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QUANTUM LOGIC UNDER SEMICLASSICAL LIMIT: INFORMATION LOSS

We consider the quantum computation efficiency from a new perspective. The efficiency is reduced to its classical counterpart by imposing the semiclassical limit. We show that this reduction is caused by the fact that any elementary quantum logic operation (gate) suffers the information loss during the transition to its classical analog. Amount of the information lost is estimated for any gate from the complete set. We demonstrate that the largest loss is obtained for non-commuting gates. This allows us to consider the non-commutativity as the quantum computational speed-up resource. Our method allows us to quantify advantages of a quantum computation as compared to the classical one by the direct analysis of the involved basic logic. The obtained results are illustrated by the application to a quantum discrete Fourier transform and Grover search algorithms.

Keywords: quantum logic, quantum algorithms, complexity.

1. Introduction

The construction of a quantum computer is an important open problem in modern physics. The interest in this endeavour is mainly due to a high efficiency of quantum algorithms such as (but not only) the Grover search and Shor's factoring, and the fact that their classical analogs are much less efficient. But why are quantum calculations much more efficient than the classical ones? The common answer to this question is as follows: the speed-up is based on the quantum parallelism and, probably, on the entanglement. However, this is only the qualitative explanation, and it is reasonable to try to explain the gap in efficiency from the basic principles of quantum and classical computations. Every computation, either it is quantum or classical, can be decomposed into a set of elementary operations. We impose the semiclassical limit and study how the complete set of quantum gates is reduced to the classical counterpart. We focus our analysis on the formal rules of the quantum and classical logics, which were first formulated by G. Birkhoff and J. von Neumann in their seminal paper [1].

In order to explore the quantum supremacy over the classical computations at the basic level, much progress has been done to date. For the reviews, see [2, 3] and references therein. There is a wide variety of papers covering different approaches to the problem, the short list of which is presented below. Some possible quantum computational structures are presented in [4]. [5] is devoted to investigations in the algebraic structure of logic within the framework of a non-commutative geometry. In [6], the so-called measurement algebras, the formalism of which is weaker than that of Hilbert spaces, were explored. A description of the orthomodular lattices via the Sasaki projection was presented in [7]. [8, 9] are devoted to the analysis of contexts, i.e., the maximal sets of commuting logic statements. Different approaches in the formal representation and formalism initiation for quantum logic (QL) have been explored. Investigations in the categorical QL were presented in [10, 11]. As for the measurement-based QL and computations, which are strongly connected to the logic representation [12], we refer to [13, 14], where the first one is devoted to the "reversible measurement" – a hypothetical operation allowing one to "look inside" the quantum computation, and the second one describes measurement-based computations on graph

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states. The theory of finite automata based on QL can be found in [15]. QL may be interpreted as a language of "pragmatically" decidable assertive formulas, thus formalizing the statements about physical quantum systems [16]. [17] is devoted to a QL representation based on the alternative set of logic operations. In [18], the author expanded the λ-calculus onto quantum computations. For some extensions of QL, we refer to [19–22]. Computational complexity in quantum and classical logic (CL) calculi are explored in [23] or others, such as [24–27]. Attempts in bridging semantic space and QL are considered in [28]. Quantum language investigation was made in [29].

Thus, we conclude that investigations in logic (especially, the quantum one) are rather actual and are tightly interconnected spanning different areas of research.

Considering the efficiency as a number of elementary logical operations necessary to execute some algorithm, it is legitimate to conclude that QL is much more efficient than CL. Such a conclusion is confirmed by the fact that the algebraic structure of QL is constructed with the help of weaker conditions than that of CL, thus allowing a wider class of operations to be processed. Obviously, one may provide computations with QL or CL. But the latter choice requires more operations to simulate the circuits from QL having no classical analog.

This present work aims to ascertain reasons for the QL being much more efficient than the CL in terms of the quantitative, rather than qualitative (which is mentioned above) explanation of its superiority. The main motivation of our research is that the existing approaches and techniques can not completely explain the efficiency gap. Till now, there is no complete theory of the classical and quantum complexity classes and of interrelations between them. We hope that the approach presented here will be helpful in this challenging problem.

Estimation of the information difference between QL and CL can be made by means of the well-known Kolmogorov complexity or quantum complexities [23–25]. Some common properties of them and their possible applications were studied in [26]. Alternatively, algorithmic entropies can be applied [27].

Despite having much in common with the Kolmogorov complexity, our method differs from it. We estimate the information loss of every elementary logic operation during *reduction* of QL to CL and then generalize it to an arbitrary calculation (for brevity, we use the term 'dequantization'). Such an approach allows us to estimate the contribution to the quantum (classical) calculation of any subspace (domain) of Hilbert (phase) space correspondingly.

[30, 31] extend QL proposed in [1]. We build upon these studies by providing dequantization of the complete set of logic operations. To do this, we use the path integral formalism together with the von Neumann and Shannon entropy definitions. It allows us to estimate the information loss of any quantum algorithm in the semclassical limit. Compared to [32], we go further and formalize the approach for any logic gate.

The interrelation between Abelian QL subalgebras and the CL algebra has been explored in [33]. In [34], some aspects of the dequantization of measurement and of entanglement, which is noted as lifting, were considered with the help of the logic entropy. Instead, we consider the dequantization of any QL statement using the von Neumann and Shannon entropy definitions. We demonstrate that the non-commuting propositions play a significant role in the QL efficiency. Compared to the results of the Gottesman–Knill theorem, we exactly demonstrate the significant efficiency contribution of non-commuting statements, which may be outside the Clifford group, while making the transition from QL to CL.

In the following, we consider any computation as some expression assessing its truth value. The method we propose estimates the amount of information loss (IL) for every elementary logical operation after its processing through the semiclassical limit, but not in the register itself. As soon as the number of elementary logical operations does not change while taking the limit, the complexity in its common interpretation does not change under the procedure.

In this work, we develop the general scheme of IL estimation for any QL proposition. The obtained results are exemplified with the dequantization and IL estimation of quantum discrete fast Fourier transform ($\mathrm{FFT_Q}$) and Grover search ($\mathrm{Gr_Q}$) algorithms.

In Sections 2 and 3, we briefly introduce the CL and QL formalisms, correspondingly. We refer those who are interested in details to origins [1, 12]. Some basics of the path integral formalism and dequantization of QL operations can be found in Section 4. After introducing all the necessary formalisms, we present the major technical details of the approach. Estimation of the information loss during the transition from QL to CL is presented in Section 5. We formulate and prove theorem, which is necessary for the application of the technique to any quantum algorithm, in Section 6. Examples of how the scheme works on $\mathrm{FFT_Q}$ and $\mathrm{Gr_Q}$ are given in Section 7. Discussion of the obtained results, their relation to other approaches and open questions are given in Section 8.

