ON THE RELATIVISTIC QUANTUM MECHANICS OF A PARTICLE IN SPACE WITH MINIMAL LENGTH

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PACS 03.65.Ge, 03.65.Pm, 02.40.Gh

Dedicated to the memory of my brother, Raphael M. Scherbakov

A noncommutative space and the deformed Heisenberg algebra $[X, P] = i\hbar\sqrt{1 - \beta P^2}$ are investigated. The quantum mechanical structures underlying this commutation relation are studied. The rotational group symmetry is discussed in detail.

1. Introduction

The existence of a minimal length is one of the most important predictions in theoretical physics. Such a minimal length is due to the fact that the high-energy particles for probing small scales of the order of the Planck length disturb the very space-time they are probing. Otherwise, the very energetic particles cannot probe distances smaller than the minimal length size. The existence of a minimal length leads to a modification of the standard commutation relation between position and momentum in usual quantum mechanics.

There are many realizations of the noncommutative Snyder space, but only two particular realizations of its algebra are known: the Snyder [1] and the Maggiore [2] ones. The well-known Snyder commutation relation reads

$$[X, P] = i\hbar\left(1 + \beta P^2\right), \tag{1}$$

where βP^2 is a small correction. On the other hand, the Maggiore algebra reads

$$[X, P] = i\hbar\sqrt{1 - \beta P^2} \tag{2}$$

and is widely used in various fields of physics. In what follows, it will be shown that the minimal length of a deformed space with the commutation relation (2) should be quantum-theoretically described by a nonzero operator Δ that can be expressed in terms of the Pauli or Gell-Mann matrices.

The study of theories with deformed Heisenberg algebra [3–16] belongs to an active area in theoretical physics due to their applications in quantum gravity [17], perturbative string theory [18–23], black holes [2], as well as noncommutative space-time [24–32]. Quantum deformations which lead to a noncommutative space-time are strictly linked with quantum groups [33–36]. the noncommutative space is also useful in describing the deformed special relativity theories, originally called the doubly special relativity [37–40].

The paper is organized as follows. In Sections 2 and 3, we study a quantum mechanical structure that underlies the commutation relation (2) and its generalization. Sections 4 and 5 are devoted to a discussion of the rotation group. In Section 6, we discuss a relativistic generalization.

2. Representation on the Momentum Space

In what follows, we use the momentum representation. A convenient possibility is

$$P\psi = p\psi,$$
$$X\psi = i\hbar\sqrt{1 - \beta P^2}\partial_p\psi \tag{3}$$

with respect to the scalar product

$$\langle\psi|\varphi\rangle = \frac{1}{\sqrt{\beta}}\int\limits_{-1}^{1}\frac{\psi^*\phi}{\sqrt{1 - \xi^2}}d\xi, \tag{4}$$

where $\xi = \sqrt{\beta}p$. The modified Heisenberg algebra (2) leads to the deformed uncertainty relation

$$\Delta X \Delta P \geq \frac{\hbar}{2}\sqrt{1-\beta(\Delta P^2)-\beta\langle P^2\rangle}. \tag{5}$$

Such an uncertainty principle is presented in Figure. We note that, in the case of the Snyder noncommutative space, there is no possibility to measure the coordinate X with accuracy more than $\hbar\sqrt{\beta}$. But, in the case of the Maggiore one, there exists the zero uncertainty in position (see Figure). Therefore, one can exactly measure the position X.

The eigenvalue problem for the coordinate operator takes, on the momentum space, the form of a differential equation

$$X^2\psi = \lambda^2\psi. \tag{6}$$

Taking the momentum representation (3) into account, this equation reads

$$(1-\beta p^2)\frac{\partial^2}{\partial p^2}\psi_\lambda(p) - \beta^2 p\partial_p\psi_\lambda(p) + \left(\frac{\lambda}{\hbar}\right)^2\psi_\lambda(p) = 0. \tag{7}$$

Using standard techniques, the solutions of this Chebyshev equation [41] are given by

$$\psi_n^{(1)} \propto (1-\beta p^2)^{1/2} F\left(-n, n+2, \frac{3}{2}, \frac{1}{2}+\frac{1}{2}\sqrt{\beta}p\right),$$
$$\lambda_n^{(1)} = \hbar\sqrt{\beta}\cdot(n+1) \tag{8}$$

and

$$\psi_n^{(2)} \propto F\left(-n, n, \frac{1}{2}, \frac{1}{2}+\frac{1}{2}\sqrt{\beta}p\right),$$
$$\lambda_n^{(2)} = \hbar\sqrt{\beta}\cdot n, \tag{9}$$

where $n = 0, 1, \ldots, \infty$.

