SCALAR COSMOLOGICAL PERTURBATIONS ON THE BRANE

A.V. VIZNYUK, YU.V. SHTANOV

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrologichna Str., Kiev 03680, Ukraine; e-mail:
viznyuk@bitp.kiev.ua, shtanov@bitp.kiev.ua)

PACS 98.80.Cq; 98.70.Vc;
98.80.Es

We derive a full system of differential equations describing the evolution of scalar cosmological perturbations on the brane in the general case where the action of the model contains the induced curvature, as well as the cosmological constants in the bulk and on the brane. This system of equations is greatly simplified in the case of ideal pressureless matter. From the brane observer viewpoint, the dynamics of perturbations of the matter on the brane is affected by an additional invisible component – perturbation of the projected Weyl tensor, or dark radiation, having purely geometric nature. The system of equations on the brane serves as boundary conditions for the perturbed bulk equations, which can be treated with the use of the Mukohyama master variable. We consider the case of a spatially closed brane universe and impose the regularity condition for perturbations in the bulk. We demonstrate that the resulting complete system of integro-differential equations is well defined.

1. Introduction

The idea that our observable Universe can be a four-dimensional manifold (the "brane"), which is embedded in a higher dimensional spacetime (the "bulk") with Standard Model particles and fields trapped on the brane, was thoroughly investigated during the last two decades. The activity in the field was triggered especially by the Randall–Sundrum (RS) braneworld model [1], in which Einstein's theory of general relativity is modified due to extra dimensional effects at relatively high energies. Apart from interesting cosmological applications, it was shown that a modified theory of gravity based on the RS braneworld model can potentially explain the observations of the galactic rotation curves and X-ray profiles of galactic clusters without invoking the notion of dark matter [2]. This theory was unable, however, to address the cosmological implications of dark matter. On the other hand, an alternative braneworld model of Dvali, Gabadadze, and Porrati (DGP), in which gravity is modified at low energies, gave rise to a cosmology with late-time acceleration without cosmological constants on the brane or in the bulk [3, 4] (although later this model was shown to contain a ghost on the self-accelerating branch [5]).

The main feature of the DGP braneworld model is the induced gravity term in the action for the brane. But the cosmological constants are absent in this theory, in contrast to the RS braneworld model. A more general braneworld model contains the induced gravity term, as well as cosmological constants, in the bulk and on the brane [6–8]. Models of such generic form can describe the late-time cosmological acceleration. In doing so, they exhibit some interesting specific features, for example, the possibility of superacceleration (supernegative effective equation of state of the dark energy $w_{\rm eff} \le -1$) [8], the possibility of cosmological loitering even in a spatially flat universe [9], and the property of cosmic mimicry, where a low-density braneworld has the expansion history of the LCDM model [10]. At the same time, this kind of the braneworld model can be used to address astrophysical observations of dark matter in galaxies [11].

Developing the theory of cosmological perturbations is a long-standing problem of the braneworld model. Structure formation, temperature anisotropy of the cosmic microwave background (CMB), and other issues that form the basis of experimental tests of any cosmological model require the knowledge of the evolution of cosmological perturbations. The main problem in the theory under investigation is the necessity of the account for the bulk gravitational effects leading to the non-locality of the resulting equations on the brane. Regardless of its computational complexity, a considerable progress has been made in this direction during the last years. A complete system of cosmological equations allowing for numerical computation was obtained in the framework of the RS [12] and DGP [13] braneworld models. Important analytical results are presented in [14, 15]. However, the problem of cosmological perturbations in the braneworld model still remains to be solved in full generality. For a modern review of this problem, see [16].

The existence of the extra dimension requires a specification of the boundary conditions in the bulk space. In the usual case of a spatially flat brane, the extra dimension is noncompact, and one has to deal with the spatial infinity of the extra dimension. This is a difficult situation with no obvious and unique choice for the boundary conditions. In the present paper, we consider the case of a spatially closed brane, an expanding three-sphere, which is bounding a four-ball in the bulk space. In this case, the boundary condition can be specified uniquely just as a regularity condition of the metric inside the ball. We obtain a complete system of equations for scalar cosmological perturbations on the brane in the braneworld theory with induced gravity, as well as cosmological constants in the bulk and on the brane, and analyze its behavior in the bulk.

2. The Theory

The braneworld action, to the lowest order in the bulk and brane curvature, can be written in the form:

$$S = M^3 \left[\int_{\text{bulk}} (\mathcal{R} - 2\Lambda) - 2 \int_{\text{brane}} K \right] +$$

$$+ \int_{\text{brane}} \left(m^2 R - 2\lambda \right) + \int_{\text{brane}} L\left(g_{\mu\nu}, \phi\right) , \qquad (1)$$

where $\mathcal{R}$ is the scalar curvature of the five-dimensional bulk metric g_{AB}, and R is the scalar curvature of the induced metric $g_{\mu\nu}$ on the brane.[1] The quantity K denotes the trace of the symmetric tensor of extrinsic curvature of the brane, and the symbol $L(g_{\mu\nu}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ, whose dynamics is restricted to the brane so that they interact only with the induced metric $g_{\mu\nu}$. All integrations over the bulk and brane are taken with the corresponding natural volume elements. The symbols M and m denote the five-dimensional and four-dimensional Planck masses, respectively, Λ is the bulk cosmological constant, and λ is referred to as the brane tension.

The action of the Randall–Sundrum braneworld model [1] is obtained after setting $m = 0$ in (1), while the special case where both the cosmological constant in the bulk and the brane tension vanish ($\Lambda = 0$ and $\lambda = 0$) describes the original model of Dvali, Gabadadze, and Porrati [3]. Finally, general relativity with the quantity $1/m^2$ playing the role of the gravitational constant is formally obtained from (1) after setting $M = 0$.

Action (1) leads to the Einstein equation with cosmological constant in the bulk,

$$\mathcal{G}_{AB} + \Lambda g_{AB} = 0\,, \qquad (2)$$

with the following equation on the brane [10, 17]:

$$G_{\mu\nu} + \frac{\Lambda_{\rm RS}}{b+1} g_{\mu\nu} = \left(\frac{b}{b+1} \right) \frac{1}{m^2} T_{\mu\nu} +$$

$$+ \frac{1}{b+1} \left(\frac{1}{M^6} Q_{\mu\nu} - C_{\mu\nu} \right) , \qquad (3)$$

where

$$b = k\ell\,, \qquad k = \frac{\lambda}{3M^3}\,, \qquad \ell = \frac{2m^2}{M^3} \qquad (4)$$

are convenient parameters of the braneworld model,

$$\Lambda_{\rm RS} = \frac{\Lambda}{2} + \frac{\lambda^2}{3M^6} \qquad (5)$$

is the value of the effective cosmological constant in the Randall–Sundrum model,

$$Q_{\mu\nu} = \frac{1}{3} E E_{\mu\nu} - E_{\mu\lambda} E^{\lambda}{}_{\nu} + \frac{1}{2} \left(E_{\rho\lambda} E^{\rho\lambda} - \frac{1}{3} E^2 \right) g_{\mu\nu} \quad (6)$$

is a quadratic expression with respect to the "bare" Einstein equation $E_{\mu\nu} \equiv m^2 G_{\mu\nu} - T_{\mu\nu}$ on the brane, and $E = g^{\rho\lambda} E_{\rho\lambda}$. The symmetric traceless tensor $C_{\mu\nu} \equiv P^{A}{}_{\mu} P^{C}{}_{\nu} n^{B} n^{D} \mathcal{C}_{ABCD}$ is the projection of the bulk Weyl tensor $\mathcal{C}_{ABCD}$, which carries information about the gravitational field outside the brane. The vector field n^A is the inner unit normal to the brane, and $P^{A}{}_{\mu}$ is the orthogonal projector to the brane. The tensor $C_{\mu\nu}$ is not freely specifiable on the brane, but it is related to the tensor $Q_{\mu\nu}$ through the conservation equation

$$\nabla^{\mu} \left(Q_{\mu\nu} - M^6 C_{\mu\nu} \right) = 0, \qquad (7)$$

which is a consequence of the Bianchi identity applied to (3).

[1] Here and below, we use upper-case Latin indices $A, B, \ldots$ for the five-dimensional bulk coordinates and Greek indices $\mu, \nu, \ldots$ for the four-dimensional coordinates on the brane.

The background cosmological evolution on the brane can be presented in the following form (see [6, 8, 18]):

$$H^2 + \frac{\kappa}{a^2} = \frac{\rho + \lambda}{3m^2} +$$

$$+ \frac{2}{\ell^2} \left[1 \pm \sqrt{1 + \ell^2 \left(\frac{\rho + \lambda}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right] . \qquad (8)$$

Here, $\rho = \rho(t)$ is the matter energy density on the brane and C is a constant resulting from the symmetric traceless tensor $C_{\mu\nu}$ in the field equations (3). The Hubble parameter $H \equiv \dot{a}/a$ describes the evolution of the Friedmann–Robertson–Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij} dx^i dx^j \, . \qquad (9)$$

A purely spatial metric γ_{ij} can be presented in the isotropic coordinates in the form

$$\gamma_{ij} = \delta_{ij} \left[1 + \frac{\kappa}{4} \sum_i \left(x^i \right)^2 \right]^{-2} , \qquad (10)$$

so that the constant κ has the dimension of inverse comoving length squared. If the spatial coordinates x^i are dimensionless (so that the dimension is placed on the scale factor), then one can choose $\kappa = 0, \pm 1$.