2. Classical Logic

Let Γ_S be the phase space describing a physical system S in some state λ. We assume that this state corresponds to some domain in Γ_S and that it is characterized by a characteristic function χ_λ which is defined on Γ_S. The statement "S possesses physical property λ" or "S is in the state λ" will be true or false for those domains in Γ_S, where $\chi_\lambda = 1$ or $\chi_\lambda = 0$, respectively.

Such characteristic functions may be used to define formal rules and elementary operations of CL on Γ_S. For example, they can describe the conjunction, implication, and negation in terms of the phase space subsets [1].

Conjunction $\wedge$ is defined as

$$\chi_\wedge = \chi_\lambda \wedge \chi_\mu = \chi_\lambda \chi_\mu \tag{1}$$

and describes the intersection subset.

Implication $\leq$ is defined as

$$\chi_\lambda \leq \chi_\mu \colon \chi_\lambda \wedge \chi_\mu = \chi_\lambda \tag{2}$$

and corresponds to rules of the subset inclusion; this operation initiates a statement ordering.

Negation $\neg$

$$\chi_{\neg\lambda} = 1 - \chi_\lambda \tag{3}$$

is equivalent to the transition to the complementing subset.

In addition, the operation of disjunction $\vee$ may be introduced. However, as $\vee$ can be expressed in terms of preliminary operations

$$\chi_\vee = \chi_\lambda + \chi_\mu - \chi_\lambda \chi_\mu, \tag{4}$$

it is not important for us in the following.

3. Quantum Logic

Let H_S be the Hilbert space of a physical system S. Let S be in a state $|\zeta\rangle$. Then, for any statement about some property λ of S, there exists a projective operator P_λ projecting its state onto the corresponding subspace of H_S. In other words, the statement "S possesses physical property λ" will be true, if $\mathrm{P}_\lambda|\zeta\rangle \neq 0$, and false, if $\mathrm{P}_\lambda|\zeta\rangle = 0$.

The projective operators on H_S have much in common with the classical characteristic functions on Γ_S. However, there is a significant difference: P_λ defines some subspace in H_S, while χ_λ defines some domain in Γ_S. As a result, two projective operators do not commute in general, but any two characteristic functions do.

To start with, let us define the quantum conjunction $\wedge$ for commuting projectors as

$$\mathrm{P}_\wedge|\zeta\rangle = (\mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle = \mathrm{P}_\lambda \mathrm{P}_\mu|\zeta\rangle = \mathrm{P}_\mu \mathrm{P}_\lambda|\zeta\rangle. \tag{5}$$

It describes the intersection of subspaces of *commuting* operators: any operator from one of the subspaces has no influence on the other subspace.

In the case of the conjunction of non-commuting operators, a situation is more complex. Any pair of non-commuting operators does not have a joint basis, i.e., the one consisting of the complete set of eigenfunctions for both of them. However, the conjunction should fulfill one of the requirements of the projective operator, namely, $\mathrm{P}_\wedge^2 = \mathrm{P}_\wedge$. It leaves the statement belonging to both subspaces, which are determined by P_λ and P_μ (see [8] for details). In order to do this, we use the following definition for the non-commuting conjunction:

$$\mathrm{P}_\wedge|\zeta\rangle = (\mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle = \lim_{n\to\infty} (\mathrm{P}_\lambda \mathrm{P}_\mu)^n\,|\zeta\rangle, \tag{6}$$

in full accord with [8] (Table IV).

Obviously, if $\mathrm{P}_\lambda \mathrm{P}_\mu = \mathrm{P}_\mu \mathrm{P}_\lambda$, then (6) transforms into (5).

Implication $\leq$ is defined as

$$\mathrm{P}_\lambda \leq \mathrm{P}_\mu \colon\ (\mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle = \mathrm{P}_\lambda|\zeta\rangle \quad \forall\,|\zeta\rangle. \tag{7}$$

It corresponds to the subspace inclusion. It initiates the statement ordering similarly to its classical analog. The same definition will also hold true for non-commuting projectors.

Negation $\neg$ (complementation) is defined as

$$\mathrm{P}_{\neg\lambda}|\zeta\rangle = (\mathrm{I} - \mathrm{P}_\lambda)\,|\zeta\rangle, \tag{8}$$

where I is the unit operator. This operation is equivalent to transition to the orthogonal subspace.

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Similarly to the CL case, the disjunction may be expressed in terms of the previously defined operations

$$\mathrm{P}_{\vee}|\zeta\rangle = (\mathrm{P}_\lambda + \mathrm{P}_\mu - \mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle \qquad (9)$$

and thus is not needed in the following.

4. Quantum Logic Dequantization

As it follows from Sections 2 and 3, mathematical fabrics of CL and QL statements differ. Classical expressions are built upon subsets from the phase space Γ_S, while their quantum counterparts are defined on the subspaces from the Hilbert space H_S. In order to compare CL and QL, we use the path integral formalism in the phase space as the one providing a common basis for both of them.

Let $|\zeta\rangle$ be any state in the Hilbert space H_S of the physical system S. The projective operator P_λ projects the state onto some subspace in H_S. Within the path integral formalism, it can be written as

$$\mathrm{P}_\lambda|\zeta\rangle = |\lambda\rangle\langle\lambda|\zeta\rangle = |\lambda\rangle \int \mathcal{D}y e^{iS_{\lambda\to\zeta}[y]/\hbar}, \qquad (10)$$

where the integration is made over all possible phase space trajectories. Action $S_{\lambda\to\zeta}[y]$ describes the transition amplitude $\langle\lambda|\zeta\rangle$ (that is underlined with a subscript $_{\lambda\to\zeta}$) along some fixed phase trajectory y with $y = \{y_1, y_2, y_3, p_{y_1}, p_{y_2}, p_{y_3}\}$; here y_i and p_{y_i} denote the i-th component of the coordinate and momentum, correspondingly.

The result of the projection $\mathrm{P}_\lambda|\zeta\rangle$ is nothing more but a state $|\lambda\rangle$ multiplied by the corresponding transition amplitude $\langle\lambda|\zeta\rangle$. The amplitude itself can be calculated by the integration over the phase space resulting in $\langle\lambda|\zeta\rangle$.

Such a representation of the projective operator has much in common with the symbol of operator. It interconnects P_λ (operator) defined in the Hilbert space H_S to the action (symbol of operator) defined in the phase space Γ_S.

