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DETECTION OF THE ENTANGLEMENT IN MANY-QUBIT QUANTUM SYSTEMS ON THE BASIS OF THE MERMIN AND ARDEHALI CRITERIA

A possibility to reveal the entanglement in generalized n-qubit two-parameter GHZ states, as well as in any n-qubit states, with the help of the Mermin and Ardehali inequalities from the collection generally called the Mermin–Ardehali–Belinskii–Klyshko inequalities has been studied. Formulas for the calculation of the Mermin and Ardehali correlation functions in any quantum n-qubit states are derived, and criteria of the violation of corresponding inequalities by specific states are obtained. A set of states that are absolutely insensitive to the Mermin and Ardehali operators is revealed. Modified Mermin and Ardehali operators are proposed, the set of which makes it possible to extend the class of n-qubit states, in which quantum correlations can be revealed.

K e y w o r d s: quantum entanglement, entanglement criteria.

1. Introduction

The multiparticle entanglement of quantum states plays an important role in both the conceptual problems of quantum-mechanical theory and the practical issues of quantum computer science, in particular, for quantum calculations, quantum cryptography, and quantum teleportation. The multiparticle entanglement has a complicated structure and, hence, is a difficult object to be studied. Numerous works were devoted to this topic in recent years. However, the creation of the theory of multiparticle entanglement still remains at an early stage. Till now, the entanglement structure has been analyzed in detail only for a few special cases of confined quantum-mechanical systems [1,2]. An interested reader can find references to papers dealing with the problem of multiparticle entanglement in works [3–5], as an example.

One of the important research directions in the theory of multiparticle entanglement is the construction of correlation functions and analogs of Bell inequalities for many-particle systems, which could be useful in the detection, both theoretically and experimentally, of the presence of the entanglement in quantum-mechanical systems. According to the Gisin theorem [6, 7], any pure two-particle entangled state violates the Bell inequality for a two-particle correlation function in the Clauser–Horne–Shimony–Holt (CHSH) form [8].

In work [9], two theorems, which generalize the Gisin theorem to the case of three-qubit systems, were formulated.

- *Theorem 1.* All generalized Greenberger–Horne–Zeilinger (GHZ) states of three-qubit systems violate the Bell inequalities for probabilities.
- *Theorem 2.* All pure two-particle entangled states of three-qubit systems violate the Bell inequalities for probabilities.

In order to generalize the Bell CHSH inequality to the case of multiqubit quantum systems, a set of inequalities of the same type was proposed [10–12]. They were called the Mermin–Ardehali–Belinskii–Klyshko (MABK) inequalities. An advantage of the

MABK inequalities consists in that their violation by n-particle GHZ states [13] grows exponentially with an increase of the number of qubits in the system, n, thereby demonstrating that, in the general case, there is no restriction on the ratio demonstrating how much the magnitude of quantum correlations can exceed the limiting values of correlations calculated in the framework of the local realism theory. This property is of large importance, because the experimental testing of the violations of MABK inequalities would ultimately resolve the issue concerning the groundlessness of the so-called "realistic local theories with latent parameters".

While attempting to apply the MABK inequalities to many-qubit systems, Scarani and Gisin [14] unexpectedly found that the generalized GHZ states

$$|\Psi\rangle = \cos\alpha|00\,...\,0\rangle + \sin\alpha|11\,...\,1\rangle \tag{1}$$

do not violate the MABK inequalities, if the parameter α satisfies the condition $\sin 2\alpha \leq \frac{1}{\sqrt{2^{n-1}}}$. This result was obtained numerically for $n = 3$, 4, and 5; and an assumption was made that it should also be true for $n > 5$. The result obtained seemed to be unexpected, because state (1) with $n = 2$ violates the CHSH inequalities at all allowed α-values. On the basis of this fact, Scarani and Gisin drew conclusion that the MABK inequalities – moreover, inequalities with two possible measurement values for every qubit – are not a natural generalization of the CHSH inequalities to the case of systems with the qubit number $n > 2$.