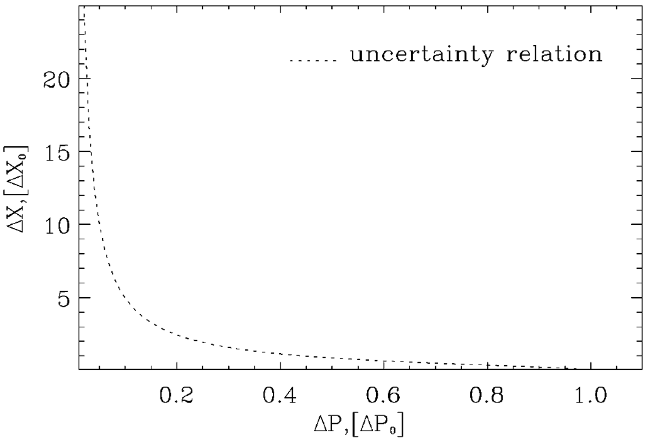
3. Generalization to n Dimensions

The formalism described in the previous section straightforwardly extends to n spatial dimensions. A natural generalization of (2) which preserves the rotational symmetry is

$$[X_i, P_j] = i\hbar\delta_{ij}\sqrt{1-\beta P^2}. \tag{10}$$

We also assume that the momenta p_μ which are transformed as vectors under the Lorentz algebra satisfy the commutation relation

$$[P_\mu, P_\nu] = 0. \tag{11}$$



Modified uncertainty relation. $\Delta X_0 = \hbar\sqrt{\beta}$, $\Delta P_0 = \sqrt{\beta}^{-1}$

Thus, the generalized representation of position and momentum reads

$$P_i\psi = p_i\psi,$$
$$X_i\psi = i\hbar\sqrt{1-\beta P^2}\frac{\partial}{\partial p_i}\psi. \tag{12}$$

Therefore, one can write the commutation relation

$$[X_i, X_j] = \frac{i\hbar\beta}{\sqrt{1-\beta P^2}}\left(X_iP_j - X_jP_i\right), \tag{13}$$

which indicates that we have a noncommutative space.

4. Representation of the Rotation Group

In quantum mechanics with a noncommutative space, the generators of rotations can be expressed in terms of the operators

$$L_{ij} = \frac{1}{\sqrt{1-\beta P^2}}(X_iP_j - X_jP_i). \tag{14}$$

There is also another convenient representation

$$L_k = \frac{1}{\sqrt{1-\beta P^2}}\epsilon_{ijk}X_iP_j, \tag{15}$$

where ϵ_{ijk} is the Levi-Civita symbol. They generalize the usual operators of orbital angular momentum. We have, therefore, the commutation relations

$$[\mathbf{X}_i, \mathbf{L}_{jk}] = i\hbar\left(\delta_{ik}\mathbf{X}_j - \delta_{ij}\mathbf{X}_k\right), \tag{16}$$
$$[\mathbf{P}_i, \mathbf{L}_{jk}] = i\hbar\left(\delta_{ik}\mathbf{P}_j - \delta_{ij}\mathbf{P}_k\right), \tag{17}$$
$$[\mathbf{L}_{ij}, \mathbf{L}_{kl}] = i\hbar\left(\delta_{ik}\mathbf{L}_{jl} - \delta_{il}\mathbf{L}_{jk} + \delta_{jl}\mathbf{L}_{ik} - \delta_{jk}\mathbf{L}_{il}\right). \tag{18}$$