The $\pm$ signs in Eq. (8) correspond to two branches which, in turn, correspond to two different ways of bounding the bulk by the brane [18]. They are usually called the normal branch (lower sign) and the self-accelerating branch (upper sign), and we will refer to them here in this way.

3. Scalar Cosmological Perturbations on the Brane

From a brane observer viewpoint, the effects of bulk gravity in the evolution of cosmological perturbations are encoded in the local ($Q_{\mu\nu}$) and nonlocal ($C_{\mu\nu}$) corrections to the Einstein equations (3). Although the tensor $C_{\mu\nu}$ cannot be completely determined, in general, by the data on the brane, the dynamics of some of its degrees of freedom can be fixed via the conservation equation (7). Note that a closed local system of equations on the brane can be obtained by specifying some restrictions on the projected Weyl tensor $C_{\mu\nu}$, which can be regarded as boundary conditions (see [19]). The simplest choice consists in setting its appropriately defined anisotropic stress to zero [see Eq. (13) below]. This condition is fully compatible with all equations of the theory, but may lead to unwanted singularities in the bulk.

In what follows, we derive a full system of differential equations describing the scalar cosmological perturbations on the brane in a general case without any simplifying assumptions.

3.1. Derivation of the main equations

Scalar cosmological perturbations of the induced metric on the brane are most conveniently described by the relativistic potentials Φ and Ψ in the so-called longitudinal gauge. The perturbed metric in the conformal coordinates reads

$$ds^2 = a^2 \left[-(1 + 2\Phi) d\eta^2 + (1 - 2\Psi)\gamma_{ij} dx^i dx^j \right] . \qquad (11)$$

We introduce the components of the linearly perturbed stress–energy tensor of matter in these coordinates:[2]

$$\delta T^\alpha{}_\beta = \begin{pmatrix} -\delta\rho \, , & -\dfrac{\nabla_i v}{a} \\ \dfrac{\nabla^i v}{a} \, , & \delta p \, \delta^i{}_j + \dfrac{\zeta^i{}_j}{a^2} \end{pmatrix} , \qquad (12)$$

where $\delta\rho$, δp, v, and $\zeta_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3} \gamma_{ij} \nabla^2 \right) \zeta$ are scalar perturbations. Similarly, we introduce the scalar perturbations $\delta\rho_{\mathcal{C}}$, $v_{\mathcal{C}}$, and $\delta\pi_{\mathcal{C}}$ of the tensor $C_{\alpha\beta}$:

$$m^2 \delta C^\alpha{}_\beta = \begin{pmatrix} -\delta\rho_{\mathcal{C}} \, , & -\dfrac{\nabla_i v_{\mathcal{C}}}{a} \\ \dfrac{\nabla^i v_{\mathcal{C}}}{a} \, , & \dfrac{\delta\rho_{\mathcal{C}}}{3} \delta^i{}_j + \dfrac{\delta\pi^i{}_j}{a^2} \end{pmatrix} , \qquad (13)$$

where $\delta\pi_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3} \gamma_{ij} \nabla^2 \right) \delta\pi_{\mathcal{C}}$.

We call v and $v_{\mathcal{C}}$ the momentum potentials for matter and dark radiation, respectively, $\delta\rho$ and $\delta\rho_{\mathcal{C}}$ are their energy density perturbations, and ζ and $\delta\pi_{\mathcal{C}}$ are the scalar potentials for their anisotropic stresses.

The perturbed version of (3) now reads (see Appendix B for a detailed derivation)

$$\frac{\delta\rho + \delta\rho_{\mathcal{C}}}{m^2} = -\beta \left[\frac{\delta\rho}{2m^2} + \frac{3\mathcal{H}}{a^2} \left(\Psi' + \mathcal{H}\Phi \right) \right] +$$

$$+ \frac{\beta}{a^2} \left(\nabla^2 + 3\kappa \right) \Psi \, , \qquad (14)$$

[2] The spatial indices $i, j, \ldots$ in purely spatially defined quantities (such as v_i and $\delta\pi_{ij}$) are always raised and lowered with the use of the spatial metric γ_{ij} ; in particular, $\gamma^i{}_j = \delta^i{}_j$. The symbol ∇_i denotes the covariant derivative with respect to the spatial metric γ_{ij}, and the spatial Laplacian is $\nabla^2 = \nabla^i \nabla_i$.

$$\frac{v+v_{\mathcal{C}}}{m^2} = -\beta\left[\frac{v}{2m^2} - \frac{1}{a}\left(\Psi' + \mathcal{H}\Phi\right)\right], \tag{15}$$

$$\frac{3\delta p + \delta\rho_{\mathcal{C}}}{3m^2} = -\frac{\beta\delta p}{2m^2} - \frac{\beta(3\gamma-1)}{3a^2}\left(\nabla^2 + 3\kappa\right)\Psi +$$

$$+\frac{\beta(3\gamma-1)}{6}\left[\frac{\delta\rho}{m^2} + \frac{6\mathcal{H}}{a^2}\left(\Psi' + \mathcal{H}\Phi\right)\right] +$$

$$+\frac{\beta}{a^2}\left[\left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \mathcal{H}\left(\Phi' + 2\Psi'\right)\right] +$$

$$+\frac{\beta}{a^2}\left[\Psi'' - \kappa\Psi + \frac{1}{3}\nabla^2 D\right], \tag{16}$$

$$\frac{\zeta + \delta\pi_{\mathcal{C}}}{m^2} = -\frac{\beta(3\gamma+1)}{4}\left(D + \frac{\zeta}{m^2}\right), \tag{17}$$

where $\mathcal{H} \equiv a'/a$ and $D \equiv \Phi - \Psi$. The time-dependent dimensionless functions β and γ are given by[3]

$$\beta \equiv \ell^2\left(\frac{\mathcal{H}^2+\kappa}{a^2} - \frac{\rho+\lambda}{3m^2}\right) - 2 =$$

$$= \pm 2\sqrt{1 + \ell^2\left(\frac{\rho+\lambda}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4}\right)}, \tag{18}$$

$$\gamma \equiv \frac{1}{3}\left(1 + \frac{\beta'}{\mathcal{H}\beta}\right) \equiv \frac{1}{3}\left(1 + \frac{\dot\beta}{H\beta}\right) =$$

$$= \frac{1}{3}\left[1 - \frac{\dfrac{\rho+p}{m^2} - \dfrac{4C}{a^4}}{2\left(\dfrac{\rho+\lambda}{3m^2} + \dfrac{1}{\ell^2} - \dfrac{\Lambda}{6} - \dfrac{C}{a^4}\right)}\right]. \tag{19}$$

These equations can be used to obtain the evolution equations for the matter and for the Weyl fluid. To derive them, we use the fact that the stress–energy of the ordinary matter on the brane is conserved, $\nabla_\mu T^\mu{}_\nu = 0$. Calculating the perturbed version of this equation, we have

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - \frac{1}{a}\nabla^2 v = 3(\rho+p)\Psi', \tag{20}$$

$$\frac{v' + 3\mathcal{H}v}{a} = \delta p + (\rho+p)\Phi + \frac{2}{3a^2}\left(\nabla^2 + 3\kappa\right)\zeta. \tag{21}$$

To derive the evolution equations for the dark radiation, we take the covariant derivative of the perturbation of (A2) to get

$$\nabla_\mu \delta C^\mu{}_\nu = \nabla_\mu\left[\frac{1}{M^6}\delta Q^\mu{}_\nu - \frac{1+b}{m^2}\delta E^\mu{}_\nu - \frac{1}{m^2}\delta T^\mu{}_\nu\right], \tag{22}$$

where the covariant derivative ∇_ν is taken with respect to the unperturbed induced metric on the brane. The right-hand side of this equation can be read-off from (14)–(17), and, after some calculation with the use of (20) and (21), we get

$$\delta\rho_{\mathcal{C}}' + 4\mathcal{H}\delta\rho_{\mathcal{C}} - \frac{1}{a}\nabla^2 v_{\mathcal{C}} = -\frac{12m^2C}{a^4}\Psi', \tag{23}$$

$$\frac{v_{\mathcal{C}}' + 3\mathcal{H}v_{\mathcal{C}}}{a} = \frac{\delta\rho_{\mathcal{C}}}{3} - \frac{4m^2C}{a^4}\Phi +$$

$$+\frac{\beta(1-3\gamma)}{6}\left(\delta\rho + \frac{3\mathcal{H}v}{a}\right) - \frac{2+\beta}{3a^2}\left(\nabla^2 + 3\kappa\right)\zeta -$$

$$-\frac{m^2\beta}{3a^2}\left(\nabla^2 + 3\kappa\right)\left[\Phi - 3\gamma\Psi\right]. \tag{24}$$

3.2. Combined system of equations on the brane

Here, we collect the equations that describe the evolution of perturbations on the brane and present them in a convenient form in terms of the physical time t.