Taking the limit $\hbar \to 0$ for the transition amplitude results in the classical action and, therefore, provides a bridge to the classical logic statements. As soon as the vector $|\lambda\rangle$ has a non-zeroth norm, $\mathrm{P}_\lambda|\zeta\rangle = 0 \Leftrightarrow \langle\lambda|\zeta\rangle = 0$. So, the situation has much in common with the characteristic functions χ_λ from the phase space Γ_S. Path integrals themselves extinct when taking the limit because of fast oscillating exponents, and only the trajectories for which the action has the extremum survive. This gives

$$\lim_{\hbar\to 0} \frac{\hbar}{i} \ln \int \mathcal{D}y e^{iS_{\lambda\to\zeta}[y]/\hbar} = \begin{cases} S_{\lambda\to\zeta}[y], & \delta S_{\lambda\to\zeta}[y] = 0, \\ 0, & \delta S_{\lambda\to\zeta}[y] \neq 0, \end{cases}$$

where δ is a variation. So, one obtains that

$$\lim_{\hbar\to 0} \frac{\hbar}{i} \ln \langle\lambda|\zeta\rangle = \chi_\lambda S_{\lambda\to\zeta}[y],$$

where

$$\chi_\lambda = \begin{cases} 1, & \delta S_{\lambda\to\zeta}[y] = 0, \\ 0, & \delta S_{\lambda\to\zeta}[y] \neq 0, \end{cases}$$

or, in the compact form,

$$P_\lambda|\zeta\rangle \overset{\hbar\to 0}{\dashrightarrow} \chi_\lambda. \qquad (11)$$

Expression (11) defines the transition from the projective operator P_λ to some characteristic function χ_λ. The notation χ_λ is used, because $|\zeta\rangle$ is any vector from H_S, and, so, there is no need in the subscript $_\zeta$. This function defines the classical action that describes the transition of S from the state with some physical property λ to a state with the property ζ. As one can see, χ_λ vanishes only for those regions in the phase space, where $\delta S_{\lambda\to\zeta}[y] \neq 0$.

Expression (11) encodes the transition of system's description from the quantum mechanical to the classical one. At the beginning, one has the Hilbert space with projectors and wavefunctions, see (10), and, at the end, one obtains the phase space with some classical trajectories fixed by the extremum of the action. The transition (we call it dequantization for brevity) is similar to the well-known semiclassical approximation, when the wavefunction is being expanded into a series in $\hbar$ up to the zeroth order.

At first, we consider QL operations for commuting projectors.

Conjunction of two commuting operators

$$(\mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle = |\lambda\rangle\langle\lambda|\mu\rangle\langle\mu|\zeta\rangle$$

after taking the limit $\hbar \to 0$ (11) transforms as

$$(\mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle \overset{\hbar\to 0}{\dashrightarrow} \chi_\lambda\chi_\mu, \qquad (12)$$

that corresponds to the classical conjunction (1).

Negation (8) can be written as

$$\mathrm{P}_{\neg\lambda}|\zeta\rangle = (\mathrm{I} - \mathrm{P}_\lambda)\,|\zeta\rangle =$$
$$= \Big(\int |\mu\rangle\langle\mu|\mathrm{d}\mu - |\lambda\rangle\langle\lambda|\Big)|\zeta\rangle,$$

thus giving the equivalent classical expression, see (3),

$$\mathrm{P}_{\neg\lambda}|\zeta\rangle \overset{\hbar\to 0}{\dashrightarrow} 1 - \chi_\lambda. \tag{13}$$

Now, we consider the conjunction of non-commuting operators. This case is more complicated because of the appearance of the commutator in expressions. Dequantization will consist of two steps: at the first one, any power of product of two non-commuting projective operators will be considered, and only then their conjunction will be dequantized.

Let P_λ, P_μ be two non-commuting projective operators such that

$$\mathrm{P}_\lambda\mathrm{P}_\mu - \mathrm{P}_\mu\mathrm{P}_\lambda = i\hbar\Pi, \tag{14}$$

where Π is Hermitian. One may argue that (14) can not describe the general case, since one may use the commutator not proportional to $\hbar$. Such generalization will be slightly discussed in the following section. Using

$$\forall\, k > 0 \quad \mathrm{P}_\lambda^k = \mathrm{P}_\lambda, \quad \mathrm{P}_\mu^k = \mathrm{P}_\mu,$$

we obtain $\forall\, n > 0$

$$(\mathrm{P}_\lambda\mathrm{P}_\mu)^n = (\mathrm{P}_\lambda\mathrm{P}_\mu)^{n-1}\,(\mathrm{P}_\mu\mathrm{P}_\lambda + i\hbar\Pi) =$$
$$= (\mathrm{P}_\lambda\mathrm{P}_\mu)^{n-1}\,(\mathrm{P}_\lambda + i\hbar\Pi) =$$
$$= \cdots = \mathrm{P}_\lambda\mathrm{P}_\mu\,(\mathrm{P}_\lambda + i\hbar\Pi)^{n-1},$$

where n is integer. From

$$\forall\, k \geq 0 \quad \begin{cases} \mathrm{P}_\lambda\,(i\hbar\Pi)^{2k} = (i\hbar\Pi)^{2k}\,\mathrm{P}_\lambda, \\ \mathrm{P}_\lambda\,(i\hbar\Pi)^{2k+1} = (i\hbar\Pi)^{2k+1}\,(\mathrm{I} - \mathrm{P}_\lambda), \end{cases}$$

we obtain then $\forall\, k \geq 0$

$$(\mathrm{P}_\lambda + i\hbar\Pi)^{2k} = \Big[\mathrm{P}_\lambda + (i\hbar\Pi)^2 + i\hbar\Pi\Big]^k =$$
$$= \sum_{s=0}^{k} \frac{k!}{(k-s)!s!}\Big[\mathrm{P}_\lambda + (i\hbar\Pi)^2\Big]^s (i\hbar\Pi)^{k-s} =$$
$$= \sum_{s=0}^{k} \frac{k!}{(k-s)!s!}\Bigg[\mathrm{P}_\lambda \sum_{l=0}^{s} \frac{s!}{(s-l)!l!}\,(i\hbar\Pi)^{2(s-l)} +$$
$$+ (\mathrm{I} - \mathrm{P}_\lambda)\,(i\hbar\Pi)^{2s}\Bigg]\,(i\hbar\Pi)^{k-s} =$$
$$= \mathrm{P}_\lambda\,\Big[\mathrm{I} + (i\hbar\Pi)^2 + i\hbar\Pi\Big]^k +$$
$$+ (\mathrm{I} - \mathrm{P}_\lambda)\,(i\hbar\Pi)^k\,(\mathrm{I} + i\hbar\Pi)^k =$$
$$= \mathrm{P}_\lambda\,(\mathrm{I} + \alpha)^k + (\mathrm{I} - \mathrm{P}_\lambda)\,\alpha^k,$$

and, finally, $\forall\, n > 0$

$$(\mathrm{P}_\lambda\mathrm{P}_\mu)^n =$$
$$= \begin{cases} \beta\,(\mathrm{I}+\alpha)^k + \mathrm{P}_\lambda i\hbar\Pi\alpha^k, & n = 2k+1, \\ \beta\Big[(\mathrm{I}+\alpha)^k + i\hbar\Pi\alpha^k\Big] + \gamma_k, & n = 2\,(k+1), \end{cases} \tag{15}$$

where

$$\alpha = i\hbar\Pi\,(\mathrm{I} + i\hbar\Pi),$$
$$\beta = \mathrm{P}_\mu\mathrm{P}_\lambda + (\mathrm{I} - \mathrm{P}_\lambda)\,i\hbar\Pi,$$
$$\gamma_k = \mathrm{P}_\lambda\,(i\hbar\Pi)^2\,(\mathrm{I} + \alpha)^k.$$