However, we also have

$$[X_i, X_j] = i\hbar\beta L_{ij} \tag{19}$$

leading to a noncommutative geometry. The algebra generated by (16)–(19) satisfies the Jacobi identities. One can verify that

i $[X_i,[X_j,X_k]]+[X_j,[X_k,X_i]]+[X_k,[X_i,X_j]]=$
$= i\hbar\beta\Big([X_i,L_{jk}]+[X_j,L_{ki}]+[X_k,L_{ij}]\Big)=$
$=-\beta\hbar^2\Big(\delta_{ik}X_j-\delta_{ij}X_k+\delta_{ji}X_k-\delta_{jk}X_i+$
$+\delta_{kj}X_i-\delta_{ki}X_j\Big)=0,$

ii $[X_i,[X_j,L_{ks}]]+[X_j,[L_{ks},X_i]]+[L_{ks},[X_i,X_j]]=$
$=i\hbar[X_i,\delta_{js}X_k-\delta_{jk}X_s]-i\hbar[X_j,\delta_{is}X_k-\delta_{ik}X_s]+$
$+i\hbar\beta[L_{ks},L_{ij}]=-\hbar^2\beta\Big(\delta_{js}L_{ik}-\delta_{jk}L_{is}-\delta_{is}L_{jk}+$
$+\delta_{ik}L_{js}-\delta_{ik}L_{js}+\delta_{is}L_{jk}-\delta_{js}L_{ik}+\delta_{jk}L_{is}\Big)=0,$

iii $[X_i,[L_{jk},L_{sl}]]+[L_{jk},[L_{sl},X_i]]+[L_{sl},[X_i,L_{jk}]]=$
$=i\hbar\Big([X_i,\delta_{js}L_{kl}-\delta_{jl}L_{ks}+\delta_{kl}L_{js}-\delta_{ks}L_{jl}]+$
$+[\delta_{il}X_s-\delta_{is}X_l,L_{jk}]-[\delta_{ik}X_j-\delta_{ij}X_k,L_{sl}]\Big)=$
$-\hbar^2\Big(\delta_{js}(\delta_{il}X_k-\delta_{ik}X_l)-\delta_{jl}(\delta_{is}X_k-\delta_{ik}X_s)+$
$+\delta_{kl}(\delta_{is}X_j-\delta_{ij}X_s)-\delta_{ks}(\delta_{il}X_j-\delta_{ij}X_l)+$
$+\delta_{il}(\delta_{sk}X_j-\delta_{sj}X_k)-\delta_{is}(\delta_{lk}X_j-\delta_{lj}X_k)-$
$-\delta_{ik}(\delta_{jl}X_s-\delta_{js}X_l)+\delta_{ij}(\delta_{kl}X_s-\delta_{ks}X_l)\Big)=0,$

iv $[L_{ij},[L_{kl},L_{mn}]+[L_{kl},[L_{mn},L_{ij}]]+$
$+[1mm][L_{mn},[L_{ij},L_{kl}]]=0$

v $[P_i,[P_j,P_k]]+[P_j,[P_k,P_i]]+[P_k,[P_i,P_j]]=0,$

vi $[X_i,[P_j,P_k]]+[P_j,[P_k,X_i]]+[P_k,[X_i,P_j]]=0,$

vii $[P_i,[P_j,L_{ks}]]+[P_j,[L_{ks},P_i]]+[L_{ks},[P_i,P_j]]=$
$=i\hbar\Big([P_i,\delta_{js}P_k-\delta_{jk}P_s]-[P_j,\delta_{is}P_k-\delta_{ik}P_s]\Big)=0,$

viii $[X_i,[X_j,P_k]]+[X_j,[P_k,X_i]]+[P_k,[X_i,X_j]]=$
$=\hbar^2\beta\Big(\delta_{jk}P_i-\delta_{ik}P_j\Big)-\hbar^2\beta\Big(\delta_{jk}P_i-\delta_{ik}P_j\Big)=0,$