It is convenient to use the notation

$$\Delta_{\rm m} = \delta\rho + 3Hv, \qquad \Delta_{\mathcal{C}} = \delta\rho_{\mathcal{C}} + 3Hv_{\mathcal{C}}. \tag{25}$$

The system of field equations then reads

$$\frac{1}{a^2}\left(\nabla^2 + 3\kappa\right)\Psi = \left(1 + \frac{2}{\beta}\right)\frac{\Delta_{\rm m}}{2m^2} + \frac{\Delta_{\mathcal{C}}}{m^2\beta}, \tag{26}$$

$$m^2\beta\left(\dot\Psi + H\Phi\right) = \left(1 + \frac{\beta}{2}\right)v + v_{\mathcal{C}}, \tag{27}$$

$$\ddot\Psi + 3(1+\gamma)H\dot\Psi + H\dot\Phi + \left[2\dot H + 3H^2(1+\gamma)\right]\Phi -$$

$$-\frac{\gamma}{a^2}\nabla^2\Psi - \frac{\kappa(1+3\gamma)}{a^2}\Psi + \frac{1}{3a^2}\nabla^2(\Phi - \Psi) =$$

[3] Everywhere, the prime denotes the derivative with respect to the conformal time η. The overdot is reserved to denote the derivative with respect to the physical time t defined as $dt = a(\eta)d\eta$, so we will have, for example, $\mathcal{H} \equiv a'/a = \dot a \equiv aH$.

$$= -\left(\gamma + \frac{2}{3\beta}\right)\frac{\delta\rho}{2m^2} + \left(1 + \frac{2}{\beta}\right)\frac{\delta p}{2m^2}\,, \tag{28}$$

$$\frac{\delta\pi_{\mathcal{C}}}{m^2} = -\frac{\beta(3\gamma+1)}{4}\left(\Phi - \Psi + \frac{\zeta}{m^2}\right) - \frac{\zeta}{m^2}\,. \tag{29}$$

The system of conservation equations is

$$\delta\dot\rho + 3H(\delta\rho + \delta p) = \frac{1}{a^2}\nabla^2 v + 3(\rho + p)\dot\Psi\,, \tag{30}$$

$$\dot v + 3Hv = \delta p + (\rho + p)\Phi + \frac{2}{3a^2}\left(\nabla^2 + 3\kappa\right)\zeta\,, \tag{31}$$

$$\delta\dot\rho_{\mathcal{C}} + 4H\delta\rho_{\mathcal{C}} = \frac{1}{a^2}\nabla^2 v_{\mathcal{C}} - \frac{12m^2C}{a^4}\dot\Psi\,, \tag{32}$$

$$\dot v_{\mathcal{C}} + 3Hv_{\mathcal{C}} = \frac{1}{3}\delta\rho_{\mathcal{C}} - \frac{4m^2C}{a^4}\Phi + \frac{1}{6}\beta(1-3\gamma)\Delta_{\rm m} -$$

$$-\frac{2+\beta}{3a^2}\left(\nabla^2 + 3\kappa\right)\zeta - \frac{m^2\beta}{3a^2}\left(\nabla^2 + 3\kappa\right)\left[\Phi - 3\gamma\Psi\right]\,. \tag{33}$$

3.3. The special case of ideal pressureless matter

Using Eqs. (26)–(29) and (30)–(33), one can derive the following useful system for perturbations in *pressureless* matter ($p = 0$, $\zeta = 0$) and for dark radiation in the important case $C = 0$:

$$\ddot\Delta + 2H\dot\Delta = \left(1 + \frac{6\gamma}{\beta}\right)\frac{\rho\Delta}{2m^2} + (1+3\gamma)\frac{\delta\rho_{\mathcal{C}}}{m^2\beta}\,, \tag{34}$$

$$\delta\dot\rho_{\mathcal{C}} + 4H\delta\rho_{\mathcal{C}} = \frac{1}{a^2}\nabla^2 v_{\mathcal{C}}\,, \tag{35}$$

$$\dot v_{\mathcal{C}} + 4Hv_{\mathcal{C}} = \gamma\Delta_{\mathcal{C}} + \left(\gamma - \frac{1}{3}\right)\Delta_{\rm m} +$$

$$+\frac{4}{3(1+3\gamma)a^2}\left(\nabla^2 + 3\kappa\right)\delta\pi_{\mathcal{C}}\,, \tag{36}$$

where $\Delta \equiv \Delta_{\rm m}/\rho$ is a conventional dimensionless variable describing the matter perturbations.

Equation (34) can be compared to its counterpart in general relativity:

$$\ddot\Delta + 2H\dot\Delta = \frac{\rho\Delta}{2m^2}\,. \tag{37}$$

One can see that the braneworld model leads to three effects as regards the evolution of matter perturbations: (i) it modifies the Hubble expansion law $H(t)$ via (8); (ii) it produces a time-dependent renormalization of the effective gravitational coupling by the factor $\Theta = 1 + 6\gamma/\beta$; and (iii) it introduces the gravity of the Weyl fluid on the right-hand side of (34).

The system of equations (34)–(36) is not closed because the evolution equation for the anisotropic stress $\delta\pi_{\mathcal{C}}$ is missing. As we have noted at the beginning of this section, one can avoid this problem by setting some restrictions on this part of the Weyl tensor directly on the brane. The simplest choice would be to set $\delta\pi_{\mathcal{C}} = 0$. In this case, system (34)–(36) becomes a closed system of differential equations describing the evolution of cosmological perturbations on the brane [19]. However, this type of boundary conditions, although simple, is not well motivated from the bulk viewpoint. Physically, the evolution of the Weyl tensor should be derived from the perturbed bulk equation (2) after setting some natural boundary conditions in the bulk [12–16, 20]. In this paper, we adopt the following simple approach to the boundary conditions. We consider a spatially closed braneworld model which bounds the interior bulk space with the spatial topology of a ball (resulting in the physically plausible normal branch) and demand that the bulk metric be regular in the brane interior.

In the next section, we describe the perturbed bulk equations in terms of the Mukohyama master variable and study the special case of the flat background bulk metric in more details.

4. Perturbations of the Bulk

4.1. Mukohyama master variable

In the natural static coordinates, the background bulk metric can be written in the form

$$ds^2_{\rm bulk} = -f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2\gamma_{ij}dx^i dx^j\,, \tag{38}$$

where γ_{ij} is the metric of a maximally symmetric space with coordinates x^i (10), and the function $f(r)$ is given by[4]

$$f(r) = \kappa - \frac{\Lambda}{6}r^2\,. \tag{39}$$

[4] The possible term C/r^2 in $f(r)$ is absent because of our regularity condition in the bulk bounded by the brane. This implies that the background bulk has zero Weyl tensor, $\mathcal{C}_{ABCD} = 0$.

In these coordinates, the FRW brane moves radially along the trajectory $r = a(\tau)$, and the relevant part of the bulk is given by $r \leq a(\tau)$. In what follows, we are interested in the case $\kappa = 1$ and $\Lambda \leq 0$.

It is convenient to present the first part of metric (38) in the form $\sigma_{ab} dx^a dx^b$, where x^a, $a = 1, 2$, are arbitrary coordinates equivalent to (τ, r). Thus, for the background metric, we have

$$ds^2_{\text{bulk}} = \sigma_{ab} dx^a dx^b + r^2 \gamma_{ij} dx^i dx^j \,, \quad (40)$$

where $r = r(x^a)$.

The scalar (with respect to the isometries of γ_{ij}) perturbations of this metric can be described as in [21]:

$$\delta g_{AB} dx^A dx^B = \sum_k h_{ab} Y dx^a dx^b + \sum_k 2 h_a V_i dx^a dx^i +$$

$$+ \sum_k \left[h_L T^{(L)}_{ij} + h_Y T^{(Y)}_{ij} \right] dx^i dx^j \,, \quad (41)$$

where Y, $V_i \equiv \nabla_i Y$, $T^{(L)}_{ij} \equiv 2 \nabla_i \nabla_j Y - \frac{2}{3} \gamma_{ij} \nabla^2 Y$, and $T^{(Y)}_{ij} \equiv \gamma_{ij} Y$ are the harmonics depending on the coordinates x^i and all expressible through the scalar harmonics Y defined on a unit three-sphere, and h_{ab}, h_a, h_L, and h_Y are the perturbation coefficients depending on x^a. Here, as before, ∇_i is the covariant derivative with respect to the metric γ_{ij}. The number k characterizes the Laplacian eigenvalue of the scalar harmonics Y.

Infinitesimal coordinate transformations of the scalar type are described by the vector field ξ^A that has the form

$$\xi_A dx^A = \sum_k \left(\xi_a Y dx^a + \xi V_i dx^i \right) \,. \quad (42)$$

Under diffeomorphisms, the perturbations are transformed as follows:

$$h_{ab} \to h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a \,, \quad (43)$$

$$h_a \to h_a - \xi_a - r^2 \nabla_a \left(r^{-2} \xi \right) \,, \quad (44)$$

$$h_L \to h_L - \xi \,, \quad (45)$$

$$h_Y \to h_Y - \xi^a \nabla_a r^2 + \frac{2}{3} k^2 \xi \,. \quad (46)$$

Here, ∇_a is the covariant derivative in the two-dimensional space spanned by (τ, r) and compatible with the metric σ_{ab}. From these quantities, one can construct the gauge-invariant variables

$$F_{ab} = h_{ab} - \nabla_a X_b - \nabla_b X_a \,, \quad (47)$$

$$F = h_Y - X^a \nabla_a r^2 + \frac{2}{3} k^2 h_L \,, \quad (48)$$

where $X_a = h_a - r^2 \nabla_a \left(r^{-2} h_L \right)$ is a gauge-dependent combination that is transformed as $X_a \to X_a - \xi_a$.