This gives

$$\forall\, n > 0 \ \lim_{\hbar\to 0}\,(\mathrm{P}_\lambda\mathrm{P}_\mu)^n = \lim_{\hbar\to 0} \mathrm{P}_\mu\mathrm{P}_\lambda, \tag{16}$$

where the limit should be interpreted in a bit formal sense, as soon as we apply it to the tensors of the non-zeroth rank. Using (11) and (16), one gets

$$\lim_{\hbar\to 0} \frac{\hbar}{i} \ln\,(\mathrm{P}_\lambda \wedge \mathrm{P}_\mu)\,|\zeta\rangle = \lim_{\hbar\to 0} \frac{\hbar}{i} \ln\,(\mathrm{P}_\mu\mathrm{P}_\lambda)\,|\zeta\rangle =$$
$$= S_{\mu\to\lambda} + S_{\lambda\to\zeta} = S_{\lambda\to\mu} + S_{\mu\to\zeta}, \tag{17}$$

for which the following variations are true:

$$\delta S_{\mu\to\lambda} = \delta S_{\lambda\to\zeta} = 0$$
$$\delta S_{\lambda\to\mu} = \delta S_{\mu\to\zeta} = 0.$$

Expression (17) determines the conjunction dequantization for the non-commuting projectors.

Implication (7) by virtue of the previous result also transforms into the classical one (2):

$$\mathrm{P}_\lambda \leq \mathrm{P}_\mu \overset{\hbar\to 0}{\dashrightarrow} \chi_\lambda \leq \chi_\mu. \tag{18}$$

5. Dequantization: Information Loss Estimation

5.1. Elementary logical statements

Suppose that S is in the pure quantum state $|\zeta\rangle$. The von Neumann entropy H_{N} of the state

$$H_{\mathrm{N}}(|\zeta\rangle) = -\mathrm{Tr}\rho\ln\rho = 0. \tag{19}$$

Here, $\rho = |\zeta\rangle\langle\zeta|$ is the density matrix of the system.

After the dequantization, the system S can be described by the corresponding characteristic function χ_λ, see (11), splitting the phase space Γ_S into two domains. As a result, S can be characterized with the Shannon entropy H_{Sh}

$$\begin{gathered} H_{\mathrm{Sh}}(\chi_\lambda) = -\phi_\lambda\ln\phi_\lambda - (1-\phi_\lambda)\ln(1-\phi_\lambda), \\ \phi_\lambda = \frac{\int\mathcal{D}x\chi_\lambda}{\int\mathcal{D}x}. \end{gathered} \tag{20}$$

In the following, the argument of H_{Sh} may be denoted with the characteristic function or the corresponding projectors with no change in the expression meaning.

After the dequantization, the entropy depends on how χ_λ splits Γ_S. It is non-zero except for two cases: $\phi_\lambda = 0$ or $\phi_\lambda = 1$. One may notice that the entropy is upper bounded, i.e.,

$$\forall\,\lambda \quad H_{\mathrm{Sh}}(\chi_\lambda) \le \ln 2. \tag{21}$$

The existence of the upper bound (21) means that some quantum states after the dequantization procedure lose all quantum correlations causing the maximal information loss possible.

Any logic statement consisting of commuting projectors is equivalent to some projector. Consequently, any pure quantum state under the statement transforms to another pure state leaving the von Neumann entropy H_{N} unchanged. However, after the statement dequantization, the entropy will change because of the re-splitting Γ_S. To show this, the entropy of dequantized logic operations should be explored.

Conjunction entropy of *commuting* projectors after taking the limit $\hbar\to 0$, see (12), is defined as

$$\begin{gathered} H_{\mathrm{Sh}}(\chi_\wedge) = -\phi_\wedge\ln\phi_\wedge - \\ -(1-\phi_\wedge)\ln(1-\phi_\wedge) \le \ln 2, \\ \phi_\wedge = \frac{\int\mathcal{D}x\chi_\wedge}{\int\mathcal{D}x}, \end{gathered} \tag{22}$$

where $\chi_\wedge = \chi_\lambda\chi_\mu$. Since $\chi_\wedge$ is nothing more but some characteristic function, we used expression (21) to define the upper bound on $H_{\mathrm{Sh}}(\chi_\wedge)$.

For the quantum negation after the dequantization (13) we obtain

$$H_{\mathrm{Sh}}(\chi_{\neg\lambda}) = H_{\mathrm{Sh}}(1-\chi_\lambda) = H_{\mathrm{Sh}}(\chi_\lambda). \tag{23}$$

Expression (23) means that, because of the symmetry of (20) the negation does not change the entropy even after the dequantization.

For the implication of commuting projectors, one gets that, according to (2) and (7), after the logic conversion (18) the entropy will have the following property:

$$\mathrm{P}_\lambda \le \mathrm{P}_\mu \Rightarrow H_{\mathrm{Sh}}(\chi_\wedge) = H_{\mathrm{Sh}}(\chi_\lambda), \tag{24}$$

where $\chi_\wedge = \chi_\lambda\chi_\mu$.

As before, in the case of non-commuting projectors, it is enough to consider the entropy of the corresponding conjunction (6) after the logic conversion (17).

Let P_λ, P_μ be two non-commuting projectors satisfying (14). The initial state $|\zeta\rangle$ of the system can be expanded into a series in the eigenstates of commutator Π:

$$|\zeta\rangle = \sum_\pi^{\dim\Pi}\zeta_\pi|\pi\rangle, \quad \Pi|\pi\rangle = \pi|\pi\rangle.$$

Terms containing non-zero powers of Π will vanish in accordance with (15) while taking the limit $\hbar\to 0$ in the conjunction (6). Thus, the density matrix ρ should be traced over the eigenstates of Π. Under this averaging, the pure state transforms into a mixture for which the von Neumann entropy is non-zero:

$$\begin{gathered} H_{\mathrm{N}}(|\zeta\rangle) \to H_{\mathrm{N}}(\rho_\Pi) = -\mathrm{Tr}\rho_\Pi\ln\rho_\Pi = \\ = -\sum_\pi^{\dim\Pi}|\zeta_\pi|^2\ln|\zeta_\pi|^2 \le \ln\dim\Pi, \end{gathered} \tag{25}$$

where $\rho_\Pi = \mathrm{Tr}_\Pi|\zeta\rangle\langle\zeta|$.

