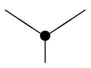
Fig. 8. About half of the category of 5-leafed stable trees

sist of just contractions of internal edges, without insertions of new vertices.

The first few categories $\mathbf{St}\mathtt{Tree}_n$ are trivial:

$$\mathbf{St}\mathtt{Tree}_0 = \emptyset,$$

$$\mathbf{St}\mathtt{Tree}_1 = \{|\},$$

$\mathbf{St}\mathtt{Tree}_2 = \left\{ \right.$  $\left. \right\}$

where, in each case, there are no arrows except for identities. The cases $n = 3$, 4, and 5 are illustrated in Figs. 6,*a*, 7,*a*, and 8,*a*.

Identity arrows are not shown, and the categories $\mathbf{St}\mathtt{Tree}_n$ are ordered sets: all diagrams commute. Vertices are also omitted; since the trees are stable, this causes no ambiguity. Parts (*b*) of the figures show the classifying spaces of these categories, solid polytopes of dimensions 1, 2, and 3. In the case of 5-leafed trees (Fig. 8), only about half of the category is shown, corresponding to the front faces of the polytope; the back faces and the terminal object of the category (the 5-leafed corolla), which sits at the center of the polytope, are hidden. The whole polytope has 6 pentagonal faces, 3 square faces, and the 3-fold rotational symmetry about the central vertical axis.

For $n \leq 5$, the classifying space $B(\mathtt{StTree}_n)$ is homeomorphic to the associahedron$\mathcal{A}_n$ (see Fig. 3 above and Fig. 9 below), and it seems very likely that this persists for all $n \in \mathbb{N}$. Indeed, the family of categories $(\mathtt{StTree}_n)_{n\in\mathbb{N}}$ forms a sub-**Cat**-operad **STTR** of **Cat**-operad **TR**, and the classifyingspace functor B : **Cat** $\longrightarrow$ **Top** preserves finite products, so there is a (non-symmetric) topological operad $B(\mathbf{STTR})$, whose nth part is the classifying space of $\mathtt{StTree}_n$. (To make B preserve finite products, we must interpret **Top** as a category of compactly generated or Kelley spaces: see [20] and [21].) This operad $B(\mathbf{STTR})$ is presumably isomorphic to Stasheff's operad $K = (K_n)_{n\in\mathbb{N}}$. A K-algebra is called an A_∞*-space*and should be thought of as an up-to-homotopy version of a topological semigroup; the basic example is a loop space.

The categories $\mathtt{StTree}_n$ also give rise to the notion of an A_∞-algebra (see [22]). For each $n \in \mathbb{N}$, there is a chain complex $P(n)$, whose degree k part is the free Abelian group on the set of n-leafed stable trees with $(n-k-1)$ vertices.

When the signs are chosen appropriately, this defines an operad P of chain complexes. A P-algebra is called an A_∞*-algebra*, to be thought of as an up-to-homotopy differentialgraded non-unital algebra; the usual example is the singular chain complex of an A_∞-space.

A P-category is called an A_∞-category(see [3]) and consists of a collection of objects, a chain complex $\mathrm{Hom}(a,b)$ for each pair (a,b) of objects, maps defining binary composition, chain homotopies witnessing that this composition is associative up to homotopy, further homotopies witnessing that the previous homotopies obey the pentagon law up to homotopy, and so on. Finally, since the polytopes $K_n = B(\mathtt{StTree}_n)$ describe higher associativity conditions, they also arise in definitions of higher-dimensional category. For example, the pentagon K_4 occurs in the classical definition of bicategory [3], and the polyhe-

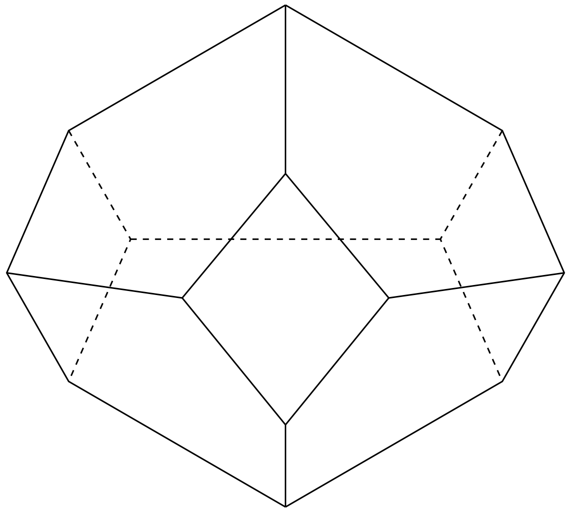


Fig. 9. Classifying space of the whole category of 5-leafed stable trees

dron K_5 occurs as the "non-Abelian 4-cocycle condition" in Gordon, Power, and Street's definition of tricategory [23].