Note that linear perturbations of the tensors with zero background values are gauge-invariant: if $T^{\cdots}_{\cdots}$ is any such tensor, then, under the infinitesimal coordinate transformations, its components are transformed as $\delta T^{\cdots}_{\cdots} = \mathcal{L}_\xi T^{\cdots}_{\cdots} = 0$. In particular, perturbations of the Weyl tensor $\mathcal{C}^A{}_{BCD}$ (and all its contractions and derivatives), as well as perturbations of the Eistein–De Sitter tensor $\mathcal{E}_{AB} = \mathcal{G}_{AB} + \Lambda g_{AB}$, are gauge-invariant because these tensors are identically equal to zero for the background solution (38).

Using the gauge transformations (44), (45), one can fix the gauge in such a way that the coefficients h_L and h_a become zero (at least, this is possible to do locally). In this gauge, the coefficients h_{ab} and h_Y coincide with the gauge invariants F_{ab} and F, respectively, and then the metric perturbation simplifies to

$$\delta g_{AB} dx^A dx^B = \sum_k Y \left(F_{ab} dx^a dx^b + F \gamma_{ij} dx^i dx^j \right) \,. \quad (49)$$

Expression in this gauge can be used whenever one is to calculate gauge-invariant perturbations such as perturbations of the Weyl tensor $\mathcal{C}^A{}_{BCD}$.

Another set of simplifications can be made with regard for the equations of motion in the bulk (2), which can be presented as $\mathcal{R}_{AB} = \frac{2\Lambda}{3} g_{AB}$. Using these relations, one can express the Weyl tensor in the bulk as follows:

$$\mathcal{C}_{ABCD} = \mathcal{R}_{ABCD} - \frac{\Lambda}{3} g_{A[C} g_{D]B} \,. \quad (50)$$

However, when calculating the curvature tensor $\mathcal{R}_{ABCD}$ to get the perturbed equations of motion in the bulk, one needs to deal with the complete metric perturbation (49).

Using (50) and (C1), (C3), (B3), one can easily compute the components of the perturbed bulk Weyl tensor $\mathcal{C}_{ABCD}$ in the gauge $h_L = 0$, $h_a = 0$. Then the coefficients h_{ab} and h_Y can be replaced by the gauge invariant variables F_{ab} and F, respectively. Thus, the perturbed bulk Weyl tensor can be expressed as

$$\delta \mathcal{C}_{abcd} = \frac{\Lambda}{6} \sum_k \left(\sigma_{d[a} F_{b]c} - \sigma_{c[a} F_{b]d} \right) Y +$$

 ISSN 2071-0194. Ukr. J. Phys. 2012. Vol. 57, No. 12

$$+\sum_k \left(\nabla_d\nabla_{[a}F_{b]c} - \nabla_c\nabla_{[a}F_{b]d}\right) Y\,, \tag{51}$$

$$\delta\mathcal{C}_{iabc} = \sum_k \left(\nabla_{[c}F_{b]a} - \frac{1}{r}F_{a[b}\nabla_{c]}r\right)\nabla_i Y\,, \tag{52}$$

$$\delta\mathcal{C}_{abij} = 0\,, \tag{53}$$

$$\delta\mathcal{C}_{aibj} = -\frac{1}{2}\sum_k F_{ab}(\nabla_i\nabla_j Y) +$$

$$+\frac{1}{2}\gamma_{ij}\sum_k r\nabla_e r\left(\nabla_a F^e{}_b + \nabla_b F^e{}_a - \nabla^e F_{ab}\right) Y -$$

$$-\frac{1}{2}\gamma_{ij}\sum_k \left[\frac{\Lambda r^2}{3}F_{ab} + \frac{\Lambda}{6}\sigma_{ab}F + r\,\nabla_a\nabla_b\left(\frac{F}{r}\right)\right] Y\,, \tag{54}$$

$$\delta\mathcal{C}_{aijk} = \sum_k \left[r^2\nabla_a\left(\frac{F}{r^2}\right) - r\nabla^b r F_{ab}\right]\gamma_{i[j}\nabla_{k]}Y\,, \tag{55}$$

$$\delta\mathcal{C}_{ijkl} = \sum_k F\left(\gamma_{i[l}\nabla_{k]}\nabla_j Y - \gamma_{j[l}\nabla_{k]}\nabla_{i=}Y\right) +$$

$$+2\gamma_{i[k}\gamma_{l]j}\sum_k \left[r^2\nabla_a r\nabla_b r F^{ab} + \left(\kappa - \frac{\Lambda r^2}{3}\right)F\right] Y -$$

$$-2\gamma_{i[k}\gamma_{l]j}\sum_k r\nabla_a r\nabla^a F\,Y\,. \tag{56}$$

Expressed in this way, the perturbed bulk Weyl tensor is not obviously traceless. In fact, as follows from (50), $\delta\mathcal{C} = \delta\mathcal{R}$, and $\delta\mathcal{C} = 0$ only if the perturbed equations of motion in the bulk are taken into account.

It was shown by Mukohyama [21] that the gauge invariants F_{ab} and F, satisfying the perturbed bulk equations of motion, can be expressed through a scalar master variable Ω:

$$rF_{ab} = \nabla_a\nabla_b\,\Omega - \frac{2}{3}\nabla^2\,\Omega\,\sigma_{ab} - \frac{\Lambda}{18}\,\Omega\,\sigma_{ab}\,, \tag{57}$$

$$F = \frac{r}{3}\left(\nabla^2\,\Omega + \frac{\Lambda}{3}\,\Omega\right)\,. \tag{58}$$

The variable Ω satisfies the master equation

$$\nabla^2\,\Omega - \frac{3}{r}\,\nabla_a r\nabla^a\Omega - \left(\frac{k^2-3\kappa}{r^2} + \frac{\Lambda}{6}\right)\Omega + \frac{U}{r^2} = 0 \tag{59}$$

with some function U, which is, in general case, a solution of the equation

$$\nabla_a\nabla_b\,U + \frac{\Lambda}{6}\sigma_{ab}\,U = 0\,. \tag{60}$$

One can verify that the trace $\delta\mathcal{C}$ of the perturbed bulk Weyl tensor $\delta\mathcal{C}_{ABCD}$, defined by (51)–(56), can be expressed through the Mukohyama master variable Ω as

$$\delta\mathcal{C} = -\frac{1}{3r^3}\sum_k\left[\nabla^2\left(r^2\Sigma\right) + \frac{\Lambda r^2}{3}\,\Sigma\right] Y\,, \tag{61}$$

where

$$\Sigma \equiv \nabla^2\,\Omega - \frac{3}{r}\,\nabla_a r\nabla^a\Omega - \left(\frac{k^2-3\kappa}{r^2} + \frac{\Lambda}{6}\right)\Omega\,. \tag{62}$$

Obviously, the Mukohyama master equation (59) implies the condition $\delta\mathcal{C} = 0$.

4.2. Perturbations on the flat background bulk geometry

The general problem of solving the Mukohyama master equation and the further projection of the bulk Weyl tensor to the brane is greatly simplified if the background bulk geometry is simply a Minkowski spacetime. Considering a spatially closed brane ($\kappa = 1$) in the theory with $\Lambda = 0$, we have $\sigma_{ab} = \eta_{ab}$. In this case, the Mukohyama master equation (59) takes the form

$$-\partial_\tau^2\Omega + \partial_r^2\Omega - \frac{3}{r}\partial_r\Omega - \frac{(n^2+2n-3)}{r^2}\Omega = 0\,, \tag{63}$$

where we have used the discrete Laplacian eigenvalues on the three-sphere: $k_n^2 = n(n+2)$, $n = 0, 1, 2\ldots$ We have also restricted ourselves to the case $n \ge 2$, for which the function U from (59) can be set to zero [21].

Equation (63) is a partial differential equation of the hyperbolic type. Its simple form allows one to separate variables: $\Omega(\tau, r) = \xi(\tau)\,\chi(r)$ with the functions $\xi(\tau)$ and $\chi(r)$ satisfying the ordinary differential equations

$$\frac{d^2\xi(\tau)}{d\tau^2} + W\xi(\tau) = 0\,, \tag{64}$$

$$\frac{d^2\chi(r)}{dr^2} - \frac{3}{r}\frac{d\chi(r)}{dr} + \left[W - \frac{(n^2+2n-3)}{r^2}\right]\chi(r) = 0\,, \tag{65}$$

where W is some constant, which can be chosen arbitrary until some boundary or regulatory conditions are specified.

