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# MIRROR SYMMETRY AS A BASIS FOR CONSTRUCTING A SPACE-TIME CONTINUUM

*By mirroring a one-dimensional oriented set in a complex space specially created on the basis of a symmetry, a mirror n-dimensional space with $n > 1$ has been constructed. The geometry of the resulting space is described by the Clifford algebra. On the basis of the algebra of hyperbolic hypercomplex numbers, a pseudo-Euclidean space has been constructed with the metric of the Minkowski space. The conditions for a function of a hyperbolic hypercomplex argument to be analytic (h-analyticity) are obtained. The conditions implicitly contain the Maxwell equations for the 4-potential in a free space.*

*Keywords*: mirror transformation, Clifford algebra, hyperbolic hypercomplex numbers, Minkowski space.

## 1. Introduction

Within the last decades, there emerged the necessity in the search for the physical ideas and a mathematical apparatus that would be capable of describing the variety of physical phenomena from unique positions. For instance, a tendency is observed to reconsider the classical space-time concepts in favor of their treatment using various methods of algebra. In particular, the binary geometrophysics (the relational theory) [1], the algebraic theory of space-time on the basis of quaternions [2,3], and the algebraic geometry on the basis of the Clifford algebra [4, 5, 12], quaternions being examples of the latter.

The geometrical Clifford algebra pretends to play the role of the unified language in mathematical physics [17] owing to a powerful mathematical apparatus of all known complex and hypercomplex numbers, which are naturally included into this algebra. Regular conferences and numerous publications in the framework of ICCA (International Conference on Clifford Algebras and their Applications in Mathematical Physics) [1] testifies to the outlook of this direction, and the scope of application permanently extends from mechanics and signal processing to chemistry and biology. Concerning physical applications, it should be noted that even the formulation of traditional physical problems in the language of the Clifford algebra leads more often to a simplification of mathematical calculations and/or new unexpected results.

As an example, the analysis of the equations of motion for a material point in an inhomogeneous anisotropic space in the Clifford algebra basis partially agrees with the Einstein equations. Moreover, the very principle of construction of equations is an alternative to variational methods [18]. The equations of motion for a classical particle with spin in an electromagnetic field become substantially simpler in the Clifford basis. In particular, instead of the nonlinear

[1] The 10-th ICCA was held in Tartu on 4–8 August, 2014. The ICCA proceedings are published in the ICCA journal "Advances in Applied Clifford Algebras".

equations of perturbation theory, linear equations are to be solved [19], and the analysis of the specific features in the neutron-nucleus interaction, as well as the possibility for neutron-nucleus molecules to exist, on the basis of the Dirac equation [22] can be carried out, by using the Clifford algebra language [2].

In turn, the Clifford algebra is connected with spinors [10]. The latter are specific geometric objects sometimes called "semivectors", because only spin-tensors of rank $n > 1$ are associated with observables. Such a role of spinors and the spinor space in the geometry is similar to that played by the wave functions in quantum mechanics and requires an additional analysis. Since the Clifford algebra pretends to be a unified language of mathematical physics, the issue concerning its interrelation with fundamental laws of the nature arises as well.

The principles of symmetry that form the basis of nature's laws [6, 20] have been substantially corrected recently. Recent researches in the string theory [7, 8] unexpectedly drew a close attention to extended capabilities of the mirror symmetry. In those researches, two sets, $X$ and $X'$, were connected with each other as mirror pairs with the use of an auxiliary space. The properties of the pairs obtained completely depend on the properties of this space at special points and are not confined to the right-to-left substitution, as in the conventional geometrical mirror symmetry. In this work, the idea of mirror symmetry with the help of an auxiliary space is used to construct, from a 1-dimensional oriented set, a vector space with dimensionality $n > 1$ in the mirror space. As an auxiliary space, the complex space specially created on the basis of a symmetry is used, in which the role of singular points is played by certain planes (the planes of "mirrors"), relative to which the geometrical mirror symmetry is obeyed. An analogy between the obtained space and the properties of complex numbers is consistently drawn, and the algebra, geometry, and physical properties of the obtained vector space are analyzed. For the presentation of the material to be logic and comprehensive, both the known and original results are discussed. This work should be considered as a first part – the substantiation of the method – of a wider research, which is planned to be presented in the future.

[2] In the Internet, one should pay attention to A.A. Ketsaris's lectures (http://toe-physics.org/ru/lectures.htm) or to the works by R. Dahm, D. Hestenes, and N.G. Marchuk (applications to the field theory), W.E. Baylis, B. Jancewicz, and P. Puska (electrodynamics), and D. Hestenes and D.S. Shirokov (Clifford algebra).

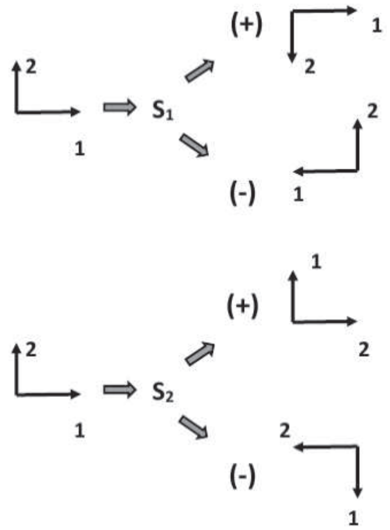


***Fig. 1.*** Action of the symmetry operators on the basis vectors of a coordinate system

## 2. Construction of an Auxiliary Space on the Basis of the Complex-Plane Symmetry

According to É. Cartan [9], symmetry is defined as the operation $\mathbf{S}$ of geometrical mirroring with respect to a hyperplane $M$ that passes through the coordinate origin, in the direction of an anisotropic vector $\boldsymbol{s}$ orthogonal to this hyperplane. Let us define the operations of mirror symmetry $\pm\boldsymbol{s}_1$ and $\pm\boldsymbol{s}_2$ in the coordinate system on the plane that change the directions of the basis vectors and the orientation of the coordinate system in accordance with the diagram depicted in Fig. 1. For an arbitrary point $P = (x, y)$ on the complex plane, the symmetry operation in the vector basis $1 \to (1, 0)$ and $2 \to (0, i)$ can be presented by the expression

$$(\boldsymbol{s}_1 + \boldsymbol{s}_2)\binom{x}{iy} = \binom{x}{-iy} + \binom{iy}{x} = \binom{x+iy}{x-iy} = \binom{z}{\bar{z}}, \tag{1}$$

where $i$ is the imaginary unit, and $\bar{z}$ means the complex conjugate value. As a result of the mirror symmetry (1), the complex plane in the basis $(1, i)$ is mapped in the 3-dimensional basis $(1, 0), (0, 1), (i, \bar{i})$ as a plane that passes through the axis of imaginary