In this work, we come back to the MABK inequalities in order to *ultimately* divide all pure n-qubit states into two groups: the states that violate the MABK inequalities and the states that may probably be entangled, but do not violate them. Using a simple transparent formalism for the description of the operators of corresponding correlation functions, we study firstly the class of generalized GHZ states. Then, we extend our analysis on any n-qubit states. In so doing, under the term "generalized GHZ states", we understand not one-parameter states (1), but a wider class of two-parameter states

$$|\chi\rangle = \cos\frac{\Theta}{2}|00\,...\,0\rangle + e^{i\phi}\sin\frac{\Theta}{2}|11\,...\,1\rangle$$

or, equivalently,

$$|\chi\rangle = \cos\frac{\Theta}{2}|\uparrow\uparrow\,...\,\uparrow\rangle + e^{i\phi}\sin\frac{\Theta}{2}|\downarrow\downarrow\,...\,\downarrow\rangle. \tag{2}$$

The state parametrization ([2]) is similar to that of one-qubit states, in which every pair value (Θ, ϕ) corresponds to a point on the Bloch sphere. State (2) is often called the "logic qubit" in the theory of error correction at the qubit passage in quantum channels.

Generally speaking about the MABK inequalities, we will study the Mermin inequality in the form proposed in work [10], as well as the inequality proposed by Ardehali [11]. Those inequalities are similar to each other, although they are different by the corresponding specific expressions for the operators of correlation functions and the forms of GHZ states, which they will be applied to.

Our task consists in the following: on the basis of quantum-mechanical theory,

1) to order the structure of operators of the Mermin and Ardehali correlation functions and to derive the results in the analytic form;

2) to calculate the values of the Mermin and Ardehali correlation functions in generalized GHZ states [2];

3) to compare the obtained results with the results of calculations carried out in the framework of the so-called local realism theory and to clarify, at which values of the parameters Θ and ϕ, as well as to what extent, the corresponding inequalities are violated;

4) to repeat analogous researches in the case where any n-qubit states rather than states (2) are considered.

In essence, the Mermin or Ardehali correlation function determines the planning of an experiment aimed at the registration of particle states. Therefore, our calculations should answer the following question: Can such experiments reveal, in principle, the presence or absence of the entanglement in a prepared many-qubit quantum state?

Concerning each of the Bell inequality proposed for many-qubit systems, the attempt to obtain an irrefragable answer to the question "What can and what cannot the operator of correlation function (the operator $\hat{A}$) for the corresponding inequality reveal?" is quite natural. More specifically:

• Is there a simple criterion to determine, which states from the whole set of multiqubit states violate the given inequality and which do not? Furthermore, if the analyzed state violates the inequality, what is the degree of this violation?

• Is it possible to reveal a class of wave functions for qubits, for which the operator A cannot reveal correlations, although the latter do exist?

• Is it possible to indicate those states, which violate the inequality as much as possible?

• Can the given inequality be considered as a generalization of the Bell inequality in the CHSH form to many-qubit systems?

• Full answers to the questions formulated above will allow conclusions to be made concerning the expediency of the application of the corresponding inequality to specific many-qubit systems.

2. Mermin Inequality

In his work [10], N.D. Mermin proposed a generalization of the Bell inequality for a system of n particles with the spin $s = \frac{1}{2}$. He proved that quantum mechanics violates this inequality, with the violation degree growing exponentially with the number of particles in the system for GHZ states. The state of a n-qubit quantum system proposed by Mermin looks like

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow ... \uparrow\rangle + i|\downarrow\downarrow ... \downarrow\rangle) \tag{3}$$

or, in a more modern notation,

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00 ... 0\rangle + i|11 ... 1\rangle).$$

State (3) is a special case of state (2) for the parameters $\Theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$. On the Bloch sphere, this is a point, where the sphere is intersected by the axis Oy.

In quantum mechanics, the correlation function in the Mermin inequality $\langle\hat{A}\rangle < 1$ corresponds to the Hermitian operator

$$\hat{A} = \frac{1}{2i}\left\{\prod_{j=1}^{n}(\sigma_x^j + i\sigma_y^j) - \prod_{j=1}^{n}(\sigma_x^j - i\sigma_y^j)\right\}, \tag{4}$$

where σ_x and σ_y are the Pauli operators, and the superscript j enumerates qubits (particles).