In addition, the contribution of every eigenstate $|\pi\rangle$ from the mixture ρ_Π to the whole entropy should be included. Any such term is expressed similarly to (20)

$$\begin{gathered} H_{\mathrm{Sh}}\left(\chi_{\wedge_\Pi|\pi}\right) = -\phi_{\wedge|\pi}\ln\phi_{\wedge|\pi} - \\ -\left(1-\phi_{\wedge|\pi}\right)\ln\left(1-\phi_{\wedge|\pi}\right), \end{gathered} \tag{26}$$

where $\chi_{\wedge_\Pi}$ is the characteristic function corresponding to the conjunction of our projectors and

$$\phi_{\wedge|\pi} = \frac{\int \mathcal{D}x_{|\pi}\chi_\lambda\chi_\mu}{\int \mathcal{D}x_{|\pi}}.$$

Here and in the following, the subscript $_{|\pi}$ means that the transition starting from the state $|\lambda\rangle$ or $|\mu\rangle$ results in the corresponding state $|\pi\rangle$ but not in $|\zeta\rangle$ as before, see (10).

Summarizing, the whole entropy for the dequantized conjunction of two non-commuting projectors is

$$H\left(\chi_{\wedge_\Pi}\right) = H_{\rm N}\left(\rho_\Pi\right) + \sum_\pi^{\dim\Pi} |\zeta_\pi|^2 H_{\rm Sh}\left(\chi_{\wedge_\Pi|\pi}\right). \qquad (27)$$

There is no subscript $_{\rm Sh}$ nor $_{\rm N}$ on the lhs of (27), since it is a sum of both von Neumann and Shannon entropies.

As one may notice, (27) is the total entropy for the mutual distribution over both degrees of freedom encoded with Π and transition trajectories in Γ_S corresponding to $|\pi\rangle$.

Expression (22) is easily obtained via the formal setting $\dim\Pi = 1$ in (27).

The upper bound of $H\left(\chi_{\wedge_\Pi}\right)$, see (21) and (25), is

$$H\left(\chi_{\wedge_\Pi}\right) \leq \ln\dim\Pi + \ln 2. \qquad (28)$$

Now, we can consider the case of the commutators not proportional to $\hbar$ in details, see (14) and the text right after it. To do it, we can replace $\hbar\Pi$ in (14) by some Hermitian operator C. Such an operator can be diagonalized, i.e., represented in the form $C = \sum_1^{\dim C} cP_c$, where P_c is the projector on the eigenstate of C with eigenvalue c. Now, following the dequantization, the procedure for such an operator (see (4)) will result in the re-definition of coefficients α, β, γ in (15) without any change in (16) and in (27). In other words, one again will meet with the information loss while taking the semiclassical limit $\hbar \to 0$ without any change at the end and, hence, with no additional entropy, but except (27). It is completely consistent with the fact that a language choice does not influence the complexity class of an algorithm, see [37] for details.

Implication of the non-commuting operators is similar to the analysis of commuting ones, see (24). The only difference is that the non-commuting conjunction entropy (27) should be used, i.e.,

$$\mathrm{P}_\lambda \leq \mathrm{P}_\mu \Rightarrow H\left(\chi_{\wedge_\Pi}\right) = H_{\rm Sh}\left(\chi_\lambda\right), \qquad (29)$$

where the projectors satisfy (14). However, (29) is a generalization of (24); the latter is obtained by setting $\dim\Pi = 1$ in (29), as we did it before.

The obtained results define the entropy increase for any elementary logical statement under the logic conversion. Such elementary statements are atomic and are equivalent to the one-qubit register. But, for the complete analysis of the information gap, registers of arbitrary length should be observed.

5.2. Compound logical expressions

Let $|\zeta\rangle^{\otimes N_{\rm I}}$ be an $N_{\rm I}$-qubit register. Any calculation with it is equivalent to the construction of some logical expression $\mathbb{E}_{\rm I}$ from the elementary logical operations defined on projectors. Suppose that $\mathbb{E}_{\rm I}$ has no implications inside (that's underlined with index $_{\rm I}$) and consists of $n_{\rm I}$ negations $\neg$ and $c_{\rm I}$ conjunctions $\wedge$. The expression

$$N_{\rm I} \leq n_{\rm I} + c_{\rm I}$$

must be true, since one can apply the encoding, where $N_{\rm I} - n_{\rm I} - c_{\rm I}$ qubits will be obsolete otherwise.

Conjunctions $c_{\rm I}$ are defined on the non-commuting projectors in general. Thus, one has to include all commutator contributions (14) while estimating the entropy. After neglecting the first such commutator, all subsequent elementary statements will operate on the mixture, but not on the pure state. However, as the negation does not influence the entropy, the conjunctions operating on the mixture should be observed only.

Suppose that the expression $\mathbb{E}_{{\rm I},\Pi_2\Pi_1}$ consists of two conjunctions characterized with commutators Π_1 (corresponds to the first calculated conjunction) and Π_2 (the second one). After the dequantization, the entropy of the expression will be

$$H\left(\mathbb{E}_{{\rm I},\Pi_2\Pi_1}|\zeta\rangle\right) = H\left(\chi_{\wedge_{\Pi_1}}\right) + \sum_{\pi_1}^{\dim\Pi_1} |\zeta_{\pi_1}|^2 H\left(\chi_{\wedge_{\Pi_2}|\pi_1}\right).$$

In general, for $\mathbb{E}_{\rm I}$ on the register $|\zeta\rangle^{\otimes N_{\rm I}}$, the whole entropy will be estimated by the recurrent formula

$$H\left(\mathbb{E}_{\rm I}|\zeta\rangle^{\otimes N_{\rm I}}\right) = \sum_{i=1}^{q_{\rm I}} H_{\rm Sh}\left(\chi_{\lambda_i}\right) + H\left(\chi_{\wedge_{\Pi_1}}\right) + \\ + \sum_{\pi_1}^{\dim\Pi_1} |\zeta_{\pi_1}|^2 H\left(\chi_{\wedge_{\Pi_2}|\pi_1}\right). \qquad (30)$$

Here, q_I is the number of qubits equipped in no conjunction. Using (21) and (28), one may obtain the upper bound for the entropy:

$$H\left(\mathbb{E}_\mathrm{I}|\zeta\rangle^{\otimes N_\mathrm{I}}\right) \leq (q_\mathrm{I}+c_\mathrm{I})\ln 2 + \sum_{k=1}^{c_\mathrm{I}} \ln\dim\Pi_k. \qquad (31)$$

To estimate the entropy of some general expression $\mathbb{E}$, one must count over all implications made during the calculation. This means that, for $\mathbb{E}$ containing subexpressions $\{\mathbb{E}_\mathrm{I}\}_\mathrm{I}$ on the register $|\zeta\rangle^{\otimes N}$, the total entropy $H\left(\mathbb{E}|\zeta\rangle^{\otimes N}\right)$ must consist of contributions from all the subexpressions, each of which is defined by (31).