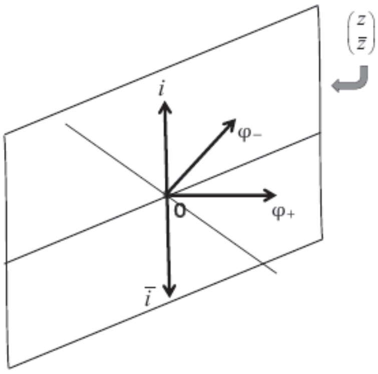


***Fig. 2.*** Auxiliary complex space of mirror images

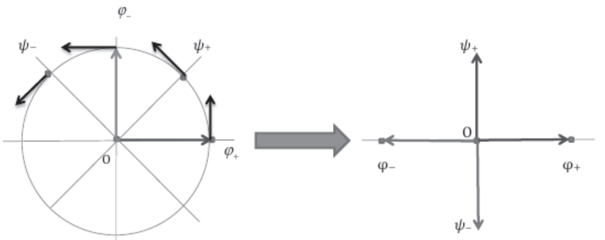


***Fig. 3.*** Construction of an orthogonal coordinate system in the mirror space

coordinates along the bisector of the orthogonal axes $(1,0)$ and $(0,1)$ (Fig. 2). The axis of imaginary coordinates is represented as a combination of two oppositely oriented imaginary axes with the common point 0, which are orthogonal to the plane of basis vectors $(1,0)$ and $(0,1)$. Let us denote those orthogonal vectors as $\varphi_+ = (1,0)$ and $\varphi_- = (0,1)$. The complexification of the real-valued Euclidean plane by a combination of symmetric imaginary axes generates a space $\mathcal{E}_2^{++}$ of two orthogonal complex planes in the bases $(\varphi_+, i\varphi_+)$ and $(\varphi_-, \bar{i}\varphi_-)$, respectively. The planes intersect each other along the imaginary axes, possess one common point 0, and determine, in the general case, a complex vector $\boldsymbol{\gamma} = (\xi, \bar{\eta})$. If $\mathcal{E}_2^{++}$ is considered as a 3-dimensional space, any of its points $P$ has real-valued coordinates: $P = (x_1, x_2, \pm y)$, where the coordinate $\pm y$ corresponds to a combination of the imaginary axes, and the vector $\boldsymbol{\gamma}$ has the coordinates $\xi = x_1 + iy$ and $\bar{\eta} = x_2 - iy$. The representation of a complex number by a 2-dimensional complex vector $(z, \bar{z})$ has its logic substantiation. Really, the product of two complex numbers can be expressed in the form

$$z_1 z_2 = (z_1 \bullet \bar{z}_2) + i[z_1 \times \bar{z}_2], \tag{2}$$

where $z_1 \bullet \bar{z}_2$ stands for the scalar and $[z_1 \times \bar{z}_2]$ for the vector product of the vectors $z_1$ and $\bar{z}_2$. The real and imaginary parts of Eq. (2) are presented in the symmetric and antisymmetric forms,

$$\begin{aligned} \mathrm{Re}(z_1 z_2) &= (z_1 \bullet \bar{z}_2) = \frac{1}{2}(z_1\bar{z}_2 + z_2\bar{z}_1), \\ \mathrm{Im}(z_1 z_2) &= [z_1 \times \bar{z}_2] = -\frac{i}{2}(z_1\bar{z}_2 - z_2\bar{z}_1). \end{aligned} \tag{3}$$

Below, it will be shown that expressions (2) and (3) are true for hypercomplex numbers as well.

## 3. Construction of a Coordinate System in the Mirror Space

The coordinate system is constructed in the mirror space by applying the operation of mirror symmetry to the basis vector $\varphi_+$ in the plane $(\varphi_+, \varphi_-)$ of the space $\mathcal{E}_2^{++}$. For this purpose, let us write down the operator of mirror symmetry in the general form as follows:

$$\mathbf{S}(\alpha) = \cos\frac{\alpha}{2}\boldsymbol{s}_1 + \sin\frac{\alpha}{2}\boldsymbol{s}_2, \tag{4}$$

where the angle $\alpha/2$ is reckoned counterclockwise from the basis vector $\varphi_+$, and the angle $\alpha$ itself is an angle between the new basis vectors in the mirror space and the mirror image of the vector $\varphi_+$. The factor $1/2$ follows from the properties of mirror images. It should be noted, first of all, that, owing to this factor, the symmetry transformation within the angular interval $0 \le \alpha/2 < 2\pi$ covers the mirror space twice. Therefore, the transformations within the intervals $0 \le \alpha/2 < \pi$ and $\pi \le \alpha/2 < 2\pi$ have to be analyzed separately.