In terms of a real experiment, Mermin describes the situation as follows [10]: n particles fly away from a common source, in which the quantum system of particles has been prepared in state (3). The state of each particle is considered in its own coordinate system, where the axis Z can be taken along the particle motion direction, whereas the axes X and Y have arbitrary directions orthogonal to each other and to the particle motion direction (one may consider that the state of each particle is examined in its arbitrary own Cartesian coordinate system). For each particle, there is a device that measures the value of spin projection on the axis X or Y. The results of measurements are used to calculate the correlation function, which corresponds to operator (4) in quantum mechanics.

Mermin showed [10] that, in the framework of the local realism theory, the absolute value of correlation function F satisfies the inequality

$$F \le F_{\max} = \begin{cases} 2^{\frac{n}{2}} & \text{if } n \text{ is even}, \\ 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd}. \end{cases}$$

At the same time, the calculation carried out according to the rules of quantum-mechanical theory brings about the value

$$F_{\Phi_1} = |\langle\Phi_1|\hat{A}|\Phi_1\rangle| = 2^{n-1}.$$

Therefore, for all $n > 2$, the inequality $F_{\Phi_1} > F_{\max}$ takes place, and the ratio between those quantities,

$$K = \frac{F_{\Phi_1}}{F_{\max}} = \begin{cases} 2^{\frac{n}{2}-1} & \text{if } n \text{ is even}, \\ 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd}, \end{cases}$$

exponentially grows with the increase of n.

In this work, Mermin's results obtained for the F- and $F_{\max}$-values are taken as a fact. Below, they are compared with the results obtained in the framework of quantum mechanics.

In order to calculate the correlation function in the generalized GHZ and arbitrary states, as well as to make expressions more compact and calculations more convenient, let us introduce the following notations:

$$\sigma_+ = \sigma_x + i\sigma_y, \quad \sigma_- = \sigma_x - i\sigma_y,$$

$$\Sigma_+ = \prod_{j=1}^{n}\sigma_+^j, \qquad \Sigma_- = \prod\nolimits_{j=1}^{n}\sigma_-^j,$$

$$|\Uparrow\rangle = |\uparrow\uparrow ... \uparrow\rangle, \ |\Downarrow\rangle = |\downarrow\downarrow ... \downarrow\rangle.$$

Note that the orthonormal character of the $|\uparrow\rangle$ and $|\downarrow\rangle$ states gives rise to the orthonormal character of the $|\Uparrow\rangle$ and $|\Downarrow\rangle$ states:

$$\langle\Uparrow|\Uparrow\rangle = \langle\Downarrow|\Downarrow\rangle = 1, \quad \langle\Downarrow|\Uparrow\rangle = \langle\Uparrow|\Downarrow\rangle = 0.$$

In terms of new notations, the state $|\Phi_1\rangle$ [Eq. (3)] and the operator $\hat{A}$ [Eq. (4)] look like

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + i|\Downarrow\rangle),$$
$$\hat{A} = \frac{1}{2i}(\Sigma_+ - \Sigma_-).$$

From the properties of the operators σ_+ and σ_-,

$$\sigma_+|\uparrow\rangle = 0, \quad \sigma_-|\uparrow\rangle = 2|\downarrow\rangle,$$
$$\sigma_+|\downarrow\rangle = 2|\uparrow\rangle, \quad \sigma_-|\downarrow\rangle = 0,$$

we obtain that

$$\Sigma_+|\Uparrow\rangle = \prod_{j=1}^{n}(\sigma_+^j|\uparrow_j\rangle) = 0,$$
$$\Sigma_+|\Downarrow\rangle = \prod_{j=1}^{n}(\sigma_+^j|\downarrow_j\rangle) = 2^n|\Uparrow\rangle,$$
$$\Sigma_-|\Uparrow\rangle = 2^n|\Downarrow\rangle, \quad \Sigma_-|\Downarrow\rangle = 0.$$

Now, it is easy to show that $|\Phi_1\rangle$ is an eigenvector of the operator $\hat{A}$ and corresponds to the eigenvalue $\lambda = 2^{n-1}$:

$$\hat{A}|\Phi_1\rangle = \frac{1}{2i}(\Sigma_+ - \Sigma_-)\frac{1}{\sqrt{2}}(|\Uparrow\rangle + i|\Downarrow\rangle) =$$
$$= \frac{1}{2\sqrt{2}}(\Sigma_+|\Downarrow\rangle + i\Sigma_-|\Uparrow\rangle) = 2^{n-1}|\Phi_1\rangle.$$

Hence, the corresponding correlation function in the state $\Phi_1\rangle$ equals

$$|F_{\Phi_1}| = |\langle\Phi_1|\hat{A}|\Phi_1\rangle| = 2^{n-1},$$

which coincides with the result obtained by Mermin in a different way.