We have already described the operad of trees as a set $\mathbf{tr}_n$ of n-leafed trees. Maps $\sigma \longrightarrow \tau$ between trees are described by induction on the structure of τ:

- if $\tau = |$, then there is only one map into τ; it has domain $|$ and we write it as $1_| : | \longrightarrow |$
- if $\tau = (\tau_1, \ldots, \tau_n)$ for $\tau_1 \in \mathbf{tr}_{k_1}, \ldots, \tau_n \in \mathbf{tr}_{k_n}$, then a map $\sigma \longrightarrow \tau$ consists of trees $\rho \in \mathbf{tr}_n), \rho_1 \in \mathbf{tr}_{k_1}, \ldots, \rho_n \in \mathbf{tr}_{k_n}$ such that $\sigma = \rho\circ(\rho_1, \ldots, \rho_n)$, together with maps

  $$\rho_1 \xrightarrow{\theta_1} \tau_1, \quad \ldots, \quad \rho_n \xrightarrow{\theta_n} \tau_n,$$

  and we write this map as

  $$\sigma = \rho\circ(\rho_1, \ldots, \rho_n) \overset{!_\rho * (\theta_1, \ldots, \theta_n)}{\longrightarrow} (\tau_1, \ldots, \tau_n) = \tau. \qquad (12)$$

It follows easily that the n-leafed corolla $\nu_n = (\,|, \ldots, |\,)$ is the terminal object of $\mathtt{Tree}_n$: the unique map from $\sigma \in \mathbf{tr}_n$ to ν_n is written as $!_\sigma * (1_|, \ldots, 1_|)$. The rest of the structure of a **Cat**-operad **TR** can be described in a similarly explicit recursive fashion.

To make precise the intuition that a map of trees is a function of some sort, the functors

$$V : \mathtt{Tree} \longrightarrow \mathbf{Set}, \qquad E : \mathtt{Tree}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

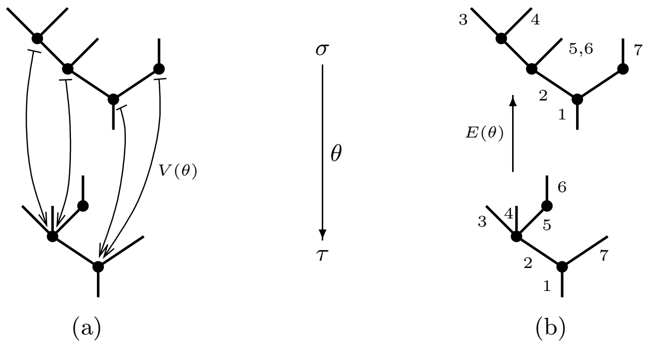


Fig. 10. Effect of a certain map of 4-leafed trees on (a) vertices and (b) edges

can be defined, encoding what happens on vertices and edges, respectively. Both functors turn out to be faithful, which means that a map of trees is completely determined by its effect on either vertices or edges. The following account of V and E is just a sketch.

The more obvious of the two is the vertex functor V defined on objects by

- $V(\,|\,) = \emptyset$
- $V((\tau_1, \ldots, \tau_n)) = 1 + V(\tau_1) + \cdots + V(\tau_n)$.