In the mirror space, let us construct an orthogonal basis coordinate system on a plane corresponding to the angles $\alpha = 0,\ \pi/2,\ \pi$, and $3\pi/4$. Substituting those values into Eq. (4), we obtain, in the mirror space, two pairs of oppositely oriented basis vectors:

$$\begin{aligned} \varphi_+ &= \mathbf{S}(0)\varphi_+ = \begin{pmatrix}1\\0\end{pmatrix},\ \psi_+ = \mathbf{S}(\pi/2)\varphi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \\ \varphi_- &= \mathbf{S}(\pi)\varphi_+ = \begin{pmatrix}0\\1\end{pmatrix},\ \psi_- = \mathbf{S}(3\pi/4)\varphi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}. \end{aligned} \tag{5}$$

The operations of mirror symmetry $\mathbf{S}(\alpha)$ with respect to the selected planes in the space $\mathcal{E}_2^{++}$ map (Fig. 3) the initial 1-dimensional vector space $R_1$ onto a 2-dimensional vector one in the mirror space, the orthogonal coordinate axes of which are presented as a bundle of two oppositely oriented basis vectors with a common zero point (the coordinate origin) (right panel in Fig. 3). The obtained system of basis vectors agrees well with the concept of arbitrary choice for the direction and the orientation of a coordinate system. However, it can be considered only as a possible realization of the coordinate system, because the coordinates of arbitrary point have an uncertainty connected with the sign of numerical values. A possibility for the basis vector to have the opposite orientation can be formalized by introducing the vector-operator of symmetry $\boldsymbol{e}$, for which the oppositely oriented vector pairs are eigenvectors, and its eigenvalues $\pm 1$ are responsible for that or another orientation of the basis vector. Since the eigenvectors are determined by expressions (5), it is easy to obtain the matrix representation of the vector-operators in the form

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \boldsymbol{e}_i \varphi_{i\pm} = \pm 1 \varphi_{i\pm}, \ \boldsymbol{e}_i \boldsymbol{e}_i = \boldsymbol{e}_i^2 = \mathbf{1}, \tag{6}$$

where $\mathbf{1}$ is a diagonal matrix, which can be considered as the identity symmetry operator. The right panel of Fig. 3 can be considered as a geometric image of basis vector-operators.

From Eq. (4), we obtain the following relation for the $\mathbf{S}(\alpha)$ transformations in the interval of angles $2\pi \le \alpha < 4\pi$:

$$\mathbf{S}(2\pi \le \alpha < 4\pi) = -\mathbf{S}(0 \le \alpha < 2\pi). \tag{7}$$

It allows us to obtain two more pairs of orthogonal basis vectors in the mirror space, $\bar{\varphi}_\pm$ and $\bar{\psi}_\pm$, which are connected with vectors (5) by the relations

$$\bar{\varphi}_\pm = -\varphi_\pm; \ \bar{\psi}_\pm = -\psi_\pm. \tag{8}$$

It is evident that the basis vectors (8) define the vector-operators $\bar{\boldsymbol{e}}_i = -\boldsymbol{e}_i$. Expressions (7) and (8) illustrate a well-known problem of sign uncertainty for transformations in the spinor space [3]. However, it can be considered from another perspective. Really, taking advantage of the analogy with the vector representation of a complex number together with its conjugate value in the auxiliary complex space, we will consider the vector $\bar{\boldsymbol{e}}_i$ to be a conjugate value of the vector $\boldsymbol{e}_i$. Such approach allows us to consider transformations in the spin space within the angular interval of $0 \le \alpha < 2\pi$ only, provided that the conjugation operation is introduced as an independent basic operation. Hence, formally, the mirror symmetry operations (5) performed the operation $\mathbf{S}$: $R_1 \mapsto R_{2(2)}$ in the auxiliary complex space, where $R_{2(2)}$ denotes a 2-dimensional vector space in the mirror space with the bundle of oppositely oriented basis vectors. In turn, $R_{2(2)}$ is a geometric image of the algebra of matrix operators $\boldsymbol{e}_i$, which will be used below to construct the Euclidean vector space $E_2$.

## 4. Algebra and Geometry of $E_2$ in the Mirror Space

In order to construct an Euclidean space, we must define a scalar product. Since vectors are presented by matrices, their product is noncommutative in the general case. Therefore, formally, we may write

$$\boldsymbol{e}_1 \boldsymbol{e}_2 = \frac{1}{2}(\boldsymbol{e}_1 \boldsymbol{e}_2 + \boldsymbol{e}_2 \boldsymbol{e}_1) + \frac{1}{2}(\boldsymbol{e}_1 \boldsymbol{e}_2 - \boldsymbol{e}_2 \boldsymbol{e}_1). \tag{9}$$

The symmetric bilinear form defines the internal, or scalar, product of vectors,

$$\boldsymbol{e}_1 \bullet \boldsymbol{e}_2 = \frac{1}{2}(\boldsymbol{e}_1 \boldsymbol{e}_2 + \boldsymbol{e}_2 \boldsymbol{e}_1), \tag{10}$$

and the antisymmetric form defines their external product,

$$\boldsymbol{e}_{12} = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \frac{1}{2}(\boldsymbol{e}_1 \boldsymbol{e}_2 - \boldsymbol{e}_2 \boldsymbol{e}_1). \tag{11}$$

The condition of vector orthogonality is $\boldsymbol{e}_1 \bullet \boldsymbol{e}_2 = 0$, so that Eqs. (9) and (10) yield

$$\boldsymbol{e}_1 \boldsymbol{e}_2 = -\boldsymbol{e}_2 \boldsymbol{e}_1, \ \boldsymbol{e}_{12} = \boldsymbol{e}_1 \boldsymbol{e}_2, \ \boldsymbol{e}_{21} = -\boldsymbol{e}_{12}. \tag{12}$$

From Eq. (12), it follows that the orthogonal vectors anticommute with each other. The external product generates a bivector $\boldsymbol{e}_{12}$, which is a simple vector product for orthogonal vectors and changes its sign if the order of multipliers (the order of indices) changes. The latter property is the property of an antisymmetric form for orthogonal vectors rather than that of the noncommutativity of a matrix product. The bivector $\boldsymbol{e}_{12}$ also determines the symmetry of the coordinate system, because it is a combination of symmetries. To find it, let us use an expression for

[3] A description of this problem can be found both in textbooks (see, e.g., lecture 24 in: M.M. Postnikov, Lectures in Geometry. Semester II. Linear Algebra (Mir, Moscow, 1983) and special literature (see, e.g., p. 33 in [15]).