Using expressions (64) and (65) and definitions (57) and (58), one can easily compute the components of the perturbed bulk Weyl tensor (51)–(56). Once this operation is done, the projection $\delta C_{\mu\nu} = P^A{}_\mu P^C{}_\nu n^B n^D \delta C_{ABCD}$ of the bulk Weyl tensor to the brane can be easily calculated.[5] Setting the brane trajectory to be $r = a(\tau)$, we get the components of $\delta C_{\mu\nu}$ [see Definition (13)] as

$$\frac{\delta\rho_{\mathcal{C}}}{m^2} = -\frac{n(n+2)(n^2+2n-3)}{3a^5}\,\Omega_{\rm b}\,, \tag{66}$$

$$\frac{v_{\mathcal{C}}}{m^2} = \frac{(n^2+2n-3)}{3a^3}\Big[aH\left(\partial_r\Omega\right)_{\rm b} - H\Omega_{\rm b} + + \sqrt{1+a^2H^2}\left(\partial_\tau\Omega\right)_{\rm b}\Big]\,, \tag{67}$$

$$\frac{\delta\pi_{\mathcal{C}}}{m^2} = -\frac{\left(1+2a^2H^2\right)}{2a}\left(\partial^2_\tau\Omega\right)_{\rm b} - \frac{\left(1+3a^2H^2\right)}{2a^2}\left(\partial_r\Omega\right)_{\rm b} - -H\sqrt{1+a^2H^2}\left(\partial^2_{\tau r}\Omega\right)_{\rm b} - -\frac{(n^2+2n-3)\left(1+3a^2H^2\right)}{6a^3}\,\Omega_{\rm b}\,. \tag{68}$$

Here, $a = a(t)$ is a scale factor of the background Friedmann–Robertson–Walker metric on the brane [the same as in (9)], $H = \dot a/a$ is the Hubble parameter on the brane, and the function $\tau = \tau(t)$ is defined by the differential equation $d\tau/dt = \sqrt{1+a^2H^2}$. The subscript $\{\}_{\rm b}$ means that the value of the corresponding quantity is taken at the brane. For example, $\Omega_{\rm b}(t) \equiv \Omega(\tau(t), a(t))$.

Using the rule of differentiation

$$\dot\Omega_{\rm b} = \sqrt{1+a^2H^2}\left(\partial_\tau\Omega\right)_{\rm b} + aH\left(\partial_r\Omega\right)_{\rm b}\,, \tag{69}$$

one can rewrite (67) and (68) in the following form:

$$\frac{v_{\mathcal{C}}}{m^2} = \frac{(n^2+2n-3)}{3a^3}\left[\dot\Omega_{\rm b} - H\Omega_{\rm b}\right]\,, \tag{70}$$

$$\frac{\delta\pi_{\mathcal{C}}}{m^2} = -\frac{1}{2a}\left[\ddot\Omega_{\rm b} - \frac{a^2H(\dot H + H^2)}{(1+a^2H^2)}\,\dot\Omega_{\rm b} + + \frac{(1-a^2\dot H)}{a(1+a^2H^2)}\left(\partial_r\Omega\right)_{\rm b} + \frac{(n^2+2n-3)}{3a^2}\,\Omega_{\rm b}\right]. \tag{71}$$

[5] The perturbations δn^A of the unit vector n^A normal to the brane do not contribute to $\delta C_{\mu\nu}$ because the Weyl tensor $\mathcal{C}_{ABCD}$ vanishes for the background solution.

We observe that the function $v_{\mathcal{C}}(t)$ can be related to the function $\delta\rho_{\mathcal{C}}(t)$ defined in (66) as

$$v_{\mathcal{C}} = -\frac{a^2}{n(n+2)}\left(\delta\dot\rho_{\mathcal{C}} + 4H\delta\rho_{\mathcal{C}}\right)\,, \tag{72}$$

which is in accordance with Eq. (32) obtained as one of the conservation equations on the brane. The relation between the functions $\delta\pi_{\mathcal{C}}(t)$ and $\delta\rho_{\mathcal{C}}(t)$ is not so trivial due to the presence of the third term in the square brackets on the right-hand side of (71). To establish the relation between $\delta\pi_{\mathcal{C}}(t)$ and $\delta\rho_{\mathcal{C}}(t)$, one should find the general solution of the master equation (63). This can be easily done. As one can see from (64), the master variable Ω demonstrates an oscillatory or exponential behavior depending on the sign of the constant W. In what follows, we consider these two cases separately.

4.3. Oscillatory behavior

Setting $W \equiv \omega^2 > 0$, we get a solution of (65) for a given ω in the form

$$\chi(r) = r^2\left[A_\omega J_{n+1}(\omega r) + B_\omega Y_{n+1}(\omega r)\right]\,, \tag{73}$$

where A_ω and B_ω are some constants that can be chosen arbitrary until the boundary conditions are specified, and $J_{n+1}(\omega r)$ and $Y_{n+1}(\omega r)$ are the Bessel and Neumann functions, respectively.

The asymptotic behavior of the function $\chi(r)$ in the neighborhood of the point $r = 0$ is determined in the leading order by the asymptotics of Neumann functions:

$$\chi(r) \to -\frac{2^{n+1}n!\,B_\omega}{\pi\omega^{n+1}}\frac{1}{r^{n-1}}\,, \quad r \to 0\,. \tag{74}$$

The requirement of regularity of the solution at $r = 0$ leads to the condition $B_\omega = 0$ for all modes with $n \geq 2$.

The general solution of the master equation (63) can be written in the form of an integral over all possible values of the parameter ω:

$$\Omega(\tau, r) = r^2\int\limits_{-\infty}^{\infty} d\omega\,\Omega(\omega)\,J_{n+1}(\omega r)\,e^{i\omega\tau}\,, \tag{75}$$

where $\Omega(\omega)$ is some complex function which is expected to be specified from the boundary equations on the brane. We would like to note that the same result can be obtained by applying the method of Fourier transformation to Eq. (63).

4.4. Exponential growth

In this case, we set $W \equiv -\omega^2 < 0$ and obtain a solution of (65) for a given ω in the form

$$\chi(r) = r^2 \left[A_\omega I_{n+1}(\omega r) + B_\omega K_{n+1}(\omega r)\right] , \tag{76}$$

where A_ω and B_ω are again some constants, and $I_{n+1}(\omega r)$ and $K_{n+1}(\omega r)$ are the modified Bessel functions of the first and second kinds, respectively.

The absence of singularities at the point $r = 0$ yields $B_\omega = 0$ for all modes with $n \geq 2$. The general solution of the master equation (63) can be written in this case as:

$$\Omega(\tau, r) = r^2 \int\limits_0^\infty d\omega \; I_{n+1}(\omega r) \left[\alpha(\omega) e^{\omega\tau} + \beta(\omega) e^{-\omega\tau}\right] . \tag{77}$$

4.5. Spatially flat brane geometry

Our solution of the Mukohyama master equation in the bulk, derived for the case $\kappa = 1$ under the assumption of the regularity at $r = 0$, tends to a solution for the spatially flat universe ($\kappa = 0$) in the limit of large values of the scale factor or, more specifically, $H^2, |\dot{H}| \gg 1/a^2$. In this approximation, our general expressions (66) and (71) read

$$\frac{\delta\rho_{\mathcal{C}}}{m^2} = -\,\frac{n(n+2)(n^2+2n-3)}{3a^5}\,\Omega_{\rm b}\,, \tag{78}$$

$$\frac{\delta\pi_{\mathcal{C}}}{m^2} \approx -\,\frac{1}{2a}\left[\ddot{\Omega}_{\rm b} - \left(H + \frac{\dot{H}}{H}\right)\dot{\Omega}_{\rm b} \; - \right.$$

$$\left. -\;\frac{\dot{H}}{aH^2}\left(\partial_r\Omega\right)_{\rm b} + \frac{(n^2+2n-3)}{3a^2}\,\Omega_{\rm b}\right] , \tag{79}$$

while we can still use expression (75) or (77) for the master variable.

5. Closed System of Equations on the Brane

Solutions of the perturbation equations in the bulk should be used jointly with the system of equations describing the cosmological perturbations on the brane. In this section, we use the general solution for the master variable obtained in Sec. 4 to derive a closed system of equations on the brane in various particular cases. For definiteness, to demonstrate how this closed system of equations can be constructed and solved, we use expression (75) for the master variable, which corresponds to the oscillatory behavior of this quantity in the fifth dimension.

5.1. The special case of ideal pressureless matter

Perturbations in a pressureless matter without anisotropic stress ($p = 0$, $\zeta = 0$) are described by the system of equations (34)–(36). Eliminating the variable $v_{\mathcal{C}}$ from this system, we obtain

$$\ddot{\Delta} + 2H\dot{\Delta} = \left(1 + \frac{6\gamma}{\beta}\right)\frac{\rho\Delta}{2m^2} + (1+3\gamma)\frac{\delta\rho_{\mathcal{C}}}{m^2\beta}\,, \tag{80}$$

$$\delta\ddot{\rho_{\mathcal{C}}} + (10 - 3\gamma)\,H\delta\dot{\rho_{\mathcal{C}}} \; +$$

$$+ \left[4\dot{H} + 24H^2 - 12H^2\gamma + \frac{n(n+2)\gamma}{a^2}\right]\delta\rho_{\mathcal{C}} \; -$$

$$-\,\frac{4n(n+2)(n^2+2n-3)}{3(1+3\gamma)a^4}\,\delta\pi_{\mathcal{C}} \; =$$

$$= -\,\frac{n(n+2)}{a^2}\left(\gamma - \frac{1}{3}\right)\rho\Delta\,. \tag{81}$$