The edge functor E can be defined by firstly defining a functor E_n: $\mathtt{Tree}(n)^{\mathrm{op}} \longrightarrow (n+1)/\mathbf{Set}$ for each $n \in \mathbb{N}$, where $(n+1)/\mathbf{Set}$ is the category of sets equipped with $(n+1)$ ordered marked points. This definition is again by induction, the idea being that E_n associates to a tree its edge-set with the n input edges and the one output edge (root) distinguished. Figure 10 illustrates a map θ: $\sigma \longrightarrow \tau$ in $\mathtt{Tree}(4)$; part (a) (= Figure 4,c) shows its effect $V(\theta)$ on vertices; part (b) shows $E(\theta)$, taking $E(\tau) = \{1, \ldots, 7\}$ and labelling the image of $i \in \{1, \ldots, 7\}$ under $E(\theta)$ by an i on the edge $(E(\theta))(i)$ of σ.

A map of trees will be called surjective if it is built up from contractions of internal edges. Formally, the *surjective* maps in `Tree` are defined by:

- $1_| : | \longrightarrow |$ is surjective
- with notation as in (12), $!_\rho * (\theta_1, \ldots, \theta_n)$ is surjective if and only if each θ_i is surjective and $\rho \neq \,|$.

The crucial part is the last: the unique map $!_\rho$ from $\rho \in \mathbf{tr}_n$ to the corolla ν_n is made up of edge-contractions just as long as ρ is not the unit tree $|$.

Dually, a map of trees is *injective*if, informally, it is built up from adding vertices to the middle of edges. Formally,

- $1_| : | \longrightarrow |$ is injective
- with notation as above, $!_\rho * (\theta_1, \ldots, \theta_n)$ is injective if and only if each θ_i is injective, and ρ is either ν_n or $|$ (the latter is possible only if $n = 1$).

7. Morphisms in Classifying Space of the Category of VKSS-Trees

In the rotation group of the n-dimensional space, it is possible to choose $n(n-1)/2$ different one-parametric subgroups carrying out, for example, the transformation of only variables x_i and $x_k (i \neq k)$ with the other variables to be unchanged [24]:

$$g_{ik}(t)=\begin{array}{cc}
\begin{array}{cccccccccc} & & & & (i) & & (k) & & & \end{array} & \\
\left\{\begin{array}{cccccccccc}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \cos t & \ldots & \sin t & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & -\sin t & \ldots & \cos t & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 1
\end{array}\right\} &
\begin{array}{l} \\ \\ \\ \\ (i). \\ \\ (k) \\ \\ \\ \\ \end{array}
\end{array} \qquad (13)$$

For the sake of convenience, let us construct a matrix of finite rotations in the space of VKSS-trees corresponding to the canonical reduction of the space [25]. Because the functor to other types of the category of VKSS-trees is known [11], we shall construct the morphism in any space of VKSS-trees, by doing so.

The canonical reduction of the space $R_n \supset R_{n-1} \supset \cdots \supset R_1$ involves the tree (Fig. 11) and the solution of the Laplace equation [9]

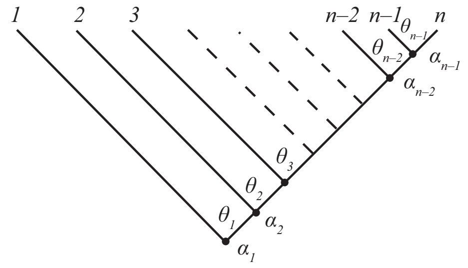
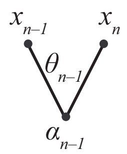


Fig. 11

$$= \prod_{i=1}^{n-2} \{N_{n_i}^{l_{i+1},l_{i+1}}\}^{-\frac{1}{2}} (1-y_i^2)^{\frac{\alpha_i+1}{2}} \mathfrak{P}_{n_i}^{l_{i+1},l_{i+1}}(y_i) \frac{e^{i\alpha_{n-1}\theta_{n-1}}}{\sqrt{2\pi}}$$
$$= \prod_{i=1}^{n-2} \psi_i(y_i) \frac{e^{i\alpha_{n-1}\theta_{n-1}}}{\sqrt{2\pi}}, \qquad (14)$$

where $n_i = \alpha_i - \alpha_{i+1}$; $y_i = cos\theta_i$, $l_i = \alpha_i + (n-i-1)/2 = 2j_i + 1$.