One can easily get convinced that the state

$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle - i|\Downarrow\rangle)$$

is also an eigenvector of the operator $\hat{A}$ and corresponds to the eigenvalue $\lambda = -2^{n-1}$:

$$\hat{A}|\Phi_2\rangle = -2^{n-1}|\Phi_2\rangle$$

so that

$$|F_{\Phi_2}| = |\langle\Phi_2|\hat{A}|\Phi_2\rangle| = 2^{n-1}.$$

Now, let us consider the average value of the operator $\hat{A}$ in an arbitrary generalized GHZ state (2):

$$F_\chi = \bar{A}_\chi = \langle\chi|\hat{A}|\chi\rangle =$$
$$= (\cos\frac{\Theta}{2}\langle\Uparrow| + e^{-i\phi}\sin\frac{\Theta}{2}\langle\Downarrow|)\frac{1}{2i}(\Sigma_+ - \Sigma_-) \times$$
$$\times (\cos\frac{\Theta}{2}|\Uparrow\rangle + e^{i\phi}\sin\frac{\Theta}{2}|\Downarrow\rangle) = 2^{n-1}\sin\Theta\sin\phi. \quad (5)$$

The obtained general expression gives the separate variants obtained above:

at $\Theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$,

$$|\chi\rangle \to |\Phi_1\rangle, \; F_\chi \to F_{\Phi_1} = 2^{n-1};$$

and, at $\Theta = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$,

$$|\chi\rangle \to |\Phi_2\rangle, \; F_\chi \to F_{\Phi_2} = -2^{n-1}.$$

Now, let us consider the GHZ states:

$$|\chi_1\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + |\Downarrow\rangle),$$
$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle).$$

They are special cases of state (2) and correspond to the parameters ($\Theta_1 = \frac{\pi}{2}, \phi_1 = 0$) and ($\Theta = \frac{\pi}{2}$, $\phi_2 = \pi$), respectively. Hence, Eq. (5) yields

$$F_{\chi_1} = \langle\chi_1|\hat{A}|\chi_1\rangle = 2^{n-1}\sin\frac{\pi}{2}\sin 0 = 0,$$
$$F_{\chi_2} = \langle\chi_2|\hat{A}|\chi_2\rangle = 2^{n-1}\sin\frac{\pi}{2}\sin\pi = 0.$$

The value of F also equals zero for all Θ-values, if $\phi = 0$ or π.

Now, the first conclusions can be drawn:

1. The quantum-mechanical values of the Mermin correlation function in the generalized GHZ state (2) are determined by formula (5).
2. In the class of all generalized GHZ states, the Mermin correlation function is maximum by the absolute value for the states

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + i|\Downarrow\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle - i|\Downarrow\rangle).$$

3. The Mermin operator is absolutely insensitive to the correlations in states (2) with $\phi = 0$ or π.

For state (2), the domain of Θ- and ϕ-values, for which the Mermin inequality is not obeyed, is determined by the inequalities

$$\begin{aligned} &|\sin\Theta\sin\phi| > \frac{1}{\sqrt{2^{n-1}}} \quad \text{if } n \text{ is odd,} \\ &|\sin\Theta\sin\phi| > \frac{1}{\sqrt{2^{n-2}}} \quad \text{if } n \text{ is even.} \end{aligned} \tag{6}$$

One should pay attention that, for two n-values that differ from each other by unity, e.g., $n_1 = 2k-1$ and $n_2 = 2k$, inequality (6) takes the same form:

$$|\sin\Theta\sin\phi| > \frac{1}{\sqrt{2^{2(k-1)}}}.$$

For example, for $n = 3$ and 4, inequalities (6) are the same:
$|\sin\Theta\sin\phi| > \frac{1}{2}$.