Using Eqs. (66), (68), and (75), we can rewrite (81) in the form

$$\frac{1}{m^2}\,\Delta(t) = \int\limits_{-\infty}^{\infty} d\omega\,\Omega(\omega)\,D(\omega, t)\,, \tag{82}$$

where $D(\omega, t)$ is a somewhat complicated function, which is well defined if the background cosmological evolution is known:

$$D(\omega, t) \equiv \frac{\left(n^2+2n-3\right)}{(3\gamma-1)\,\rho a}\,d(\omega, t)\,e^{i\omega\tau(t)}\,, \tag{83}$$

$$d(\omega, t) \equiv \omega a\,J_n(\omega a)\left[\dot{H} + (4-3\gamma)\,H^2 - \frac{2\left(1+3a^2H^2\right)}{(1+3\gamma)\,a^2} \; + \right.$$

$$\left. + \; 2i\,\omega H\left(\frac{3\gamma-1}{3\gamma+1}\right)\sqrt{1+a^2H^2}\,\right] +$$

$$+J_{n+1}(\omega a)\Bigg[-n\dot{H}+n\,(n-2+3\gamma)\,H^2+\frac{n\,(n+2)\,\gamma}{a^2}-$$

$$-\frac{2n\,(n-1)\big(1+3a^2H^2\big)}{3\,(1+3\gamma)\,a^2}-\omega^2\left(\frac{3\gamma-1}{3\gamma+1}\right)\big(1+2a^2H^2\big)-$$

$$-i\,\omega H\sqrt{1+a^2H^2}\left[3\gamma+2\,(n-1)\left(\frac{3\gamma-1}{3\gamma+1}\right)\right]+$$

$$+\frac{i\,\omega a^2H(\dot{H}+H^2)}{\sqrt{1+a^2H^2}}\Bigg]\,. \qquad (84)$$

Equations (80) and (82) together with (66) and (75) form a system of *integro-differential* equations, which reflects the *nonlocality* property of the evolution of cosmological perturbations in a braneworld scenario [21].[6]

To analyze the properties of the obtained system of equations, it may be useful to apply the method of Fourier transformation. We decompose the functions $\Delta(t)$ and $D(\omega,t)$ as

$$\Delta(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}d\varpi e^{\mathrm{i}\varpi t}\Delta(\varpi)\,, \qquad (85)$$

$$D(\omega,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}d\varpi e^{\mathrm{i}\varpi t}D(\omega,\varpi)\,, \qquad (86)$$

where $\Delta(\varpi)$ and $D(\omega,\varpi)$ are the Fourier transforms of the corresponding functions with respect to the time variable. Then Eq. (82) reads

$$\frac{1}{m^2}\,\Delta(\varpi)=\int\limits_{-\infty}^{\infty}d\omega\,\Omega(\omega)\,D(\omega,\varpi)\,. \qquad (87)$$

We assume the existence of the inverse kernel $D^{-1}(\varpi,\omega)$ such that

$$\int\limits_{-\infty}^{\infty}d\varpi\,D(\omega,\varpi)D^{-1}(\varpi,\omega')=\delta(\omega-\omega')\,. \qquad (88)$$

[6] The nature of the brane–bulk interaction in a cosmological context was thoroughly investigated in [22] on the examples of some simple toy models. The nonlocal bulk effects in realistic braneworld models were under investigation in [20, 23].

In this case, Eq. (87) can be inverted to give

$$\Omega(\omega)=\frac{1}{m^2}\int\limits_{-\infty}^{\infty}d\varpi\,\Delta(\varpi)\,D^{-1}(\varpi,\omega)\,. \qquad (89)$$

Having this result in mind, one can now return to Eq. (80). After the Fourier transformation, it can be formulated as an integral equation for the Fourier transform $\Delta(\varpi)$ of the function $\Delta(t)$:

$$\int\limits_{-\infty}^{\infty}d\varpi e^{\mathrm{i}\varpi t}\Bigg[-2\pi\varpi^2\Delta(\varpi)\,+$$

$$+\int\limits_{-\infty}^{\infty}d\varpi'\Delta(\varpi')K(\varpi,\varpi')\Bigg]=0\,, \qquad (90)$$

where the kernel function $K(\varpi,\varpi')$ is specified by

$$K(\varpi,\varpi')\equiv 2\mathrm{i}\,\varpi' H(\varpi-\varpi')-\frac{1}{2m^2}R(\varpi-\varpi')+$$

$$+\frac{2\pi n(n+2)(n^2+2n-3)}{3m^2}\int\limits_{-\infty}^{\infty}d\omega D^{-1}(\varpi',\omega)\,E(\omega,\varpi)\,. \qquad (91)$$

Here, $H(\varpi)$ is the Fourier transform of the Hubble function $H(t)$, $R(\varpi)$ is the Fourier transform of the function

$$R(t)\equiv\left(1+\frac{6\gamma}{\beta}\right)\rho\,, \qquad (92)$$

and $E(\omega,\varpi)$ is the Fourier transform of

$$E(\omega,t)\equiv\left(\frac{1+3\gamma}{\beta\,a^3}\right)e^{\mathrm{i}\omega\tau}J_{n+1}(\omega a)\,. \qquad (93)$$

The last term on the right-hand side of (91) incorporates the nonlocal bulk effects in the Weyl fluid. We would like to emphasize that, regardless of the apparent complexity of expressions (83), (92), and (93), all the constituents of (90) are well-defined functions, as long as the background cosmological evolution is known.

To understand how Eq. (90) could be solved, we analyze its general-relativistic counterpart [obtained from (37)] in the case of constant energy density $\rho=\rho_0$ and, correspondingly, $H=H_0$. In this case, kernel (91) takes the simple form

$$K(\varpi,\varpi')=2\pi\left(2\mathrm{i}\varpi' H_0-\frac{\rho_0}{2m^2}\right)\delta(\varpi-\varpi')\,, \qquad (94)$$

and Eq. (90) can be written as

$$\int\limits_{-\infty}^{\infty} d\varpi e^{\mathrm{i}\varpi t}\Delta(\varpi)\,(\varpi - \mathrm{i}\varpi_+)\,(\varpi - \mathrm{i}\varpi_-) = 0\,, \qquad (95)$$

where

$$\varpi_\pm \equiv H_0 \pm \sqrt{H_0^2 + \frac{\rho_0}{2m^2}}\,. \qquad (96)$$

To satisfy (95), one should choose (formally)

$$\Delta(\varpi) = A_+\delta(\varpi - \mathrm{i}\varpi_+) + A_-\delta(\varpi - \mathrm{i}\varpi_-)\,, \qquad (97)$$

with $A_\pm$ being some arbitrary constants. Finally, we obtain

$$\Delta(t) = \frac{A_+}{2\pi}\, e^{-\varpi_+ t} + \frac{A_-}{2\pi}\, e^{-\varpi_- t}\,, \qquad (98)$$

which can be easily verified to be the general solution of (37).

Although the general expression (91) seems to be much more complicated than (94), we expect Eq. (90) to be solved in the same manner. In any case, Eq. (90) can be regarded as a well-defined *closed* equation describing the cosmological evolution of pressureless matter perturbations on the brane, thus confirming the general consideration of [24] about the wellposedness of scalar cosmological perturbations on the brane.

5.2. Quasistatic approximation

The quasistatic approximation was proposed by Koyama and Maartens in [14] as some reasonable simplification of the general equations describing the structure formation problem in the braneworld molel. The main assumption of this approximation is that the terms with time derivatives can be neglected relative to those with spatial gradients. Specifically, this assumption implies $H\dot{\Omega}_{\rm b}\,,\, \ddot{\Omega}_{\rm b} \ll (n^2/a^2)\,\Omega_{\rm b}$, where the values of n should be taken sufficiently large ($n \gg 1$). In this case, our general expressions (66), (71) turn into the approximate ones:

$$\frac{1}{m^2}\delta\rho_{\mathcal{C}}^{(qs)} \approx -\,\frac{n^4}{3a^5}\,\Omega_{\rm b}\,, \qquad (99)$$

$$\frac{1}{m^2}\delta\pi_{\mathcal{C}}^{(qs)} \approx -\frac{1}{6a}\left[\frac{n^2}{a^2}\,\Omega_{\rm b} + \frac{3(1-a^2\dot{H})}{a(1+a^2H^2)}\,(\partial_r\Omega)_{\rm b}\right]. \qquad (100)$$

This result *exactly* coincides with that presented in [14] in the limit $H^2, \dot{H} \gg 1/a^2$ corresponding to the spatially flat brane geometry. The regulatory conditions imposed in [14] for the master variable in the bulk allowed the authors of that work to neglect the term proportional to $(\partial_r\Omega)_{\rm b}$ on the right-hand side of (100), thus deriving the relation between the functions $\delta\rho_{\mathcal{C}}(t)$ and $\delta\pi_{\mathcal{C}}(t)$ in the form

$$\delta\rho_{\mathcal{C}}^{(qs)} \approx \frac{2n^2}{a^2}\delta\pi_{\mathcal{C}}^{(qs)}\,, \qquad (101)$$

which completely closes equations on the brane in the quasistatic approximation.

6. Summary and Open Issues

In this article, first of all, we have derived the full set of differential equations (26)–(33) describing the dynamics of scalar cosmological perturbations on the brane in a general case where the induced gravity as well as cosmological constants in the bulk and on the brane are included in the action of the model. This set of equations is greatly simplified if we are interested only in the evolution of perturbations of the ideal pressureless matter without anisotropic stress on the background which does not include the contribution from dark radiation (34)–(36). From a brane observer viewpoint, the evolution of the metric and matter perturbations is affected by an additional invisible component – the perturbation of the projected Weyl tensor, or dark radiation – with its own nontrivial dynamics.