The normalized solution $\exp(i\alpha_{n-1}\theta_{n-1})/\sqrt{2\pi}$ corresponds to the fork formed by coordinates x_{n-1} and x_n:



$$= \exp(i\alpha_{n-1}\theta_{n-1})/\sqrt{2\pi}. \qquad (15)$$

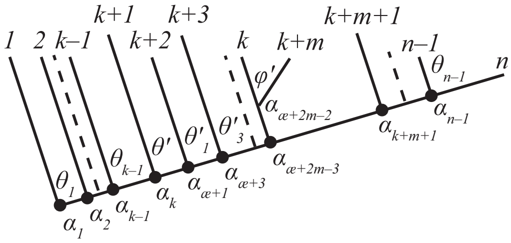
It is obvious that, at the rotation in the $(x_{n-1}x_n)$-plane, fork (15) will be multiplied by an exponent. Therefore, in order to construct the matrix of transformations of the functions of VKSS-trees arising at the rotation of coordinates, it is necessary to construct the fork from the coordinates, in the plane of which the rotation is carried out (in other words, to make transition to the other VKSS-tree by replanting the branches (see [11])), to implement the rotation through the angle φ, and then to come back to the initial VKSS-tree, by using the inverse replantation.

Let us suppose that the rotation through an angle φ is carried out in the plane $(x_k x'_k)$ $(k' = k+m)$. By carrying out the consecutive replantation of the k-th branch from the k-th place to the $(k+m)$-th one, we come to the formula

$$\psi_{\rm can}^{l_1,\ldots,l_{n-1}}(y_1,y_2,\ldots,y_{n-1}) = \sum_{l_{\varkappa+1},l_{\varkappa+3},\ldots,l_{\varkappa+2m-3}} \begin{pmatrix} l_k, & l_{k+1}, & \ldots, & l_{k+m+1} & \\ & & & & l_{\varkappa+2m-2} \\ l_{\varkappa+1}, & l_{\varkappa+3}, & \ldots, & l_{\varkappa+2m-3}, & \end{pmatrix} \times$$

$$\times \quad \psi_{{\rm per}\, l_{\varkappa+2m-2},l_{k+m+1},\ldots,l_{n-1}}^{l_1,\ldots,l_k;l_{\varkappa+1},\ldots,l_{\varkappa+2m-3}}(y_1,\ldots,y_{k-1};y',y'_{\varkappa+1},y_{\varkappa+3},\ldots,y'_{\varkappa+2m-3},\varphi';y_{k+m+1},\ldots,y_{n-1}), \qquad (16)$$

where $\psi_{\rm can}^{l_1,\ldots,l_{n-1}}(y_1,\ldots,y_{n-1})$ is defined by (14), and $\psi_{{\rm per}_{\{l\}}}^{\{l\}}(\{y\})$ by



$$= \prod_{i=1}^{k-1} \psi_i(y_i) \{N^{l_{\varkappa+1},l_{\varkappa+1}}_{\alpha_k-\alpha_{\varkappa+1}}\}^{-1/2}(1-y'^2)^{\frac{\alpha_{\varkappa+1}}{2}} \mathfrak{P}^{l_{\varkappa+1},l_{\varkappa+1}}_{\alpha_k-\alpha_{\varkappa+1}}(y') \times$$
$$\times \prod_{j=3,5,\dots}^{2m-3} \{N^{l_{\varkappa+j},l_{\varkappa+j}}_{\alpha_{\varkappa+j-2}-\alpha_{\varkappa+j}}\}^{-1/2}(1-y'^2_{j-2})^{\frac{\alpha_{\varkappa+j}}{2}} \times$$
$$\times \mathfrak{P}^{l_{\varkappa+j},l_{\varkappa+j}}_{\alpha_{\varkappa+j-2}-\alpha_{\varkappa+j}}(y'_{j-2}) 2^{\frac{n-k-m+2}{4}} \{N^{l_s,l_c}_{n'}\}^{-1/2} \times$$
$$\times (1-y_{\varkappa+2m-3})^{\frac{\alpha_{k+m+1}}{2}} (1+y_{\varkappa+2m-3})^{\frac{\alpha_{\varkappa+2m+2}}{2}} \mathfrak{P}^{l_s,l_c}_{n'}(y_{\varkappa+2m-3}) \times$$
$$\times \prod_{i=k+m+1}^{n-2} \psi_i(y_i) \frac{e^{i\alpha_{n-1}\theta_{n-1}}}{\sqrt{2\pi}} \frac{e^{i\alpha_{\varkappa+2m-2}\varphi'}}{\sqrt{2\pi}}. \qquad (17)$$