Let the point M on the Bloch sphere be determined by its Cartesian coordinates $\{x_m, y_m, z_m\}$. Then, from Eq. (6), it follows that the Mermin inequality becomes violated at

$$\begin{aligned} &|y_m| > \frac{1}{\sqrt{2^{n-1}}} \quad \text{if } n \text{ is odd,} \\ &|y_m| > \frac{1}{\sqrt{2^{n-2}}} \quad \text{if } n \text{ is even.} \end{aligned}$$

In other words, in the case of odd n, the planes $y_m = \frac{1}{\sqrt{2^{n-1}}}$ and $y_m = -\frac{1}{\sqrt{2^{n-1}}}$ cut off regions from the Bloch sphere, and the points in those regions correspond to states (2) that violate the Mermin inequality. For even n, this role is played by the planes $y_m = \frac{1}{\sqrt{2^{n-2}}}$ and $y_m = -\frac{1}{\sqrt{2^{n-2}}}$.

Figures 1 to 3 demonstrate the corresponding regions on the Bloch sphere for $n = 3$ and 4 (Fig. 1), 5 and 6 (Fig. 2), and 9 and 10 (Fig. 3). A comparison of the figures makes it evident that the distinguished regions increase with the growth of n, so that the whole Bloch sphere becomes filled at $n \to \infty$.

Let us express the generalized GHZ states in the form

$$|\chi\rangle = \alpha|\Uparrow\rangle + \beta|\Downarrow\rangle, \; |\alpha|^2 + |\beta|^2 = 1, \tag{7}$$

where the coefficient α is considered to be a real number. Then the correlation function looks like

$$F_\chi = \langle\chi|\hat{A}|\chi\rangle = 2^n \alpha \, \mathrm{Im}(\beta).$$

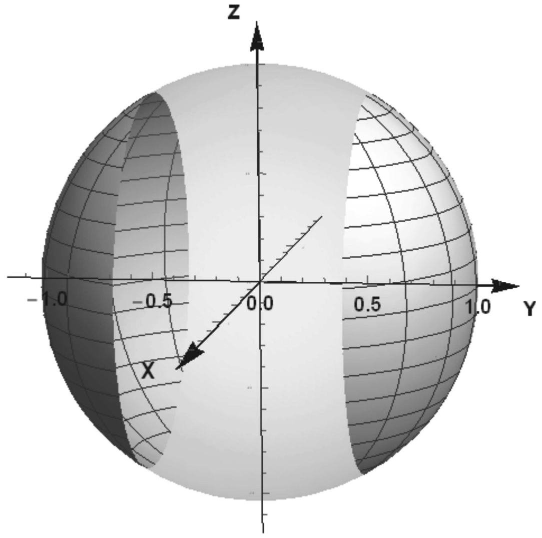

Fig. 1. Regions on the Bloch sphere, where the corresponding states violate the Mermin inequality for the case of 3 and 4 qubits

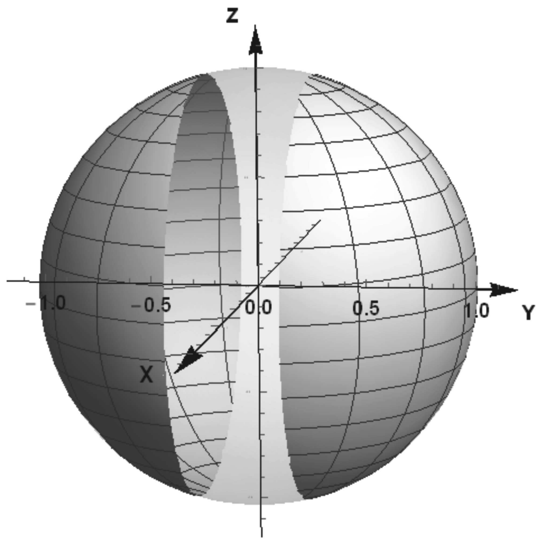

Fig. 2. The same as in Fig. 1, but for the case of 5 and 6 qubits

Hence, we obtain that F_χ is distinct from zero, only if the parameter β has an imaginary part.