the mirror image of the vector $x$ in the basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ in the direction of the unit vector $\boldsymbol{s}$ [10]: $\boldsymbol{x}' = -\boldsymbol{sxs}$. Let $\boldsymbol{x} = \boldsymbol{e}_i$ and let the mirroring be performed twice: first at $\boldsymbol{s} = \boldsymbol{e}_2$, and then at $\boldsymbol{s} = \boldsymbol{e}_1$. We obtain

$$\boldsymbol{e}'_i = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_i\boldsymbol{e}_2\boldsymbol{e}_1 = -\boldsymbol{e}_i, \tag{13}$$

where properties (12) were taken into account. Since the even images generate rotations, Eq. (13) corresponds to a rotation of the basis coordinate system by an angle of 180°. In the case of $R_{2(2)}$, this operation corresponds to the swapping of the basis vectors with the subscripts "+" and "−". The bivector $\boldsymbol{e}_{12}$ determines the oriented area of a parallelogram constructed on the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Therefore, the rotation axis in $R_{2(2)}$ is considered to be the axis of bivector coordinates. Then, similarly to Eq. (8), we have

$$\boldsymbol{e}_{12}\varphi_{12\pm} = \pm i\varphi_{12\pm},\ \boldsymbol{e}_{12}^2 = -\mathbf{1}. \tag{14}$$

The complexification $R_{2(2)} \to R_{2(2)}^{+(+)}$ as a result of introducing a bundle of imaginary rotation coordinate axes becomes possible owing to the extension of the symmetry concept to the case of a plane by including the ±180°-rotations. The bivector $\boldsymbol{e}_{21} = -\boldsymbol{e}_{12}$ has the opposite orientation and changes the orientation of the system of basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_{21}$, as a whole, to $R_{2(2)}^{+(+)}$. If this system of symmetry vector-operators is appended by the operator of identical symmetry, $\mathbf{1}$, we obtain a system of linearly independent vectors, on which the Clifford algebra $\mathbf{C}_2$ over the field of real or complex numbers is built,

$$\mathbf{A} = a^0\mathbf{1} + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^{12}\boldsymbol{e}_{12}. \tag{15}$$

Expression (15), where $a^0$ is a scalar, $a^1$ and $a^2$ are vector components, and $a^{12}$ is a bivector component,

**Multiplication tables of quaternion basis vectors for right- and left-handed orientations of the coordinate system**

| | $\boldsymbol{i}$ | $\boldsymbol{j}$ | $\boldsymbol{k}$ | | $\boldsymbol{i}^*$ | $\boldsymbol{j}^*$ | $\boldsymbol{k}^*$ |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{i}$ | $-\mathbf{1}$ | $-\boldsymbol{k}$ | $\boldsymbol{j}$ | $\boldsymbol{i}^*$ | $-\mathbf{1}$ | $\boldsymbol{k}^*$ | $-\boldsymbol{j}^*$ |
| $\boldsymbol{j}$ | $\boldsymbol{k}$ | $-\mathbf{1}$ | $-\boldsymbol{i}$ | $\boldsymbol{j}^*$ | $\boldsymbol{k}^*$ | $-\mathbf{1}$ | $\boldsymbol{i}$ |
| $-\boldsymbol{k}$ | $-\boldsymbol{j}$ | $\boldsymbol{i}$ | $-\mathbf{1}$ | $\boldsymbol{k}^*$ | $\boldsymbol{j}^*$ | $-\boldsymbol{i}^*$ | $-\mathbf{1}$ |

will be referred to as an aggregate. The $\mathbf{C}_2$ algebra contains all three known systems of complex numbers with the basis pairs $(1, i)$, $(1, e)$, and $(1, \epsilon)$, which have their analogs in $\mathbf{C}_2$:

• ordinary complex numbers $a + ib$, $i^2 = -1$ $1 \to \mathbf{1}$; $i \to \boldsymbol{e}_{12}$;

• binary numbers $a + eb$, $e^2 = 1$ $1 \to \mathbf{1}$; $e \to \boldsymbol{e}_1, \boldsymbol{e}_2$;

• dual numbers $a + \epsilon b$, $\epsilon^2 = 0$ $1 \to \mathbf{1}$; $\epsilon \to (\boldsymbol{e}_1 + \boldsymbol{e}_{12})$.

According to the properties of their absolute values $|z|^2 = z\bar{z}$, which are defined by the expressions $|z|^2 = a^2 + b^2$, $|z|^2 = a^2 - b^2$, and $|z|^2 = a^2$, those numbers are sometimes called elliptic, hyperbolic, and parabolic complex numbers, respectively [11].

But the capabilities of $\mathbf{C}_2$ are not restricted to that. If we formally change from the basis $(\varphi_{1\pm}, \varphi_{2\pm})$ to the basis $(i\varphi_{1\pm}, i\varphi_{2\pm})$ in $R_{2(2)}^{+(+)}$ in the case of $\mathbf{C}_2$, we will obtain a new basis $(i\boldsymbol{e}_1, i\boldsymbol{e}_2, \boldsymbol{e}_{12})$ in the right-handed coordinate system and a basis $(i\boldsymbol{e}_1, i\boldsymbol{e}_2, \boldsymbol{e}_{21})$ in the left-handed one. Let us introduce new notations for those vectors: $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ for the right-handed coordinate system and $(\boldsymbol{i}^*, \boldsymbol{j}^*, \boldsymbol{k}^*)$ for the left-handed one. The Clifford algebras (15) constructed on those basis vectors form the systems of elliptic hypercomplex numbers, which are known as quaternions. Historically, things so happened that the majority of authors use the system with the left-handed orientation of basis vectors. By expressing a quaternion in terms of the scalar and vector parts, $\boldsymbol{Q}(a, \boldsymbol{u})$, it is easy to verify with the help of Table for the multiplication of basis vectors that the product of two quaternions, $\boldsymbol{Q}(a, \boldsymbol{u})\boldsymbol{Q}(b, \boldsymbol{v})$, looks like

$$\boldsymbol{Q}_R(ab - \boldsymbol{u} \bullet \boldsymbol{v}, a\boldsymbol{v} + b\boldsymbol{u} - \boldsymbol{u} \times \boldsymbol{v}) \tag{16a}$$

for the right-handed coordinate system and

$$\boldsymbol{Q}_L(ab - \boldsymbol{u} \bullet \boldsymbol{v}, a\boldsymbol{v} + b\boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{v}) \tag{16b}$$

for the left-handed one, where $\boldsymbol{u} \bullet \boldsymbol{v}$ denotes the scalar product of vectors, and $\boldsymbol{u} \times \boldsymbol{v}$ stands for their vector product. The difference in expressions (16) consists in different signs before the vector product.