Fig. 12

where $y' = \cos\theta'$; $l_{\varkappa+i} = \alpha_{\varkappa+i} + (n-k-i-1)/2$; $y'_i = \cos\theta'_i$; $y_{\varkappa+2m-3} = \cos 2\theta_{\varkappa+2m-3}$; $l_s = \alpha_{k+m+1} + (n-k-m-2)/2$; $l_c = |\alpha_{\varkappa+2m-2}|$; $n' = (\alpha_{\varkappa+2m-3} - \alpha_{\varkappa+2m-2} - \alpha_{k+m+1})/2$.

It can be seen from (14) and (17) that the functions corresponding to these VKSS-trees differ from each other from the k-th to the $(k+m)$-th branch for the VKSS-tree depicted in Fig. 11, and from the $(k+1)$-th to the $(k+m)$-th branch for the VKSS-tree depicted in Fig. 12. In other words, in order to calculate the transition matrix from (16), it is sufficient to calculate only the overlapping integral between these portions.

Starting from the rules set out in [10, 11], we obtain the matrix of transition to expression (16) in the j-representation in the form

$$\begin{pmatrix} j_k, & j_{k+1}, & \dots, & j_{k+m+1} & \\ & & & & j_{\varkappa+2m-2} \\ j_{\varkappa+1}, & j_{\varkappa+3}, & \dots, & j_{\varkappa+2m-3}, & \end{pmatrix} =$$
$$= \underbrace{\sum_{j_\varkappa, j_{\varkappa+2}, \dots, j_{\varkappa+2m-4}}}_{m-1} \left\| \begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_{k+m+1} \\ j_{\varkappa+2m-2} & j_{\varkappa+2m-3} & j_{k+m} \end{matrix} \right\| \times$$
$$\times \prod_{i=2}^{m} \left\| \begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_{k+1} \\ j_{\varkappa+2i-4} & j_{\varkappa+2i-5} & j_{k+i-1} \end{matrix} \right\| \times$$
$$\times (-)^{\frac{2j_{\varkappa+2i-4}+1}{2}} \left\| \begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_{k+i} \\ j_{\varkappa+2i-4} & j_{\varkappa+2i-5} & j_{\varkappa+2i-3} \end{matrix} \right\|, \qquad (18)$$

for $m = 2, 3, \dots, n-2$; and for $m = 1$, if the neighboring x_k and x_{k+1} do not make a fork

$$\begin{pmatrix} j_k, & j_{k+1} & \\ & & j_{k+2} \\ j_{\varkappa-1}, & j_\varkappa & \end{pmatrix} = \left\| \begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_{k+2} \\ j_\varkappa & j_{\varkappa-1} & j_{\varkappa+1} \end{matrix} \right\|, \qquad (19)$$

and

$$(j_k,\, j_{k+1},\, j_{k+2}) \equiv 1, \qquad (20)$$

if x_k and x_{k+1} make a fork.

The quantities $\left\| \begin{matrix} j_1 & j_2 & j_3 \\ j_{12} & j & j_{23} \end{matrix} \right\|$ in (18)–(20) are T-coefficients, which are, in the particular case under consideration, the Clebsch–Gordan coefficients (up to a phase), i.e.,

$$\left\| \begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_3 \\ j_{12} & j & j_{24} \end{matrix} \right\| =$$
$$= (-)^{j-3j_{23}+j_{12}-2j_3-5/4} \mathbb{C}^{2j_{23}+1/2,2j_3+1}_{jj_3-j_{12},jj_3+j_{12}+1}. \qquad (21)$$

In general, they turn out to be $6j$-symbols analytically continued into the non-physical (from the point of view of the momentum theory) range of j [26, 27].