Let us consider a more general situation. Let an arbitrary n-qubit state $|\Psi\rangle$ be given. What is the value $F_\Psi = \langle\Psi|\hat{A}|\Psi\rangle$ of the Mermin correlation function in this state? Does this state violate the Mermin inequality?

In the 2^n-dimensional Hilbert space of n-qubit states, let us choose a basis, each state of which is

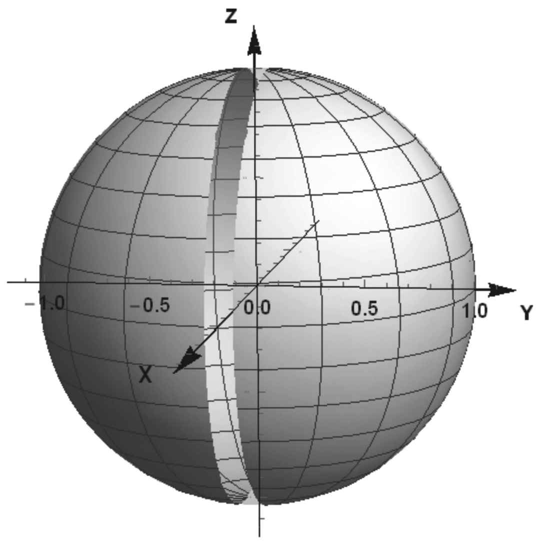

Fig. 3. The same as in Fig. 1, but for the case of 9 and 10 qubits

the product of one-qubit states $|\uparrow\rangle$ and $|\downarrow\rangle$ (or $|0\rangle$ and $|1\rangle$):

$$\begin{array}{lcl} |\xi_1\rangle = |\uparrow\uparrow ... \uparrow\uparrow\rangle, & & |\xi_1\rangle = |00\,...\,00\rangle, \\ |\xi_2\rangle = |\uparrow\uparrow ... \uparrow\downarrow\rangle, & \text{or} & |\xi_2\rangle = |00\,...\,01\rangle, \\ |\xi_3\rangle = |\uparrow\uparrow ... \downarrow\uparrow\rangle, & & |\xi_3\rangle = |00\,...\,10\rangle, \\ \cdots & & \cdots \\ |\xi_{2^n}\rangle = |\downarrow\downarrow ... \downarrow\downarrow\rangle & & |\xi_{2^n}\rangle = |11\,...\,11\rangle. \end{array}$$

This basis is usually called the standard or computational basis. The expansion of an arbitrary vector $|\Psi\rangle$ describing an n-qubit state in this complete basis can be presented in the form

$$|\Psi\rangle = a|\chi\rangle + b|\tilde{\chi}\rangle, \tag{8}$$

where

$$\begin{aligned} &|\chi\rangle = \alpha|\uparrow\uparrow ... \uparrow\rangle + \beta|\downarrow\downarrow ... \downarrow\rangle = \alpha|\Uparrow\rangle + \beta|\Downarrow\rangle, \\ &|\alpha|^2 + |\beta|^2 = 1, \end{aligned} \tag{9}$$

and

$$|\tilde{\chi}\rangle = \sum_{i=2}^{2^n-1} \gamma_i|\xi_i\rangle, \quad \sum_{i=2}^{2^n-1} |\gamma_i|^2 = 1.$$

If $|a|^2 + |b|^2 = 1$, then $|\Psi\rangle$ is normalized to unity. In each basis state $|\xi_i\rangle$ (here, $i = 2, 3, ..., 2^n - 1$), the "spin projection" of at least one particle is opposite to the "spin projections" of other particles. As a consequence, we have

$$\Sigma_+|\xi_i\rangle = 0, \quad \Sigma_-|\xi_i\rangle = 0, \quad i = 2, 3, ..., 2^n - 1.$$

Therefore,

$$\Sigma_+|\tilde{\chi}\rangle = \Sigma_-|\tilde{\chi}\rangle = \hat{A}|\tilde{\chi}\rangle = 0.$$