It should be noted that, by their properties, the basis vectors of quaternions $\boldsymbol{Q}_R$ and $\boldsymbol{Q}_L$ are bivectors and form a 3-dimensional space of bivectors $B_3$ with the right- and left-handed orientations. A necessity to consider the orientations of a coordinate system stems from the fact that the invariant of a complex or hypercomplex number is formed in the general case as a quadratic form of the initial and conjugate numbers. In this work, the conjugate number is introduced with the help of symmetry operations that change the orientation of coordinate systems. Really, considering aggregate (15) as a hypercomplex number written in the general form in the 4-dimensional space, the conjugate aggregate $\bar{\mathbf{A}}$ is constructed in the basis

$$\mathbf{1}, -\boldsymbol{e}_1, -\boldsymbol{e}_2, -\boldsymbol{e}_{12}, \tag{17}$$

and the invariant or fundamental quadratic form (FQF), which is defined as $|\mathbf{A}|^2 = \mathbf{A}\bar{\mathbf{A}}$, looks like

$$|\mathbf{A}|^2 = (a^0)^2 + (a^{12})^2 - (a^1)^2 - (a^2)^2. \tag{18}$$

This indefinite FQF corresponds to the pseudo-Euclidean space $H_4^{(2)}$. It is easy to verify that, for quaternions, the FQF is positive definite and corresponds to the Euclidean space $H_4$. A necessity to be attentive to the orientation of the systems of basis vectors and bivectors substantially grows, while changing from the 2-dimensional space to a 3-dimensional one, which is considered in the next section. Really, the 3-dimensional space formally includes 3 oriented planes, in which the orthogonal polar basis vectors are determined, and their orientation is given by the corresponding bivector. Therefore, a common orientation of the basis of polar vectors is closely related to the orientation of basis bivectors. Conventionally, the bases with the right-hand orientation are used by default, if any special reasons do not force this rule to be violated. In this case, a special remark is made.

## 5. Algebra and Geometry of $E_3$ in the Mirror Space

To obtain the third vector of the Clifford basis, let us perform the mirror symmetry operation (4) for the basis vector $\varphi_+$ in the plane $(\varphi_+, \bar{i}\varphi_-)$. In view of the orientation of this plane (Fig. 2), a 3-dimensional Clifford basis with the right-hand orientation is obtained, if the angle $\alpha = -\pi/2$ or $-3\pi/4$. In this case, the eigenvectors $\varphi_{3\pm}$ and the basis vector $\boldsymbol{e}_3$ look like

$$\begin{gathered}\varphi_{3+} = \mathbf{S}_3(-\pi/2)\varphi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix};\\ \varphi_{3-} = \mathbf{S}_3(-3\pi/4)\varphi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ i\end{pmatrix};\ \boldsymbol{e}_3 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.\end{gathered} \tag{19}$$

The complete Clifford algebra on $\mathbf{C}_3$ includes 8 basis vectors:
- the scalar (0-vector) $a^0\mathbf{1} = A_S$,
- the vector $a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3 = A_V$,
- the bivector $a^{23}\boldsymbol{e}_{23} + a^{31}\boldsymbol{e}_{31} + a^{12}\boldsymbol{e}_{12} = A_B$,
- the 3-vector $a^{123}\boldsymbol{e}_{123} = a^{123}\boldsymbol{i} = A_{PS}$.

The basis 3-vector or pseudo-scalar $\boldsymbol{i} = \boldsymbol{e}_{123}$ commutes with all vectors and bivectors, and its properties are determined by the expressions

$$\boldsymbol{i}^2 = -\mathbf{1},\ \boldsymbol{i}^{-1} = \boldsymbol{e}_{321} = -\boldsymbol{e}_{123},\ \boldsymbol{i}\boldsymbol{i}^{-1} = \mathbf{1},\ \boldsymbol{i} = i\mathbf{1}. \tag{20}$$

The geometric image of a 3-vector is the volume of a parallelepiped constructed on the vectors $\boldsymbol{e}_i$. The multiplication table for the basis vectors can be obtained rather easily, if we take into account that the even permutations of subscripts result in a multiplier of $+1$, and odd ones in $-1$. The $\mathbf{C}_3$ algebra includes all operations of the ordinary vector algebra in the description of classical mechanics [4]. However, the properties of $\mathbf{C}_3$ are not confined to that. Let us form a system of hyperbolic hypercomplex numbers $\mathbf{H}$ and a conjugate one, $\bar{\mathbf{H}}$, as follows:

$$\begin{aligned}\mathbf{H} &= a^0\boldsymbol{e}_0 + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3,\\ \bar{\mathbf{H}} &= a^0\boldsymbol{e}_0 - a^1\boldsymbol{e}_1 - a^2\boldsymbol{e}_2 - a^3\boldsymbol{e}_3,\end{aligned} \tag{21}$$

where $\boldsymbol{e}_0 = \mathbf{1}$. The FQF $|\mathbf{H}|^2 = \mathbf{H}\bar{\mathbf{H}}$ looks like

$$|\mathbf{H}|^2 = (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2. \tag{22}$$