Acting by the operator $R_{k,k+m}(\varphi)$ (operator of rotation at the place $((k, k+m)$ through an angle φ) on the function $\psi^{l_1,\dots,l_{n-1}}_{\text{can}}(y_1,\dots,y_{n-1})$ with regard for its connection with the function

$$\psi^{l_1,\dots,l_k;l_{\varkappa+1},\dots,l_{\varkappa+2m-3}}_{\text{per}\, l_{\varkappa+2m-2},l_{k+m+1},\dots,l_{n-1}}(y_1,\dots,y_{k-1},y',\varphi'),$$

we obtain the following expression for the rotation matrix

$$R_{k,k+m}^{\{j_{k+1},\dots,j_{k+m}\},\{j'_{k+1},\dots,j'_{k+m}\}}(\varphi)=\sum_{j_{\varkappa+1}j_{\varkappa+3}\cdots j_{\varkappa+2m-3}}\begin{pmatrix} j_k, & j_{k+1}, & \dots, & j_{k+m+1} & \\ & & & & j_{\varkappa+2m-2} \\ j_{\varkappa+1}, & j_{\varkappa+3}, & \dots, & j_{\varkappa+2m-3}, & \end{pmatrix}\times$$

$$\times e^{il_{\varkappa+2m-2}\varphi}\begin{pmatrix} j_k, & j'_{k+1}, & \dots, & j'_{k+m}, & j_{k+m+1} & \\ & & & & & j_{\varkappa+2m-2} \\ j_{\varkappa+1}, & j_{\varkappa+3}, & \dots, & j_{\varkappa+2m-3}, & & \end{pmatrix}. \tag{22}$$

The quantities $\begin{pmatrix} \dots \\ . \\ \dots \end{pmatrix}$ in (22) are defined by (18)–(20) and are one of the types of $3nj$-symbols (as for $3nj$-symbols, see, e.g., [28]).

The product of $n(n-1)/2$ such matrices with coefficients $\begin{pmatrix} \dots \\ . \\ \dots \end{pmatrix}$, defined by (18)–(20), gives the function of a "symmetric top" in the $\frac{(2\alpha_1+n-2)(n+\alpha_1-3!)}{(n-2)!\alpha_1!}$-dimensional space, i.e. the matrix of finite rotations in the space of VKSS-trees. For example, in the 4-dimensional space at $m=1$, relation (22) yields

$$R_{k,k+m}^{j_{k+1},j'_{k+1}}(\varphi)=\sum_{j_\varkappa}\left\|\begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_{k+2} \\ j_\varkappa & j_{\varkappa-1} & j_{\varkappa+1}\end{matrix}\right\| e^{i(2j_\varkappa+1)\varphi}\left\|\begin{matrix} -\frac{3}{4} & -\frac{3}{4} & j_{k+2} \\ j_\varkappa & j_k & j'_{\varkappa+1}\end{matrix}\right\|^{-1}=(-)^{3(j'_{k+1}-j_{k+1})}\times$$

$$\times\sum_{i_\varkappa}\mathbb{C}^{2j_{k+1}+\frac{1}{2},2j_{k+1}+1}_{j_kj_{k+2}-j_\varkappa;j_kj_{k+2}+j_{\varkappa+1}}e^{i(2j_\varkappa+1)\varphi}\mathbb{C}^{2j'_{k+1}+\frac{1}{2},2j_{k+1}+1}_{j_kj_{k+2}-j_\varkappa;j_kj_{k+2}+j_{\varkappa+1}}.$$

The authors are especially grateful to the Austrian Academy of Sciences and the Russian Foundation for Fundamental Research which in the framework of the collaboration with the National Academy of Sciences of Ukraine co-financed this research.