Then we evidently obtain that

$$\hat{A}|\Psi\rangle = a\hat{A}|\chi\rangle$$

and

$$F_\Psi = \langle\Psi|\hat{A}|\Psi\rangle = |a|^2\langle\chi|\hat{A}|\chi\rangle.$$

Expressing the vector $|\chi\rangle$ in form (2), we have

$$F_\Psi = |a|^2 2^{n-1}\sin\Theta\sin\phi, \quad a = \langle\chi|\Psi\rangle.$$

The state $|\Psi\rangle$ can also be presented in the form

$$|\Psi\rangle = a_1|\Uparrow\rangle + a_2|\Downarrow\rangle + b|\tilde{\chi}\rangle,$$

where

$$a_1 = \langle\Uparrow|\Psi\rangle = |a_1|e^{i\phi_1}, \quad a_2 = \langle\Downarrow|\Psi\rangle = |a_2|e^{i\phi_2}. \tag{10}$$

Then

$$|\Psi\rangle = e^{i\phi_1}(|a_1|\cdot|\Uparrow\rangle + e^{i(\phi_2-\phi_1)}|a_2|\cdot|\Downarrow\rangle) + b|\tilde{\chi}\rangle. \tag{11}$$

A comparison of Eqs. (11) and (8) with regard for Eqs. (2), (9), and (10) allows the following relations between various parameters to be written down:

$$\begin{aligned} &\phi = \phi_2 - \phi_1, \quad a = e^{i\phi_1}\sqrt{|a_1|^2 + |a_2|^2}, \\ &\alpha = \frac{|a_1|}{\sqrt{|a_1|^2 + |a_2|^2}}, \quad \beta = e^{i\phi}\frac{|a_2|}{\sqrt{|a_1|^2 + |a_2|^2}}, \\ &\cos\frac{\Theta}{2} = \frac{|a_1|}{\sqrt{|a_1|^2 + |a_2|^2}}, \quad \sin\frac{\Theta}{2} = \frac{|a_2|}{\sqrt{|a_1|^2 + |a_2|^2}}, \\ &\sin\Theta = \frac{2|a_1a_2|}{|a_1|^2 + |a_2|^2}. \end{aligned}$$

Finally, let us propose a simple algorithm for the calculation of the Mermin correlation function for an arbitrary vector of the n-qubit state $|\Psi\rangle$:

1. For the given $|\Psi\rangle$, we find $a_1 = \langle\Uparrow|\Psi\rangle$ and $a_2 = \langle\Downarrow|\Psi\rangle$.
2. Then we determine $|a_1|$, $|a_2|$, $\phi_1 = \arg(a_1)$, $\phi_2 = \arg(a_2)$, and $\phi = \phi_1 - \phi_2$.
3. Afterward, we obtain $\sin\Theta = \frac{2|a_1a_2|}{|a_1|^2+|a_2|^2}$.
4. Finally, we calculate

$$\begin{aligned} &F_\Psi = \langle\Psi|\hat{A}|\Psi\rangle = (|a_1|^2 + |a_2|^2)2^{n-1}\sin\Theta\sin\phi = \\ &= 2^n|a_1a_2|\sin\phi. \end{aligned}$$

In this case, the criterion that the state $|\Psi\rangle$ violates the Mermin inequality looks like

$$\left|\frac{F_\Psi}{F_{\max}}\right| > 1 \Rightarrow \begin{cases} 2^{\frac{n+1}{2}}\,|a_1a_2\sin\phi| > 1 & \text{if } n \text{ is odd,} \\ 2^{\frac{n}{2}}\,|a_1a_2\sin\phi| > 1 & \text{if } n \text{ is even.} \end{cases} \quad (12)$$

The obtained relations completely resolve the problem about a possibility to reveal the entanglement (or non-locality) in n-qubit systems with the use of the Mermin inequality. From Eq. (12), it follows that, if n is large enough, every state $|\Psi\rangle$ with $\phi \neq 0$ or π, but simultaneously with $a_1 \neq 0$ and $a_2 \neq 0$, violates the Mermin inequality. If at least one of the equalities $\phi = 0$, $\phi = \pi$, $a_1 = 0$, or $a_2 = 0$ takes place, the Mermin operator (4) cannot reveal correlations. Of all n-qubit states, the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ violate the Mermin inequality to the largest extent. If the state $|\Psi\rangle$ does not violate the Mermin inequality, this fact does not testify that it is free of entanglement. This is only the evidence that the Mermin operator $\hat{A}$ cannot reveal the presence or absence of the entanglement in this state in principle.