ix $[P_i,[L_{jk},L_{mn}]+[L_{jk},[L_{mn},P_i]]+[L_{mn},[P_i,L_{jk}]]=$
$=i\hbar[P_i,\delta_{jm}L_{kn}-\delta_{jn}L_{km}+\delta_{kn}L_{jm}-\delta_{km}L_{jn}]-$
$-i\hbar[L_{jk},\delta_{in}P_m-\delta_{im}P_n]-i\hbar[\delta_{ik}P_j-\delta_{ij}P_k,L_{mn}]=$
$=-\hbar^2\Big(\delta_{jm}(\delta_{in}P_k-\delta_{ik}P_n)-\delta_{jn}(\delta_{im}P_k-\delta_{ik}P_m)+$
$+\delta_{kn}(\delta_{im}P_j-\delta_{ij}P_m)-\delta_{km}(\delta_{in}P_j-\delta_{ij}P_n)+$
$+\delta_{in}(\delta_{mk}P_j-\delta_{mj}P_k)-\delta_{im}(\delta_{nk}P_j-\delta_{nj}P_k)-$
$-\delta_{ik}(\delta_{jn}P_m-\delta_{jm}P_n)+\delta_{ij}(\delta_{kn}P_m-\delta_{km}P_n)\Big)=0.$

Now, instead of the operators X_i and L_{ij}, we prefer to use new operators $\mathbf{X}_i$ and $\mathbf{J}_k$ defined in a way such that we have the commutation relations

$$[P_i,\mathbf{J}_j]=i\hbar\epsilon_{ijk}P_k, \tag{20}$$
$$[\mathbf{X}_i,\mathbf{J}_j]=i\hbar\epsilon_{ijk}\mathbf{X}_k, \tag{21}$$
$$[\mathbf{J}_i,\mathbf{J}_j]=i\hbar\epsilon_{ijk}\mathbf{J}_k, \tag{22}$$
$$[\mathbf{X}_i,\mathbf{X}_j]=i\hbar\beta\epsilon_{ijk}\mathbf{J}_k. \tag{23}$$

The nonzero components of operators $\mathbf{X}_i$ and $\mathbf{J}_i$ can be expressed in terms of the Pauli matrices:

$$\mathbf{J}=\mathbf{L}+\frac{\hbar}{2}\boldsymbol{\sigma}, \tag{24}$$
$$\mathbf{X}=\mathbf{x}+\frac{\hbar\sqrt{\beta}}{2}\boldsymbol{\sigma}. \tag{25}$$

There is the nonzero minimal length that can be described by the operator $\frac{\hbar\sqrt{\beta}}{2}\boldsymbol{\sigma}$.

5. A Particle with Spin 1

The generators of the rotational symmetry discussed above can be applied to the study of the SU(2)-symmetric quantum mechanics. There is also another possibility. In particular, we now consider the motion with spin $s=1$. In order to make the study easier, we prefer to consider the special case $\mathbf{x}=0$, $\mathbf{L}=0$. The following representation is possible:

$$(\mathbf{J}_i)_{jk}=-i\hbar\epsilon_{ijk}, \tag{26}$$

$$\mathbf{X}_1=\hbar\sqrt{\beta}\begin{pmatrix}0&0&0\\0&0&-1\\0&-1&0\end{pmatrix},\quad \mathbf{X}_2=\hbar\sqrt{\beta}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

$$\mathbf{X}_3=\hbar\sqrt{\beta}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}. \tag{27}$$

In addition, we also consider the vector $\mathbf{I}$ useful for the following consideration:

$$\mathbf{I}_1=\hbar^2\sqrt{\beta}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\quad \mathbf{I}_2=\hbar^2\sqrt{\beta}\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$\mathbf{I}_3=\hbar^2\sqrt{\beta}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}. \tag{28}$$

One can verify that

$$[\mathbf{I}_i, \mathbf{X}_j] = i\hbar^2\beta\theta_{ijk}\mathbf{J}_k, \tag{29}$$

$$[\mathbf{I}_i, \mathbf{J}_j] = -i\hbar^2\theta_{ijk}\mathbf{X}_k, \tag{30}$$

$$[\mathbf{I}_i, \mathbf{I}_j] = 0, \tag{31}$$

where $\theta_{ijk} = -(\epsilon_{ij} - 2\delta_{ij})\delta_{jk}$ (we do not sum here). Let us consider the matrix