It corresponds to the metric of the pseudo-Euclidean Minkowski space $M_4^{(3)}$. In order to prove that $\mathbf{H} \in M_4^{(3)}$, let us determine the metric tensor $g_{\alpha\beta}$ using the scalar product of basis vectors $\boldsymbol{e}_\alpha$ $(\alpha = 0, 1, 2, 3)$ analogously to that for complex numbers (Eq. (3)):

$$g_{\alpha\beta} = \boldsymbol{e}_\alpha \bullet \bar{\boldsymbol{e}}_\beta = \frac{1}{2}(\boldsymbol{e}_\alpha\bar{\boldsymbol{e}}_\beta + \boldsymbol{e}_\beta\bar{\boldsymbol{e}}_\alpha) \tag{23}$$

With the help of the equality $\boldsymbol{e}_0\boldsymbol{e}_n = \boldsymbol{e}_n\boldsymbol{e}_0$ $(n = 1, 2, 3)$, it is easy to check that $g_{\alpha\beta}$ has a diagonal form with the signature $\mathrm{diag}(g_{\alpha\beta}) = (+1, -1, -1, -1)$. In other words, a hyperbolic hypercomplex number $\mathbf{H}$ is a vector in the pseudo-Euclidean Minkowski space $M_4^{(3)}$ with the orthogonal basis vectors $\boldsymbol{e}_\alpha$ $(\alpha = 0, 1, 2, 3)$. The hypercomplex number $\mathbf{H}$ can be regarded as a hyperbolic quaternion or $\mathbf{H}$-quaternion, which, unlike ordinary quaternions, does not form a field and a division algebra, although some reservations should be made. Nevertheless, **H**-quaternions with the Minkowski space structure have to be studied further in physical problems. First attempts to construct the algebraic theory of space-time and matter on the basis of 2-dimensional hyperbolic numbers have already been done [13,14]. In this work, we only mark some features of differential operations over hyperbolic hypercomplex numbers.

As in the case of ordinary complex numbers, the coordinates of 4-vector are considered to be functions of 4 variables: $a^\alpha(x^0, x^1, x^2, \text{and } x^3)$. Let us introduce a space-time 4-gradient as a covariant vector in the reciprocal basis $(\boldsymbol{e}^0, \boldsymbol{e}^n)$,

$$\boldsymbol{\nabla} = \boldsymbol{e}^\alpha \partial_\alpha = \boldsymbol{\nabla}(\boldsymbol{e}^0\partial_0, \boldsymbol{e}^n\partial_n),\ \partial_0 = \frac{1}{c}\frac{\partial}{\partial t}, \quad (24)$$

where the summation over repeated indices is implied. Let us formulate the analyticity conditions for the function $\mathbf{G}(\mathbf{H})$ of a hyperbolic hypercomplex argument, similar to Cauchy–Riemann ones for ordinary complex numbers. For this purpose, let us construct the invariant of a hypercomplex number as its combination in the initial and conjugate bases (the right- and left-hand orientations of the coordinate system). For the 4-gradient, the relation between the initial and reciprocal bases is given by the metric tensor $\boldsymbol{e}_\alpha = g_{\alpha\beta}\boldsymbol{e}^\beta$. Therefore, the 4-gradient in the initial basis is defined by the expression

$$\bar{\boldsymbol{\nabla}} = \boldsymbol{\nabla}(\boldsymbol{e}_0\partial_0, \bar{\boldsymbol{e}}_n\partial_n). \quad (25)$$

The analyticity condition for the function $\mathbf{G}(\mathbf{H})$ is written in the form of the equation $\bar{\boldsymbol{\nabla}}\mathbf{G}(a^0\boldsymbol{e}_0, a^n\boldsymbol{e}_n) = 0$. Direct calculations and zeroing the components of a scalar, a vector, and a bivector give

$$\partial_0 a^0 = \operatorname{div}\vec{\boldsymbol{a}};\ \partial_0\vec{\boldsymbol{a}} = \operatorname{grad} a^0;\ \operatorname{rot}\vec{\boldsymbol{a}} = 0, \quad (26)$$

where the notation $\vec{\boldsymbol{a}} = a^n\boldsymbol{e}_n$ is used. In the simple case of binary numbers, it follows from Eq. (26) that

$$\frac{\partial a^0}{\partial x^0} = \frac{\partial a^1}{\partial x^1};\ \frac{\partial a^1}{\partial x^0} = \frac{\partial a^0}{\partial x^1}. \quad (27)$$

Expression (27) corresponds to the condition of hyperbolic analyticity or $h$-analyticity for the binary numbers [16]. Therefore, we adopt that conditions (26) are the extension of $h$-analyticity onto the case of hypercomplex hyperbolic numbers. Defining the function $\mathbf{G}(\mathbf{H})$ as a vector of 4-potential $\mathbf{A}(\varphi^0\boldsymbol{e}_0, A^n\boldsymbol{e}_n)$, calculating the divergence of the second equality in Eq. (26), and substituting the result into the first equality in Eq. (26), we obtain the Maxwell equation for the scalar potential $\varphi$. An alternative way can also be used. Namely, we calculate

$$\bar{\boldsymbol{\nabla}}\bar{\mathbf{A}} = (\partial_0\varphi^0 + \operatorname{div}\vec{\mathbf{A}})\boldsymbol{e}_0 - \partial_0\vec{\mathbf{A}} - \operatorname{grad}\varphi^0 + i\operatorname{rot}\vec{\mathbf{A}}, \quad (28)$$

where $\vec{\mathbf{A}} = A^n\boldsymbol{e}_n$. Expression (28) contains the scalar $A_S = \partial_0\varphi^0 + \operatorname{div}\vec{\mathbf{A}}$, the vector $\mathbf{E} = -\partial_0\vec{\mathbf{A}} - \operatorname{grad}\varphi^0$, and the bivector $i\mathbf{H} = i\operatorname{rot}\vec{\mathbf{A}}$. If we put $A_S = 0$ (the Lorenz gauge condition) and associate $\mathbf{E}$ and $\mathbf{H}$ with the vectors of electric and magnetic fields, respectively, then expression (28) will define a complex Riemann–Silberstein vector $\mathbf{R} = \mathbf{E} + i\mathbf{H}$. It should be emphasized that the introduction of the 4-potential $\bar{\mathbf{A}}$ conjugation is associated with the necessity to equalize the orientations of basis coordinate systems in expression (28). At last, the equation $\nabla\mathbf{R} = 0$ contains the Maxwell equation for the vector $\mathbf{R}$ in the free space ($\epsilon = \mu = 1$).