The system of equations on the brane is not closed because the evolution equation for the anisotropic stress of the Weyl fluid is missing. To remedy this situation, in our previous paper on this topic [19], in which we considered only the spatially flat case, a certain constraint on the projected Weyl tensor was imposed directly on the brane. This approach was motivated by an approximate relation of the form (101) on the brane, first derived in [14] in the quasistatic approximation in the case of flat spatial geometry. In the present paper, we took into consideration the full system of perturbed equations in the bulk. This was done for a spatially closed brane bounding a flat Minkowski bulk, using the Mukohyama master variable and the corresponding master equation for this variable. The only condition imposed on the Mukohyama master variable is the regularity condition in the bulk. The components of the projected Weyl tensor – the energy density and anisotropic stress of dark radiation – are expressed via the Mukohyama master variable in (66) and (71). At the same time, the Mukohyama master variable itself can be expressed through the Fourier (75) or Laplace (77) integrals with the kernels playing the role of unknown variables, which can be determined, for example, from Eqs. (34)–(36). From the bulk viewpoint, the equations on the brane are boundary conditions restricting the solutions of the master variable. In Sec. 5 5.1, we have shown that a problem formulated in this way is well defined and can be (at least for the case of pressureless matter) reduced to the single integral equation (90).

Using a closed system of equations describing the evolution of scalar cosmological perturbations on the brane and in the bulk, one can act in two possible ways. One can develop a numerical method for computation similar to those employed in the analysis of perturbations in the RS [12] and DGP [13] braneworld models. One can also develop analytic approximate methods similar to the quasistatic approximation [14, 15]. Both are the subject of the further investigation.

This paper was supported in part by the Cosmomicrophysics section of the Programme of the Space Research of the National Academy of Sciences of Ukraine and by the Joint Ukrainian-Russian SFFR-RFBR project F40.2/108.

APPENDICES

A. Scalar Perturbation of the Induced Brane Equations

The background and perturbed components of the Einstein tensor for metric (11) read

$$
\begin{aligned}
&G^0{}_0 = -\frac{3}{a^2}\left(\mathcal{H}^2+\kappa\right), \quad G^0{}_i = 0, \\
&G^i{}_j = -\frac{1}{a^2}\left(2\mathcal{H}'+\mathcal{H}^2+\kappa\right), \\
&\delta G^0{}_0 = \frac{2}{a^2}\left[3\mathcal{H}\left(\Psi'+\mathcal{H}\Phi\right)-\left(\nabla^2+3\kappa\right)\Psi\right], \\
&\delta G^0{}_i = -\frac{2}{a^2}\nabla_i\left(\Psi'+\mathcal{H}\Phi\right), \\
&\delta G^i{}_j = \frac{2}{a^2}\left[\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+\mathcal{H}\left(\Phi'+2\Psi'\right)+\right. \\
&\left.+\Psi''-\kappa\Psi\right]\delta^i{}_j - \frac{1}{a^2}\left[\nabla^i\nabla_j-\delta^i{}_j\nabla^2\right]\left(\Phi-\Psi\right),
\end{aligned}
\tag{A1}
$$

where $\mathcal{H} \equiv a'/a$, and the prime denotes the derivative with respect to the conformal time η.

For convenience, we rewrite Eq. (3) as follows:

$$
\frac{1}{m^2}T^\mu{}_\nu + C^\mu{}_\nu = \frac{1}{M^6}Q^\mu{}_\nu - \frac{1+b}{m^2}E^\mu{}_\nu - \Lambda_{\rm RS}h^\mu{}_\nu\,. \tag{A2}
$$

First, we calculate the components of $E^\mu{}_\nu/m^2$:

$$
\begin{aligned}
&\frac{E^0{}_0}{m^2} = \frac{\rho}{m^2} - \frac{3}{a^2}\left(\mathcal{H}^2+\kappa\right) + \\
&+\frac{\delta\rho}{m^2} + \frac{6\mathcal{H}}{a^2}\left(\Psi'+\mathcal{H}\Phi\right) - \frac{2}{a^2}\left(\nabla^2+3\kappa\right)\Psi \\
&\frac{E^0{}_i}{m^2} = \nabla_i\left[\frac{v}{am^2}-\frac{2}{a^2}\left(\Psi'+\mathcal{H}\Phi\right)\right], \\
&E^i{}_0 = -\gamma^{ij}E^0{}_j, \\
&\frac{E^i{}_j}{m^2} = -\left[\frac{p}{m^2}+\frac{1}{a^2}\left(2\mathcal{H}'+\mathcal{H}^2+\kappa\right)\right]\delta^i{}_j - \frac{\delta p}{m^2}\delta^i{}_j + \\
&+\frac{2}{a^2}\left[\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+\mathcal{H}\left(\Phi'+2\Psi'\right)+\Psi''-\kappa\Psi\right]\delta^i{}_j - \\
&-\frac{1}{a^2}\left[\nabla^i\nabla_j-\frac{1}{3}\delta^i{}_j\nabla^2\right]\left(\frac{\zeta}{m^2}+D\right)+\frac{2}{3a^2}\delta^i{}_j\nabla^2 D \equiv \\
&\equiv \frac{1}{3m^2}\tilde{E}\delta^i{}_j - \frac{1}{a^2}\left[\nabla^i\nabla_j-\frac{1}{3}\delta^i{}_j\nabla^2\right]\left(\frac{\zeta}{m^2}+D\right),
\end{aligned}
\tag{A3}
$$

where the trace of $E^i{}_j$ is denoted by $\tilde{E}$, and $D \equiv \Phi - \Psi$.

It remains to calculate the component of tensor (6). Having in mind that $E = E^0{}_0 + \tilde{E}$ and neglecting terms quadratic in perturbations, we get

$$
\begin{aligned}
&Q^0{}_0 = -\frac{1}{3}\left(E^0{}_0\right)^2, \\
&Q^0{}_i = -\frac{2}{3}E^0{}_0E^0{}_i\,, \quad Q^i{}_0 = -\frac{2}{3}E^0{}_0E^i{}_0\,, \\
&Q^i{}_j = \frac{1}{3}E^0{}_0\left(E^0{}_0-\frac{2}{3}\tilde{E}\right)\delta^i{}_j + \\
&+\frac{m^2\left(\tilde{E}-E^0{}_0\right)}{3a^2}\left[\nabla^i\nabla_j-\frac{1}{3}\delta^i{}_j\nabla^2\right]\left(\frac{\zeta}{m^2}+D\right).
\end{aligned}
\tag{A4}
$$

Collecting all terms in the perturbation of (A2), we finally obtain the required system of equations in the linear approximation (14)–(17).

B. Background Cosmological Solution in the Bulk

For the background bulk metric (38), the following relations are satisfied:

$$
\begin{aligned}
&\mathcal{R}_{abcd} = R_{abcd}\,, \quad \mathcal{R}_{aibj} = -\gamma_{ij}\,r\nabla_a\nabla_b\,r\,, \\
&\mathcal{R}_{ijkl} = r^2\left[\kappa-\left(\nabla_a r\right)^2\right]\left(\gamma_{ik}\gamma_{jl}-\gamma_{il}\gamma_{jk}\right),
\end{aligned}
\tag{B1}
$$

where the curvature tensor R_{abcd} and covariant derivative ∇_a correspond to the two-dimensional metric tensor σ_{ab} defined by

$$
ds^2_{(2)} = \sigma_{ab}dx^a dx^b = -f(r)d\tau^2 + \frac{dr^2}{f(r)}\,. \tag{B2}
$$

For the case under consideration, $f(r) = \kappa - \Lambda r^2/6$, we have

$$
\begin{aligned}
&\nabla_a\nabla_b\,r = -\frac{\Lambda r}{6}\sigma_{ab}\,, \quad \left(\nabla_a r\right)^2 = \kappa - \frac{\Lambda}{6}r^2\,, \\
&R_{abcd} = \frac{\Lambda}{6}\left(\sigma_{ac}\sigma_{bd}-\sigma_{ad}\sigma_{bc}\right).
\end{aligned}
\tag{B3}
$$

C. Scalar Perturbation of the Bulk Metric

In the gauge $h_L = 0$ and $h_a = 0$, the general expression for the perturbed bulk metric (40), (41) can be written in the form

$$
ds^2_{bulk} = \left[\sigma_{ab}+\sum_k h_{ab}Y\right]dx^a dx^b + \left[r^2+\sum_k h_Y Y\right]\gamma_{ij}dx^i dx^j\,, \tag{C1}
$$

where r, h_{ab}, and h_Y depend on x^a, while Y depends on x^i.