$$\boldsymbol{\mu}^i_j = \begin{pmatrix} \frac{1}{\hbar\sqrt{\beta}}\mathbf{X}_1 & \frac{1}{\hbar}\mathbf{J}_1 & \frac{1}{\hbar^2\sqrt{\beta}}\mathbf{I}_1 \\ \frac{1}{\hbar\sqrt{\beta}}\mathbf{X}_2 & \frac{1}{\hbar}\mathbf{J}_2 & \frac{1}{\hbar^2\sqrt{\beta}}\mathbf{I}_2 \\ \frac{1}{\hbar\sqrt{\beta}}\mathbf{X}_3 & \frac{1}{\hbar}\mathbf{J}_3 & \frac{1}{\hbar^2\sqrt{\beta}}\mathbf{I}_3 \end{pmatrix} \tag{32}$$

with the commutation relations

$$[\boldsymbol{\mu}^m_i, \boldsymbol{\mu}^n_j] = i f^{mns}_{ijk}\boldsymbol{\mu}^s_k, \tag{33}$$

where

$$f^{121}_{ijk} = f^{222}_{ijk} = f^{112}_{ijk} = \epsilon_{ijk},$$
$$f^{312}_{ijk} = -f^{321}_{ijk} = \theta_{ijk} \tag{34}$$

and the other components are zeros.

The matrices $\boldsymbol{\mu}^i_j$ have the following properties:

$$\mathrm{Tr}(\boldsymbol{\mu}^i_j) = 0, \tag{35}$$

$$\{\boldsymbol{\mu}^i_k, \boldsymbol{\mu}^j_k\} = 4\delta_{ij}. \tag{36}$$

Note that the vectors $\mathbf{A}$, $\mathbf{J}$, and $\mathbf{I}$ are orthogonal ones. The matrices $\boldsymbol{\mu}^i_j$ can be expressed in terms of the Gell-Mann matrices λ_α:

$$\lambda_1 = \boldsymbol{\mu}^1_3 \qquad \lambda_2 = \boldsymbol{\mu}^2_3 \qquad \lambda_3 = \boldsymbol{\mu}^3_3$$
$$\lambda_4 = \boldsymbol{\mu}^1_2 \qquad \lambda_5 = -\boldsymbol{\mu}^2_2 \qquad \lambda_6 = -\boldsymbol{\mu}^1_1 \tag{37}$$
$$\lambda_7 = \boldsymbol{\mu}^2_1 \quad \lambda_8 = \tfrac{1}{\sqrt{3}}(2\boldsymbol{\mu}^3_1 + \boldsymbol{\mu}^3_3).$$

6. Relativistic Generalization

This section is devoted to the description of a standard relativistic particle with a special emphasis put on the way of introducing the time. The formal and conceptual issues concerning the noncommutative Maggiore space are discussed here as well. The quantized space-time was also considered in [42–45].

The exploration of physical events by means of the features from relativistic quantum theory will hopefully bring more accurate explanations, by elucidating the observational issues and other perplexing problems in contemporary physics.

In the previous sections, we have studied the position space without time generalization. In what follows, we assume that the commutation relation between the deformed time T and Hamilton $\mathcal{H}$ operators reads

$$[T, \mathcal{H}] = -i\nu\sqrt{1 - \beta\frac{\mathcal{H}^2}{c^2}}, \tag{38}$$

where the function ν depends on β. Similarly to relations (refd) and (refe), we can calculate the eigenvalues τ_n of the time operator:

$$\tau^{(1)}_n = \frac{\nu\hbar\sqrt{\beta}}{c}\,(n+1)\,, \tag{39}$$

$$\tau^{(2)}_n = \frac{\nu\hbar\sqrt{\beta}}{c}\,n. \tag{40}$$

Let $T, \mathcal{H}$, on the one hand, and X, P, on the other one, act as operators

$$T = t, \tag{41}$$

$$\mathcal{H} = \frac{\nu c}{\sqrt{\beta}}\sin\left(\frac{\sqrt{\beta}H}{c}\right) \tag{42}$$

and

$$X = x, \tag{43}$$

$$P = \frac{1}{\sqrt{\beta}}\sin(\sqrt{\beta}p), \tag{44}$$

where $[x, p] = i\hbar$ and $[t, H] = -i\hbar$. Taking into account that the relativistic Hamiltonian is given by