6. Conjunction Theorem

Now, we are almost ready to verify our approach on real algorithms.

Any algorithm, in order to be computable, should consist of a finite amount of elementary gates. It implies that the algorithm should use some finite number of products of non-commuting projectors and seems to leave no space for non-commuting conjunction, see (6).

Useful quantum algorithms may contain products of non-commuting projectors. But, according to their computability, it seems that they can not utilize all the possibilities allowed within QL.

So, we have the problem:

1. How can we combine the computability with the full power of QL?
2. How can we translate that during the dequantization in the case of success?

We solve it with the help of the following theorem.

Conjunction theorem. *Let* $\mathrm{P}_\lambda, \mathrm{P}_\mu$ *be any two projective operators such that* $[\mathrm{P}_\lambda, \mathrm{P}_\mu] = i\hbar\Pi$, $\mathrm{P}_\wedge = \mathrm{P}_\lambda \wedge \mathrm{P}_\mu$. *Then*

$$\forall\, k>0, \quad H\big((\mathrm{P}_\lambda\mathrm{P}_\mu)^k\big) = H\left(\mathrm{P}_\wedge\right). \qquad (32)$$

Proof. Using (6), we can write

$$\left(\mathrm{P}_\lambda\mathrm{P}_\mu\right)^k \mathrm{P}_\wedge = \mathrm{P}_\wedge \left(\mathrm{P}_\lambda\mathrm{P}_\mu\right)^k = \mathrm{P}_\wedge. \qquad (33)$$

As it follows from (6), $\mathrm{P}_\wedge^2 = \mathrm{P}_\wedge$, i.e., it is a projective operator. This is not true for the product $\left(\mathrm{P}_\lambda\mathrm{P}_\mu\right)^k$, since P_λ and P_μ do not commute. But, such product defines some Hilbert subspace, and, therefore, (33) is the common implication for commuting operators, see (7). Then, using (24), we get finally (32) completing the proof. □

The theorem means that, from the CL point of view, any non-commuting conjunction behaves itself in the same way as a simple product of the non-commuting projectors, the conjunction consists of; ILs are equal.

Now, we can generalize expressions (30) and (31). Due to (6), any conjunction gives the same IL as the product of the projectors involved in it. In such a case, the implication (commuting or non-commuting) IL can be estimated directly: due to definition, it contains the conjunction or the projector replacing the conjunction itself. All we need to do is just to remove the subscript $_\mathrm{I}$ in (30) and (31). So, we finally obtain

$$H\left(\mathbb{E}|\zeta\rangle^{\otimes N}\right) = \sum_{i=1}^{q} H_\mathrm{Sh}\left(\chi_{\lambda_i}\right) + H\left(\chi_{\wedge_{\Pi_1}}\right) + $$
$$+ \sum_{\pi_1}^{\dim\Pi_1} |\zeta_{\pi_1}|^2 H\left(\chi_{\wedge_{\Pi_2}|\pi_1}\right) \qquad (34)$$

and

$$H\left(\mathbb{E}|\zeta\rangle^{\otimes N}\right) \leq (q+c)\ln 2 + \sum_{k=1}^{c} \ln\dim\Pi_k, \qquad (35)$$

where $\mathbb{E}$ is the expression being processed on the N-qubit register, q is the number of qubits equipped in no conjunction, and c is the number of conjunctions; any product of non-commuting operators should be considered as a conjunction due to (6).

7. Examples

Let us introduce some notations before we proceed. At first, we define the notation

$$\mathrm{P}_q = |q\rangle\langle q|, \quad \mathrm{P}_{\neg q} = \mathrm{I} - \mathrm{P}_q, \quad |q\rangle : \sigma_q|q\rangle = |q\rangle,$$

where I is the unit operator, $q = \{x, y, z\}$ and σ_q is the corresponding Pauli matrix. It is easy to check that $\mathrm{P}_q\mathrm{P}_{q'} \neq \mathrm{P}_{q'}\mathrm{P}_q$, if $q \neq q'$.

Since $\{\mathrm{I}, P_x, P_y, P_z\}$ are linearly independent, we can encode any qubit operator as some linear combination of these matrices. To proceed, we need the following operators:

$$\mathrm{W}_k = \frac{1}{\sqrt{2}}\left(\mathrm{P}_{zk} - \mathrm{P}_{\neg zk} + \mathrm{P}_{xk} - \mathrm{P}_{\neg xk}\right) =$$
$$= \sqrt{2}\left(\mathrm{P}_{xk} - \mathrm{P}_{\neg zk}\right),$$
$$\mathrm{C}_{k,s} = \left(1 - e^{i\phi_{k,s}}\right)\left(\mathrm{P}_{zs}\mathrm{P}_{\neg zk} + \mathrm{I}_s\mathrm{P}_{zk}\right) + e^{i\phi_{k,s}}\mathrm{I}_s\mathrm{I}_k,$$

where W_k is the Walsh–Hadamard gate on the k-th qubit, and $\mathrm{C}_{k,s}$ is the controlled-phase gate on the k-th and s-th qubits with the phase shift $\phi_{k,s} = \pi/2^{s-k}$. Here and in the following, operator's subscripts $_{k,s}$ denote the qubits these operators act on.

We emphasize that, using another basis matrices (and consequently the projectors) will not influence the result, since it may be provided by a simple unitary rotation. This is a simple consequence of the fact that changing the language (but except the unary languages consisting of one symbol only) can not significantly influence the algorithm complexity, see [37].

Expression (35) can be used for the estimation of the IL of any quantum algorithm $\mathbb{E}$. Processing the estimation, we should keep in mind that the number of conjunctions c equals the number of projector's products from the viewpoint of the IL, see Section 6.

Below, we estimate IL for two different quantum algorithms: quantum discrete Fourier transform $\mathrm{FFT_Q}$ and the Grover search algorithm $\mathrm{Gr_Q}$. We refer those who are interested in the details of the algorithms to [38]. These algorithms provide the essential speed-up compared to their classical analogs, which are exponentially complex, and one expects that to be reflected by IL in some way.