Using (C1), one can find the following expressions for the Christoffel symbols $\Upsilon^A{}_{BC}$ of the five-dimensional geometry:

$$\Upsilon^a{}_{bc} = \Gamma^a{}_{bc} + \frac{1}{2}\sum_k \left(\nabla_b h^a{}_c + \nabla_c h^a{}_b - \nabla^a h_{bc}\right) Y\,,$$

$$\Upsilon^a{}_{bi} = \frac{1}{2}\sum_k h^a{}_b \nabla_i Y\,,$$

$$\Upsilon^a{}_{ij} = -\frac{1}{2}\gamma_{ij}\nabla^a r^2 + \frac{1}{2}\gamma_{ij}\left[\left(\nabla^b r^2\right)\sum_k h^a{}_b Y - \sum_k \nabla^a h_Y Y\right],$$

$$\Upsilon^i{}_{ab} = -\frac{1}{2r^2}\sum_k h_{ab}\nabla^i Y\,,$$

$$\Upsilon^i{}_{aj} = \delta^i{}_j \nabla_a \ln r + \frac{1}{2r^2}\delta^i{}_j\left[-2(\nabla_a \ln r)\sum_k h_Y Y + \sum_k \nabla_a h_Y Y\right],$$

$$\Upsilon^i{}_{jk} = \gamma^i{}_{jk} + \frac{1}{2r^2}\sum_k h_Y\left(\delta^i{}_j\nabla_k Y + \delta^i{}_k \nabla_j Y - \gamma_{jk}\nabla^i Y\right)\,, \quad \text{(C2)}$$

where we use the background metrics σ_{ab} and γ_{ij} for raising and lowering the corresponding indices.

The perturbation of the five-dimensional Riemann curvature tensor $\mathcal{R}_{ABCD}$ in this case is found to be

$$\delta\mathcal{R}_{abcd} = \sum_k \left(h^e{}_{[b}R_{a]ecd} + \nabla_c\nabla_{[b}h_{a]d} - \nabla_d\nabla_{[b}h_{a]c}\right) Y\,,$$

$$\delta\mathcal{R}_{iabc} = \sum_k \left(\nabla_{[c}h_{b]a} - \frac{1}{r}h_{a[b}\nabla_{c]}r\right)\nabla_i Y\,,$$

$$\delta\mathcal{R}_{abij} = 0\,,$$

$$\delta\mathcal{R}_{aibj} = -\frac{1}{2}\sum_k h_{ab}\nabla_i\nabla_j Y +$$

$$+\frac{1}{2}\gamma_{ij}\sum_k\left[r(\nabla_e r)\left(\nabla_a h^e{}_b \nabla_b h^e{}_a - \nabla^e h_{ab}\right) -\right.$$

$$\left. - \frac{\nabla_a\nabla_b r}{r}h_Y - r\,\nabla_a\nabla_b\left(\frac{h_Y}{r}\right)\right] Y\,,$$

$$\delta\mathcal{R}_{aijk} = \sum_k\left[r^2\nabla_a\left(\frac{h_Y}{r^2}\right) - r(\nabla^b r)h_{ab}\right]\gamma_{i[j}\nabla_{k]}Y\,,$$

$$\delta\mathcal{R}_{ijkl} = \sum_k h_Y\left(\gamma_{i[l}\nabla_{k]}\nabla_j Y - \gamma_{j[l}\nabla_{k]}\nabla_{i=}Y\right) +$$

$$+2\gamma_{i[k}\gamma_{l]j}\sum_k\left[r^2(\nabla_a r)(\nabla_b r)h^{ab} +\right.$$

$$\left. + \kappa h_Y - r(\nabla_a r)(\nabla^a h_Y)\right] Y\,, \quad \text{(C3)}$$

where R_{abcd} is the background curvature tensor, defined in (B3).

1. L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), [arXiv:hep-ph/9905221]; L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999), [arXiv:hep-th/9906064].
2. M.K. Mak and T. Harko, Phys. Rev. D **70**, 024010 (2004) [arXiv:gr-qc/0404104]; S. Pal, S. Bharadwaj, and S. Kar, Phys. Lett. B **609**, 194 (2005) [arXiv:gr-qc/040902]; S. Pal and S. Kar, [arXiv:0707.0223 [gr-qc]]; T. Harko and K.S. Cheng, Astrophys. J. **636**, 8 (2006) [arXiv:astro-ph/0509576]; C. G. Böhmer and T. Harko, Class. Quantum Grav. **24**, 3191 (2007) [arXiv:0705.2496 [gr-qc]]; T. Harko and K.S. Cheng, Phys. Rev. D **76**, 044013 (2007) [arXiv:0707.1128 [gr-qc]].
3. G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000) [arXiv:hep-th/0005016].
4. G. Dvali and G. Gabadadze, Phys. Rev. D **63**, 065007 (2001) [arXiv:hep-th/0008054].
5. C. Charmousis, R. Gregory, N. Kaloper, and A. Padilla, J. High Energy Phys. **10**, 066 (2006) [arXiv:hep-th/0604086]; D. Gorbunov, K. Koyama, and S. Sibiryakov, Phys. Rev. D **73**, 044016 (2006) [arXiv:hep-th/0512097]; K. Koyama, Class. Quant. Grav. **24**, 231 (2007) [arXiv:0709.2399 [hep-th]].
6. Yu. Shtanov, *On Brane-World Cosmology*, arXiv:hep-th/0005193.
7. Yu. Shtanov, Phys. Lett. B **541**, 177 (2002) [arXiv:hep-ph/0108153].
8. V. Sahni and Yu. Shtanov, J. Cosmol. Astropart. Phys. **0311**, 014 (2003) [arXiv:astro-ph/0202346].
9. V. Sahni and Yu. Shtanov, Phys. Rev. D **71**, 084018 (2005) [arXiv:astro-ph/0410221].
10. V. Sahni, Yu. Shtanov, and A. Viznyuk, J. Cosmol. Astropart. Phys. **0512**, 005 (2005) [arXiv:astro-ph/0505004].
11. A. Viznyuk and Yu. Shtanov, Phys. Rev. D **76**, 064009 (2007) [arXiv:0706.0649 [gr-qc]].
12. A. Cardoso, T. Hiramatsu, K. Koyama, and S.S. Seahra, J. Cosmol. Astropart. Phys. **07**, 008 (2007) [arXiv:0705.1685 [astro-ph]].
13. I. Sawicki, Y.-S. Song, and W. Hu, Phys. Rev. D **75**, 064002 (2007) [arXiv:astro-ph/0606285]; A. Cardoso, K. Koyama, S.S. Seahra, and F.P. Silva, Phys. Rev. D **77**, 083512 (2008) [arXiv:0711.2563 [astro-ph]].
14. K. Koyama, and R. Maartens, J. Cosmol. Astropart. Phys. **01**, 016 (2006) [arXiv:astro-ph/0511634].
15. K. Koyama, J. Cosmol. Astropart. Phys. **03**, 017 (2006) [arXiv:astro-ph/0601220].
16. R. Maartens and K. Koyama, Living Rev. Rel. **13**, 5 (2010) [arXiv:1004.3962 [hep-th]].
17. T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000) [arXiv:gr-qc/9910076].
18. H. Collins and B. Holdom, Phys. Rev. D **62**, 105009 (2000) [arXiv:hep-ph/0003173]; C. Deffayet, Phys. Lett. B **502**, 199 (2001) [arXiv:hep-th/0010186].
19. Yu. Shtanov, V. Sahni, and A. Viznyuk, Class. Quantum Grav. **24**, 6159 (2007) [arXiv:astro-ph/0701015].
20. S. Mukohyama, Phys. Rev. D **64**, 064006 (2001) [arXiv:hep-th/0104185].
21. S. Mukohyama, Phys. Rev. D **62**, 084015 (2000) [arXiv:hep-th/0004067].
22. P. Binetruy, M. Bucher, and C. Carvalho, Phys. Rev. D **70**, 043509 (2004) [arXiv:hep-th/0403154].

23. K. Koyama, D. Langlois, R. Maartens, and D. Wands, J. Cosmol. Astropart. Phys. **0411**, 002 (2004) [arXiv:hep-th/0408222]; K. Koyama and S. Mizuno, JCAP **0607**, 013 (2006) [arXiv:gr-qc/0606056]; M. Remazeilles, Phys. Rev. D **79**, 043523 (2009) [arXiv:0807.4238 [hep-th]].
24. C. Deffayet, Phys. Rev. D **71**, 023520 (2005) [arXiv:hep-th/0409302].

Received 28.04.12

ЗБУРЕННЯ СКАЛЯРНОГО
ТИПУ В ТЕОРІЇ ГРАВІТАЦІЇ НА БРАНІ

О.В. Візнюк, Ю.В. Штанов

Р е з ю м е

Отримано повну систему диференціальних рівнянь, що описують еволюцію космологічних збурень скалярного типу в теорії гравітації на брані в загальному випадку, коли дія теорії містить як індуковану кривину, так і космологічні сталі в об'ємі і на брані. Рівняння значно спрощуються, якщо розглядати лише збурення матерії без тиску. З точки зору спостерігача на брані на динаміку збурень полів матерії впливає додаткова невидима компонента – збурення проекції тензора Вейля, або темного випромінювання, яке має суто геометричну природу. Система рівнянь на брані відіграє роль граничних умов для рівнянь, що описують динаміку збурень у п'ятивимірному об'ємі. Ці рівняння можуть бути сформульовані в термінах змінної Мукохіями. Нами розглянуто випадок просторово замкнутого Всесвіту і запропоновано умову регулярності для збурень у п'ятивимірному об'ємі. Показано, що отримана повна система інтегро-диференціальних рівнянь є добре визначеною.