$$\mathcal{H}^2 = c^2P^2 + m^2c^4 \tag{45}$$

and the expressions (42) and (44), we have

$$\nu^2\cos(2\frac{\sqrt{\beta}H}{c})\psi(x,t) - \cos(2\sqrt{\beta}p)\psi(x,t) = \tag{46}$$

$$= \left(\nu^2 - 1 - 2(\sqrt{\beta}mc)^2\right)\psi(x,t). \tag{47}$$

It can be rewritten as the equation in finite differences

$$\frac{\nu^2}{2}\left(\psi\left(x, t + 2\frac{\hbar\sqrt{\beta}}{c}\right) + \psi\left(x, t - 2\frac{\hbar\sqrt{\beta}}{c}\right)\right) - $$
$$-\frac{1}{2}\left(\psi\left(x + 2\hbar\sqrt{\beta}, t\right) + \psi\left(x - 2\hbar\sqrt{\beta}, t\right)\right) = \nu^2 - 1 - 2\beta m^2c^2. \tag{48}$$

If we present the function ν as a power series, i.e., as a sum of powers of βm^2c^2,

$$\nu = 1 + \frac{1}{2}\beta m^2c^2 + \ldots, \tag{49}$$

then we recover the usual Klein–Gordon–Fock equation as $\beta \to 0$.

7. Summary

The noncommutative space considered here has been widely discussed for a long time. We have studied a one-parameter deformed Heisenberg algebra. Such a deformation leads to the existence of a nonzero minimal length.

In the framework of the chosen deformation, it is possible to study the SU(2) and SU(3) symmetries. It is shown that a Maggiore-like deformed space can be described in terms of the noncommutative geometry.

There are many experiments trying to prove the existence of the observable minimal length (see, e.g., [46]). However, the theoretical estimates of the parameter β are far beyond the experimental precision.

I would like to thank T.V. Fityo, M.M. Stetsko, and Prof. V.M. Tkachuk for the discussions.