7.1. $\mathrm{FFT_Q}$ dequantization

As is known, $\mathrm{FFT_Q}$ on the N-qubit register may be written as the following operator:

$$\begin{aligned}&\mathrm{FFT_Q} = \Phi_0 \ldots \Phi_{N-1},\\ &\Phi_k = \mathrm{W}_k \mathrm{C}_{k,N-1} \mathrm{C}_{k,N-2} \ldots \mathrm{C}_{k,k+1}.\end{aligned} \tag{36}$$

One can notice that every $\mathrm{C}_{k,s}$ contains 2 non-reducing terms with P_{zk} which do no commute with the corresponding P_{xk} of W_k. Then the number of non-commuting projector products is $c_k = 2^{N-k-1}$ for any Φ_k and

$$c = \sum_{k=0}^{N-1} c_k = \sum_{k=0}^{N-1} 2^{N-k-1} = 2^N - 1. \tag{37}$$

Any Φ_k contains $N-k-1$ commuting projector products (commuting conjunctions); summing over k gives $N(N-1)/2$ commuting conjunctions in general. Substituting this, (37) and $\dim\Pi_k = 2^N$ in (35), we obtain

$$\begin{aligned}&H(\mathrm{FFT_Q}) \le \left[q + \frac{N(N-1)}{2} + \right.\\ &\left. + (N+1)\left(2^N - 1\right)\right] \ln 2 = \mathcal{O}\left(N2^N\right),\end{aligned} \tag{38}$$

thus meeting an exponential IL of the dequantized $\mathrm{FFT_Q}$. As is known, its classical analog $\mathrm{FFT_C}$ needs $\mathcal{O}\left(N2^N\right)$ amount of resources.

7.2. Grover dequantization

$\mathrm{Gr_Q}$, which is operating on the database containing 2^N elements, can be represented with the operator

$$\begin{aligned}&\mathrm{Gr_Q} = \left\{\left[2\left(\mathrm{WP}_z\mathrm{W}\right)^{\otimes N} - \mathrm{I}^{\otimes N}\right] \otimes \right.\\ &\left. \otimes \left(\mathrm{P}_{\neg z} - \mathrm{P}_z\right)\right\}^{\frac{\pi}{4}2^{N/2}} \mathrm{U}_\Gamma,\\ &\mathrm{U}_\Gamma\colon |x\rangle|0\rangle \to |x\rangle|\Gamma(x)\rangle,\end{aligned} \tag{39}$$

where Γ is the tested statement (i.e., $\mathrm{Gr_Q}$ determines the elements on which Γ is true). The operator U_Γ requires the number of gates depending not on the register size, but on the particular expression for Γ only and, thus, will not be considered in the following.

As for the component $(\mathrm{P}_{\neg z} - \mathrm{P}_z)$ acting on the ancillary qubit, it includes $c_{k|\Gamma} = 1$ intersections for the complementary (and, hence, commuting) projectors only, for which one can put formally $\dim\Pi_{k|\Gamma} = 1$ while estimating IL. The number of these anxillary intersections is

$$c_{|\Gamma} = \sum_{k=1}^{\frac{\pi}{4}2^{N/2}} c_{k|\Gamma} = \frac{\pi}{4}2^{N/2}. \tag{40}$$

For the operator in the square brackets, we obtain

$$\mathrm{WP}_z\mathrm{W} = 2\left(\mathrm{P}_x - \mathrm{P}_{\neg z}\right)\mathrm{P}_z\left(\mathrm{P}_x - \mathrm{P}_{\neg z}\right) = 2\mathrm{P}_x\mathrm{P}_z\mathrm{P}_x,$$

thus giving one intersection of non-commuting projectors. The number of such intersections in the square brackets is $c_{k|[\,]} = N$ (one for every qubit in the register). Since the iteration should be applied $\frac{\pi}{4}2^{N/2}$ times, we have

$$c_{|[\,]} = \sum_{k=1}^{\frac{\pi}{4}2^{N/2}} c_{k|[\,]} = \sum_{k=1}^{\frac{\pi}{4}2^{N/2}} N = \frac{\pi}{4}N2^{N/2}. \tag{41}$$

As $\dim\Pi_{k|[\,]} = 2^N$, we obtain after substituting (40) and (41) into (35)

$$\begin{aligned}&H(\mathrm{Gr_Q}) \le \left(q + c_{|\Gamma} + c_{|[\,]}\right)\ln 2 + Nc_{|[\,]}\ln 2 =\\ &= \left[q + \frac{\pi}{4}\left(N^2 + N + 1\right)2^{N/2}\right]\ln 2 =\\ &= \mathcal{O}\left(N^2 2^{N/2}\right).\end{aligned} \tag{42}$$

As is known, the classical search algorithm requires $\mathcal{O}(2^N)$ number of resources, while $\mathrm{Gr_Q}$ needs $\mathcal{O}(2^{N/2})$. Thus, we obtain non-polynomial IL in this case. It may be an example of the "incomplete" algorithm reduction, i.e., when the algorithm under the dequantization reduces to the rather complicated one.

Since the search algorithm belongs to the NP complexity class, the example demonstrates that at least some quantum algorithms being NP do not meet complete IL (i.e., IL for them does not necessarily equal in the number of resources required with their classical analogs) under the dequantization: we obtained $\mathcal{O}\left(N^2 2^{N/2}\right)$ instead of $\mathcal{O}\left(2^N\right)$ IL for $\mathrm{Gr_Q}$. The reason for such a difference is the transformation of the algorithm which is discussed in the following section.

One may argue that the obtained results may be explained by the Gottesman–Knill theorem (see [38] for the formulation and proof): (36) includes the gates outside the Clifford group, while (39) employs the gates from the Clifford group only. This is what we obtained by expressions (38) and (42). However, compared to the Gottesman–Knill theorem, we exactly show how the computational efficiency is being reduced and give a concrete recipe for the loss estimation, see (34) and (35).

8. Discussion

In our research, we studied how the set of elementary QL operations can be reduced to the classical counterpart via taking the semiclassical limit $\hbar \to 0$. We applied the projective operator representation of QL, see [12]. One may argue that the projector sets are not as wide-spread in a quantum information research as the other sets of gates (such as one-qubit gates and Toffoli gate or Controlled-NOT, for example). However, it seems to be a non-trivial task to try to find a classical analog for such gates as the Walsh–Hadamard gate. So, we focused on the similar formal sets of logic gates such as the conjunction, implication, and negation. We used the projectors to investigate the formal rules of both quantum and classical logics. Due to the algorithm complexity theory, a choice of a language is not essential up to some polynomial increase of resources. So, the consideration of the projector sets cannot influence the final result.

After that, we estimated the amount of information loss during the dequantization process, shedding light on the loss of the logic efficiency and on the efficiency gap problem itself. To quantify our approach, we used both the von Neumann and the Shannon entropies for quantum and dequantized logic gates, respectively. It implies the application of some techniques from information theory and requires the analysis of conditional distributions.

We dequantized the complete set of elementary quantum logic operations including the non-commuting conjunction. In addition, the general expression estimating the IL for any dequantized quantum algorithm has been derived, see (34) and (35). We formulated and proved the conjunction theorem (see Section 6), which is necessary for the estimation of a conjunction of non-commuting projectors. The technique was applied to $\mathrm{FFT_Q}$ and $\mathrm{Gr_Q}$ algorithms; it demonstrated exponential and non-polynomial ILs for the algorithms, correspondingly.