## 6. Coordinate Transformation in $\mathcal{E}_2^{++}$ and in the Mirror Space

The complex Euclidean space $\mathcal{E}_2^{++}$ with the mirroring operation is, in essence, an eigenvector space of symmetry operators $\boldsymbol{e}_i$. Therefore, any vector $\boldsymbol{\gamma}$ in this space can be decomposed into any pair $(\varphi_{i+}, \varphi_{i-})$ of orthogonal eigenvectors $\boldsymbol{e}_i$,

$$\boldsymbol{\gamma}_i = \xi^i\varphi_{i+} + \eta^i\varphi_{i-}, \quad (29a)$$

and the influence of $\boldsymbol{e}_i$ on $\boldsymbol{\gamma}$ is described by the expression

$$\boldsymbol{e}_i\boldsymbol{\gamma} = \xi^i\varphi_{i+} + (-\eta^i)\varphi_{i-}, \quad (29b)$$

where $\xi^i$ and $\eta^i$ are the representation of $\boldsymbol{\gamma}$ in the corresponding basis. In Section 3, the mirror images in $\mathcal{E}_2^{++}$ that form a group of improper rotations were considered. In this section, we consider eigenrotations of the coordinate system $\mathbf{V}$ in $\mathcal{E}_2^{++}$ and in the mirror space. Subjecting both Eqs. (29) to some transformation $\mathbf{V}$, it is easy to obtain a relation between the transformation in $\mathcal{E}_2^{++}$ and the basis vectors $\boldsymbol{e}_i$ in the form

$$\boldsymbol{e}_i'\boldsymbol{\gamma}' = \xi^i\varphi_{i+}' + (-\eta^i)\varphi_{i-}', \quad (30)$$

where the transformed quantities are defined by the expressions

$$\boldsymbol{e}'_i = \mathbf{V}\boldsymbol{e}_i\mathbf{V}^{-1}\boldsymbol{\gamma}' = \mathbf{V}\boldsymbol{\gamma},$$
$$\varphi'_{i+} = \mathbf{V}\varphi_{i+}\varphi'_{i-} = \mathbf{V}\varphi_{i-}. \tag{31}$$

Let us confine the consideration here to unimodular transformations $\mathbf{V}$ with the determinant equal to $+1$. Rotations will be analyzed in 5 planes of $\mathcal{E}_2^{++}$, which are defined by the pairs of basis vectors $(\varphi_+, \varphi_-)$, $(\varphi_+, i\varphi_+)$, $(\varphi_-, \bar{i}\varphi_-)$, $(\varphi_+, \bar{i}\varphi_-)$, and $(\varphi_-, i\varphi_+)$. The notation $\bar{i}$ for the imaginary unit means that the imaginary coordinate axis $\bar{i}\varphi_-$ is oriented oppositely to the imaginary axis $i\varphi_+$ (Fig. 2). The counterclockwise transformation of the basis in the real plane $(\varphi_+, \varphi_-)$ is described by the expression

$$\varphi'_{i+} = \cos\frac{\alpha}{2}\varphi_+ + \sin\frac{\alpha}{2}\varphi_-,$$
$$\varphi'_{i-} = -\sin\frac{\alpha}{2}\varphi_+ + \cos\frac{\alpha}{2}\varphi_-. \tag{32}$$

The matrix describing the basis rotation by the angle $\alpha/2$ around the bundle of imaginary axes looks like

$$\mathbf{V}(\alpha/2) = \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}. \tag{33}$$

The transformation of the basis in the plane $(\varphi_+, \bar{i}\varphi_-)$ by the angle $\beta/2$ around the axis $\varphi_-$ toward the positive direction of the axis $\bar{i}\varphi_-$ (clockwise) looks like

$$\varphi'_{i+} = \cos\frac{\beta}{2}\varphi_+ + \sin\frac{\beta}{2}(i\varphi_-),$$
$$(i\varphi_-)' = -\sin\frac{\beta}{2}\varphi_+ + \cos\frac{\beta}{2}(i\varphi_-). \tag{34}$$

This direction of a rotation was chosen for the angle $\beta$ to be positively defined. By multiplying the second equality (34) by $-i$, the rotation around the axis $\varphi_-$ in the plane $(\varphi_+, \bar{i}i\varphi_-)$ is transformed into the corresponding rotation around the bundle of imaginary axes in the plane $(\varphi_+, \varphi_-)$. The matrix of this transformation looks like

$$\mathbf{V}(\beta/2) = \begin{pmatrix} \cos\frac{\beta}{2} & i\sin\frac{\beta}{2} \\ i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}. \tag{35}$$

The transformation of the basis in the plane $(\varphi_-, i\varphi_+)$ counterclockwise by the angle $\gamma/2$ around the axis $\varphi_+$ is described, similarly to Eq. (35), and the corresponding transformation matrix looks like

$$\mathbf{V}(\gamma/2) = \begin{pmatrix} \cos\frac{\gamma}{2} & i\sin\frac{\gamma}{2} \\ i\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}. \tag{36}$$

The transformation in the planes $(\varphi_+, i\varphi_+)$ and $(\varphi_-, \bar{i}\varphi_-)$ is a common transformation of a pair of complex numbers and is described, in the general form, by the expression

$$\mathbf{V}(k, \delta/2) = \begin{pmatrix} k\exp(i\delta/2) & 0 \\ 0 & k^{-1}\exp(-i\delta/2) \end{pmatrix}. \tag{37}$$

A transformation of this type is a homothety (compression-expansion) with a rotation in the complex plane.