1. H.S. Snyder, Phys. Rev. **71**, 38 (1947).
2. M. Maggiore, Phys. Lett. B **319**, 83 (1993).
3. A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995); H. Hinrichsen and A. Kempf, J. Math. Phys. **37**, 2121 (1996).
4. L.N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65**, 125027 (2002); [Prepint hep-th/0111181].
5. I. Dadic, L. Jonke, and S. Meljanac, Phys. Rev. D **67**, 087701 (2003); Prepint [hep-th/0210264].
6. C. Quesne and V.M. Tkachuk, J. Phys. A **36**, 10373 (2003); Prepint [math-ph/0306047].
7. S. Hossenfelder, Phys. Rev. D **70**, 105003 (2004); Prepint [hep-ph/0405127].
8. S. Hossenfelder, Mod. Phys. Lett. A **19**, 2727 (2004); Prepint [hep-ph/0410122].
9. C. Quesne and V.M. Tkachuk, J. Phys. A **37**, 10095 (2004); Prepint [math-ph/0312029].
10. S. Benczik, L.N. Chang, D. Minic, and T. Takeuchi, Phys. Rev. A **72**, 012104 (2005); Prepint [hep-th/0502222].
11. C. Quesne and V.M. Tkachuk, J. Phys. A **38**, 1747 (2005); Prepint [math-ph/0412052].
12. T.V. Fityo, I.O. Vakarchuk, and V.M. Tkachuk, J. Phys. A **39**, 2143 (2006); Prepint [quant-ph/0507117].
13. S. Hossenfelder, Class. Quantum Grav. **23**, 1815 (2006); Prepint [hep-th/0510245].
14. K. Nozari and T. Azizi, Gen. Rel. Grav. **38**, 735 (2006); Prepint [quant-ph/0507018].
15. F. Buisseret, Prepint [hep-th/1011.3690].
16. R. Brout, C. Gabriel, M. Lubo, and P. Spindel, Phys. Rev. D **59**, 044005 (1999); Prepint [hep-th/9807063].
17. L.J. Garay, Internat. J. Modern Phys. A **10**, 145 (1995); Preprint [gr-qc/9403008].
18. D.J. Gross and P.F. Mende, Nucl. Phys. B **303**, 407 (1988).
19. D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B **216**, 41 (1989).
20. K. Konishi, G. Paffuti, and P. Provero, Phys. Lett. B 234, 276 (1990).
21. E. Witten, Phys. Today **49**, 24 (1997) .
22. P.F. Mende, Prepint [hep-th/9210001].
23. R. Lafrance and R.C. Myers, Phys. Rev. D **51**, 2584 (1995); Prepint [hep-th/9411018].
24. S. Doplicher, K. Fredenhagen, and J.E. Roberts, Phys. Lett. B **331**, 39 (1994).
25. S. Doplicher, K. Fredenhagen, and J.E. Roberts, Comm. Math. Phys. **172**, 187 (1995); Preprint [hep-th/0303037].
26. R. Oeckl, Nucl. Phys. B **581**, 559 (2000); Preprint [hep-th/0003018].
27. P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, and J. Wess, Classical Quantum Gravity **22**, 3511 (2005); Preprint [hep-th/0504183].
28. M. Dimitrijević, L. Jonke, L. Möller, E. Tsouchnika, J. Wess, and M. Wohlgenannt, Eur. Phys. J. C **31**, 129 (2003); Preprint [hep-th/0307149].
29. L. Freidel, J. Kowalski-Glikman, and S. Nowak, Phys. Lett. B **648**, 70 (2007); Preprint [hep-th/0612170].
30. T.R. Govindarajan, K.S. Gupta, E. Harikumar, S. Meljanac, and D. Meljanac, Phys. Rev. D **77**, 105010 (2008); Preprint [arXiv:0802.1576].
31. M. Dimitrijević, L. Jonke, Möller, and J. Wess, Eur. Phys. J. C **36**, 117 (2004); Preprint [hep-th/0310116].
32. A. Borowiec and A. Pachol, Preprint [math-ph/1005.4429]
33. V.G. Drinfeld, Quantum groups, in *Proc. of the Int. Congress of Mathematicians*, edited by A. Gleason (AMS, Providence, RI, 1987), p. 798.
34. M. Jimbo, Lett. Math. Phys. **10**, 63 (1985).
35. P.P. Kulish and N.Yu. Reshetikhin, J. Math. Sci. **23**, 2435 (1983).
36. S.L. Woronowicz, Comm. Math. Phys. **111**, 613 (1987).
37. G. Amelino-Camelia, Internat. J. Mod. Phys. D **11**, 35 (2002); Preprint [gr-qc/0012051].
38. B. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, Phys. Lett. B **522**, 133 (2001); Preprint [hep-th/0107039].
39. J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002); Preprint [hep-th/0112090].
40. F. Girelli, E. R. Livine, Braz. J. Phys. **35**, 432 (2005); Preprint [gr-qc/0412079].

41. V.I. Smirnov, *A Course of Higher Mathematics*, Vol. III (Pergamon Press, Oxford, 1964).
42. F. Casagrande, Scientia **112**, 417 (1977).
43. M. Jammer, *Concepts of Space* (Harvard Univ.Press., Cambridge, 1954).
44. T.D. Lee, Phys. Lett. B **122**, 217 (1983); J. Stat. Phys. **46**, 843 (1987).
45. R.A.H. Rarias, Erasmo Recami, Preprint [quant-ph/9706059].
46. F. Brau, J. Phys. A **36**, 1523 (2003); Preprint [quant-ph/0301167].

Received 07.12.09

ДЕЯКІ ЗАУВАЖЕННЯ ЩОДО РЕЛЯТИВІСТСЬКОЇ КВАНТОВОЇ МЕХАНІКИ ЧАСТИНКИ В ПРОСТОРІ ІЗ МІНІМАЛЬНОЮ ДОВЖИНОЮ

К.М. Щербаков

Р е з ю м е

В статті досліджено некомутативний простір з деформованою алгеброю Гайзенберга $[X, P] = i\hbar\sqrt{1-\beta P^2}$. Розглянуто також прості квантово-механічні структури, що імплементують розглянуті комутаційні співвідношення. Додатково проаналізовано симетрію групи обертань в просторі з мінімальною довжиною.