1. S. Mac Lane, *Categories for the Working Mathematician* (Springer, Berlin, 1998).
2. Y.I. Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces* (Amer. Math. Soc., Providence, RI, 1999).
3. T. Leinster, *Higher Operads, Higher categories* (Cambridge Univ. Press, Cambridge, 2004).
4. M. Markl, *Operads and PROPs*, Preprint `math/0601129` (2006).
5. M. Aguiar and F. Sottile, *Structure of the Loday–Ronco Hopf Algebra of Trees*, Preprint `math.CO/0409022` (2004).
6. S.S. Moskaliuk, in *Proceedings of the XVIth International Hutsulian Workshop "Symmetry, Cosmology and Categorification"* (TIMPANI, Kyiv, 2008), p. 1.
7. G.I. Kuznetsov, S.S. Moskaliuk, Yu.F. Smirnov, and V.P. Shelest, *Graphical Theory of Representations of Orthogonal and Unitary Groups and Its Physical Applications* (Naukova Dumka, Kiev, 1992) (in Russian).
8. N.A. Gromov, S.S. Moskaliuk, and Yu.F. Smirnov, *The Tree, Contractions and Analytical Continuation Methods in the Theory of Representations of Classical Groups and Quantum Algebras* (Naukova Dumka, Kiev, 1993) (in Russian).
9. G.I. Kuznetsov, S.S. Moskaliuk, and Yu.F. Smirnov, in *Proceedings of the XIth International Gutsulian Workshop "Group Theoretical Methods in Physics", October 25–31, 1992, Rakhiv, Ukraine* (Hadronic Press, Palm Harbor, 1995), p. 23.
10. G.I. Kuznetsov and S.S. Moskaliuk, Hadr. J. Suppl. **15**, 349 (2000).
11. G.I. Kuznetsov and S.S. Moskaliuk, Hadr. J. **24**, 17 (2001).
12. S.S. Moskaliuk and Yu.F. Smirnov, Hadr. J. **24**, 129 (2001).
13. S.S. Moskaliuk and Yu.F. Smirnov, Hadr. J. **27**, 399 (2004).
14. M. Moshinsky and C. Quesne, J. Math. Phys. **11**, 1631 (1970).

15. S.S. Moskaliuk, *From Cayley-Klein Groups to Categories* (TIMPANI, Kyiv, 2006).
16. G.M. Ziegler, *Lectures on Polytopes* (Springer, New York, 1995).
17. R.P. Stanley, *Enumerative Combinatorics* (Cambridge Univ. Press, Cambridge, 1999).
18. E. Getzler and M.M. Kapranov, in *Geometry, Topology and Physics, for Raoul Bott*, edited by S.-T. Yau (Intern. Press, Boston, 1995), p. 167.
19. M. Kontsevich and Yu. Manin, Comm. Math. Phys. **164**, 525 (1994).
20. G. Segal, Inst. des Hautes Études Sci. Publ. Mathém. **34**, 105 (1968).
21. P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory* (Springer, Berlin, 1967).
22. J.D. Stasheff, Trans. Amer. Math. Soc. **108**, 275 (1963).
23. R. Gordon, A.J. Power, and R. Street, *Coherence for Tricategories*, Memoirs of the AMS **117**, No. 558 (1995).
24. I.M. Gel'fand and M.L. Tseitlin, Dokl. Akad. Nauk SSSR **71**, 1017 (1950).
25. N.Ya. Vilenkin, G.I. Kuznetsov, and Ya.A. Smorodinsky, Yad. Fiz. **2**, 906 (1965).
26. G.I. Kuznetsov and Ya.A. Smorodinsky, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 378 (1975).
27. G.I. Kuznetsov and Ya.A. Smorodinsky, *T-coefficients and 6j-symbols*, Preprint (Inst. Atom. Energy, Moscow, 1976) (in Russian).
28. D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii, *Quantum Theory of Angular Momentum* (World Sci., Singapore, 1988).

Received 15.07.11

КАТЕГОРІЯ ДЕРЕВ ВІЛЄНКІНА–КУЗНЄЦОВА–СМОРОДІНСЬКОГО–СМІРНОВА

С.С. Москалюк, Н.М. Москалюк

Р е з ю м е

У першій частині статті дано короткий огляд означень та основних властивостей операд і дерев. Існує багато корисних типів операд, кожен з них визначається вибором двох категорій: симетричної моноїдальної категорії $(\mathcal{C}, \boxtimes)$, яка є носієм класичних лінійних операд, та категорії графів Γ, що відображає комбінаторику операдних даних і деяких аксіом. З цієї точки зору, конкретна операда є функтором $\Gamma \to \mathcal{C}$. Основною метою другої частини роботи є побудова категорії дерев Віленкіна–Кузнєцова–Смородінського–Смірнова (ВКСС), зокрема, ВКСС-дерев, як об'єктів та морфізмів перетворень ВКСС-дерев при поворотах n-вимірного простору.