Expression (34) estimates the amount of information being lost by a quantum algorithm, which is encoded with $\mathbb{E}$, after the processing through the semiclassical limit. It implies that *description* of the IL requires the additional memory of the $H\left(\mathbb{E}|\zeta\rangle^{\otimes N}\right)$ size which is upper bounded with (35). This is equivalent to the same increase of the amount of elementary logical steps (by one per each additional memory cell to write it down) at least. So, we conclude that the technique presented in the paper might shed some light on the NP problem (in the case where we consider some NP-complete algorithm, such as $\mathrm{Gr_Q}$) and on the algorithm complexity classification.

Any quantum algorithm under the dequantization keeps the number of elementary logical operations the same with no change in the complexity in its common sense. But the description of its IL requires the additional memory and, consequently, time (measured in the number of elementary logical steps). The interrelation between the estimated IL (or efficiency) and the complexity in its common sense is unclear, since the dequantized algorithm and its IL description do not coincide.

The algorithmic entropies, see [27], may be used to describe the "distance" between the desired and calculated results in the case of using a quantum or classical algorithm. The entropies are used to estimate the probability of obtaining the desired result; they are defined for the states calculated with some algorithm. Our approach differs a lot from that one, since we investigate the changes of the elementary logic operations while taking the limit $\hbar \to 0$.

The Kolmogorov complexity approach is useful for the estimation of the difference between quantum and classical calculations. It gives the program minimized in size that realizes the corresponding algorithm. Such an approach helps to define conditions on the calculations, which are easy in the quantum case, but are hard in the classical one. For more details, see [23] and other ones [24–26].

However, our approach differs from the Kolmogorov's one. Dequantization of elementary QL operations allows one to estimate the corresponding entropy for any logical expression. It gives the amount of information loss during the *reduction* of the quantum algorithm to the classical one. It has much in common with (but can not be interpreted as comparison of) the corresponding Kolmogorov complexities in the cases where quantum and classical algorithms solve *the same* problem only, since the amount of elementary operations does not change, while one takes the semiclassical limit. The similarity origins from the re-estimation of quantum gates in terms of classical ones. But, after the dequantization, the algorithm may solve another problem (like $\mathrm{FFT_Q}$). In this case, both Kolmogorov and our approaches can not be compared directly.

We illustrate this statement with the help of the classical discrete fast Fourier transform (FFT). As is known, $\mathrm{FFT_Q}$ is a polynomial time algorithm. It needs $\mathcal{O}\left(N^2+N\right)$ operations, while FFT needs $\mathcal{O}\left(N2^N\right)$. According to (7), the number of elementary operations during the dequantization remains the same, i.e., polynomial. However, some amount of information is lost, and this amount can be estimated. We suggest that the only explanation for this is the algorithm changeover. In particular, $\mathrm{FFT_Q}$ transforms into the Legendre transform (but not into FFT) [35]; for more information, see [36].

Such algorithm simplification after the dequantization is explained by the fact that the QL algebra can be split up on some Boolean subalgebras, each of which is similar to the CL algebra [8]. But the statements from different QL subalgebras do not commute, thus providing the largest IL possible.

It has been widely believed that the entanglement is a quantum resource responsible for a high efficiency of quantum algorithms. In our approach, projective operators (non-commuting in general) are only used with no direct relation to the entanglement. One may say that the non-commuting projectors project the state to different subspaces (say, H_1 and H_2) such that the basis vectors from H_1 are represented as an entangled in the basis of H_2. But there are doubts that this can be stated and proved in general. In [39], a simulation of Shor's factoring algorithm was made, and the authors found no significant role of the entanglement in providing the exponential speed-up of the algorithm. Based on this and on our own results, we suppose that the entanglement can not be considered as a resource of the computational speed-up in quantum calculations. The high computational efficiency of quantum algorithms is highly interrelated with the presence of non-commuting statements which can not be simulated efficiently by CL. However, the last item is true, if IL is strongly interconnected with the computational efficiency only and requires a further research. Both the efficiencies, in spite of having much in common, differ from each other.

In addition, we would like to emphasize that the application of the presented approach to the complexity classification requires the reverse engineering of the approach. Namely, one should be able to estimate the gain in the efficiency for any classical algorithm while transiting it to the quantum one. Such a task seems to be highly non-trivial to date.

Some questions still remain open, and it seems reasonable to solve them. Here, they are:

• How are the *complexity* and IL interconnected with each other?

• If some optimal quantum algorithm gives an exponential IL, does it imply that the classical algorithm for the same problem is exponential in time?

• Can the presented approach be used for the comparison of quantum and classical algorithm complexity classes and for the ascertainment of a correlation between these classes?

• Generally, any quantum algorithm transforms into another one, which solves another problem, under the semiclassical limit. But, what can we say about the reversion? Can one obtain some quantum algorithm (or the class of them), being given the classical one? Some investigation on the topic of the transition from subsets of CL to compatible (i.e., determined with mutually commuting operator sets) linear subspaces of QL is presented in [34]; it is called lifting in this work. In our opinion, such a reversion should be ambiguous due to the differences between subsets and linear subspaces. The point is that there is no recipe to go to the incompatible subspaces. In our opinion, the formalization and further development of the approach presented in [40] might be helpful while constructing the recipe. However, it is not known surely whether it can be solved or not for some classes of quantum algorithms at least. In particular, one can try to build the quantum analog of the Legendre transform. Due to the ambiguity, we expect to derive some class of quantum algorithms, but not $\mathrm{FFT_Q}$ only. One more interesting point is to look for the quantum analogs of inefficient classical algorithms such as FFT or the factorization and to verify whether the analogs will be inefficient in QL as well.

Summing up the questions mentioned above, we conclude that the problem of the reverse transition to the dequantization is worth of further investigations.

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КВАНТОВА ЛОГІКА
У КВАЗІКЛАСИЧНОМУ НАБЛИЖЕННІ:
ВТРАТА ІНФОРМАЦІЇ

Ми розглядаємо квантову обчислювальну ефективність з нової точки зору. Дана ефективність зводиться до класичної за допомогою квазікласичного наближення. Ми показуємо, що дане спрощення викликане тим, що кожна елементарна квантова логічна операція (вентиль) втрачає інформацію під час переходу до свого класичного аналогу. Проведено оцінку втрати інформації для всіх вентилів, що утворюють повний набір. Ми показуємо, що найбільше інформації втрачається для некомутуючих вентилів. Це дозволяє розглядати некомутативність як джерело квантового прискорення обчислень. Наш метод дозволяє кількісно оцінити переваги квантових обчислень порівняно з класичними за допомогою прямого аналізу використовуваної логіки. Отримані результати проілюстровано на прикладі квантового дискретного перетворення Фур'є та пошукового алгоритму Гровера.

Ключові слова: квантова логіка, квантові алгоритми, складність.