To analyze the basis transformations in the mirror space, let us enumerate the basis vectors $\boldsymbol{e}_i$ so that the components of the vector $\mathbf{r} = (x, y, z)$ in the mirror space, $\mathbf{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$, would correspond to the matrix representation

$$\mathbf{r} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}. \tag{38}$$

Matrices (33) and (35)–(37) of the coordinate transformation can be expanded in the Clifford basis $(\mathbf{1}, \boldsymbol{e}_i\boldsymbol{e}_j)$, and the rotation in the mirror space can be expressed in terms of quaternionic variables $\mathbf{Q}$. For instance, rotation (33) will obtain the form

$$\mathbf{V}(\alpha/2) = \mathbf{Q}(\alpha/2) = \cos\frac{\alpha}{2}\mathbf{1} + \sin\frac{\alpha}{2}\boldsymbol{e}_3\boldsymbol{e}_1. \tag{39}$$

Since $\boldsymbol{e}'_i = \mathbf{V}\boldsymbol{e}_i\mathbf{V}^{-1}$, and since $\boldsymbol{e}_{31}$ anticommutes with $(\boldsymbol{e}_1, \boldsymbol{e}_3)$ and commutes with $\boldsymbol{e}_2$, the transformation of a vector $\mathbf{r}$ in the mirror space reads

$$\mathbf{r}' = \mathbf{Q}(\alpha)(x\boldsymbol{e}_1 + z\boldsymbol{e}_3) + y\boldsymbol{e}_2. \tag{40}$$

Expression (40) describes a rotation around the basis axis $\boldsymbol{e}_2$. Analogously to transformation (35), the rotation around the basis vector $\boldsymbol{e}_1$ takes the form

$$\mathbf{r}' = x\boldsymbol{e}_1 + \mathbf{Q}(\beta)(y\boldsymbol{e}_2 + z\boldsymbol{e}_3), \tag{41}$$

where $\mathbf{Q}(\beta) = \cos\beta\,\mathbf{1} + \sin\beta\,\boldsymbol{e}_2\boldsymbol{e}_3$. Expressions (35) and (36) are an example of the "degeneration" effect induced by a symmetry; in the specific case, this is the symmetry of imaginary axes. A similar phenomenon takes place in quantum mechanics, where the energy degeneration of the spin states can be eliminated with the help of a magnetic field owing to a decrease in the symmetry. Therefore, let us describe the rotation around the basis vector $\boldsymbol{e}_3$ by expression (37). Then, putting $k = 1$ and $\delta = \gamma$, the corresponding quaternion looks like $\mathbf{Q}(\gamma) = \cos\gamma\mathbf{1} + \sin\gamma\boldsymbol{e}_1\boldsymbol{e}_2$. If $k \neq 1$, no analogs of the transformations of a 3-dimensional space are known. However, in the space with the Minkowski metric, the variable $k$ is responsible for the boost transformation in the special theory of relativity. It should be noted that the quaternion in expressions (40) and (41) can act on the coordinates in the parentheses not only from left to right, but also from right to left. In the latter case, the signs of angles in the transformation formulas have to be changed to the opposite ones.

## 7. Discussion of the Results

1. Two- and three-dimensional spaces obtained by mirroring in the auxiliary complex space have a basis of the vector Clifford algebra in the mirror space. In turn, the basis vectors $\boldsymbol{e}_i$ of the Clifford algebra are symmetry operators describing two opposite orientations of a basis vector. These properties of the mirror images emphasize a direct relationship between the Clifford algebra and the fundamental principles of symmetry. The auxiliary complex space $\mathcal{E}_2^{++}$, which was constructed with the use of symmetry operations and in which the mirroring operations are performed, is, in essence, a space of eigenvectors for the symmetry vector-operators $\boldsymbol{e}_i$ of the Clifford basis. This space has the Euclidean metric with orthogonal basis vectors and does not fit the classical definition of spin-space [21], which has an antisymmetric metric on isotropic basis vectors. At the same time, the properties of $\mathcal{E}_2^{++}$ have a certain similarity with those of the classical spin-space. This statement concerns the identity of the expressions for coordinate transformations and their sign ambiguity. However, the introduction of the Clifford basis conjugation operation at the stage of the basis construction allowed us, in a definite sense, to solve the problem of transformation ambiguity.

2. The consistent application of the vector formalism of complex numbers in the Clifford algebra made it possible to redefine the scalar product for the Clifford basis and to correctly introduce the metric tensor $g_{\alpha\beta}$ of an orthogonal basis in the Minkowski space $M_4^{(3)}$ for hyperbolic hypercomplex numbers. For this space, the $h$-analyticity conditions were obtained for functions of a hypercomplex argument, which are similar to the Cauchy–Riemann conditions for complex numbers. Moreover, the conditions of $h$-analyticity implicitly include the Maxwell equations for a 4-potential in the free space.

3. In the vector interpretation of hypercomplex numbers, the important role is played by the orientation of a coordinate system. A change in the orientation of the system of basis vectors for hypercomplex numbers corresponds to the operation of complex number conjugation. The construction of the invariant for hypercomplex numbers–for complex numbers, this is the absolute value composed of conjugated numbers–corresponds to the composition of numbers in the left- and right-hand oriented coordinate systems. Just those reasons served as a basis to obtain the conditions of $h$-analyticity. The consideration of a coordinate system orientation also made it possible to distinguish between right- and left-handed quaternions, thereby emphasizing the vector character of the Clifford algebra.

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ДЗЕРКАЛЬНА
СИМЕТРІЯ ЯК ОСНОВА ПОБУДОВИ
ПРОСТОРОВО-ЧАСОВОГО КОНТИНУУМУ

Р е з ю м е

За допомогою дзеркального відображення 1-вимірної орієнтованої множини у спеціально створеному на основі симетрії комплексному просторі будується в задзеркаллі простір розмірності $n > 1$. Геометрія отриманого простору описується векторною алгеброю Кліффорда. На основі алгебри гіперболічних гіперкомплексних чисел будується псевдоевклідів простір з метрикою простору Мінковського. Одержано умови аналітичності функції від гіперболічного гіперкомплексного аргументу ($h$-аналітичність), в яких в неявному вигляді містяться рівняння Максвелла для 4-потенціалу у вільному просторі.