3. Ardehali Inequality

An inequality for many-qubit systems was proposed in Ardehali's work [11]. It is similar to the Mermin equality considered in the previous section. Owing to this similarity, some details will be omitted in this section to avoid repetitions. Nevertheless, the notations from the previous section will be retained. Our purpose is also to find violation criteria for the Ardehali inequality: first, in the class of generalized GHZ states, and then for an arbitrary n-qubit state.

The operator of the n-qubit Ardehali correlation function looks like

$$\hat{A} = \hat{A}_1 + \hat{A}_2,$$

where

$$\begin{aligned} \hat{A}_1 = (&-\sigma_x^1\sigma_x^2\sigma_x^3\ldots\sigma_x^{n-1} + \sigma_y^1\sigma_y^2\sigma_x^3\ldots\sigma_x^{n-1} + \ldots - \\ &-\sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4\sigma_x^5\ldots\sigma_x^{n-1} - \ldots + \\ &+\sigma_y^1\ldots\sigma_y^6\sigma_x^7\ldots\sigma_x^{n-1} + \ldots - \ldots)(\sigma_a^n - \sigma_b^n), \\ \hat{A}_2 = (&\sigma_y^1\sigma_x^2\ldots\sigma_x^{n-1} + \ldots - \sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\ldots\sigma_x^{n-1} - \ldots + \\ &+\sigma_y^1\ldots\sigma_y^5\sigma_x^6\ldots\sigma_x^{n-1} + \ldots - \\ &-\sigma_y^1\ldots\sigma_y^7\sigma_x^8\ldots\sigma_x^{n-1} - \ldots + \ldots)(\sigma_a^n + \sigma_b^n), \end{aligned} \quad (13)$$

the meaning of the plus and minus signs is explained in Appendix, and σ_x^j and σ_y^j are the Pauli operators for the j-th particle. The operators σ_a and σ_b are defined by the following expressions:

$$\begin{aligned} &\sigma_a = \boldsymbol{\sigma}\cdot\mathbf{a} = \boldsymbol{\sigma}\frac{1}{\sqrt{2}}(\mathbf{e}_x + \mathbf{e}_y) = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \\ &\sigma_b = \boldsymbol{\sigma}\cdot\mathbf{b} = \boldsymbol{\sigma}\frac{1}{\sqrt{2}}(-\mathbf{e}_x + \mathbf{e}_y) = \frac{1}{\sqrt{2}}(-\sigma_x + \sigma_y), \\ &\sigma_a + \sigma_b = \sqrt{2}\sigma_y = \frac{1}{i\sqrt{2}}(\sigma_+ - \sigma_-), \\ &\sigma_a - \sigma_b = \sqrt{2}\sigma_x = \frac{1}{\sqrt{2}}(\sigma_+ + \sigma_-), \end{aligned}$$

where $\sigma_+ = \sigma_x + i\sigma_y$ and $\sigma_- = \sigma_x - i\sigma_y$ are the operators, whose properties were described in the previous section. The unit vectors $\mathbf{a}$ and $\mathbf{b}$ are oriented in the plane Oxy and form angles of 45° and 135°, respectively, with the axis Ox; i.e.

$$\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{e}_x + \mathbf{e}_y), \quad \mathbf{b} = \frac{1}{\sqrt{2}}(-\mathbf{e}_x + \mathbf{e}_y).$$

This structure of the operator $\hat{A}$ corresponds to an experiment, in which the spin projections on the directions of the axes X and Y are measured for $n-1$ particles, and the spin projections on the directions of the unit vectors $\mathbf{a}$ and $\mathbf{b}$ are measured for the n-th particle.

M. Ardehali considered an n-qubit state corresponding to the vector

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\ldots\uparrow\rangle - |\downarrow\downarrow\ldots\downarrow\rangle),$$

or

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle) \quad (14)$$

in the notations adopted above. M. Ardehali showed that, in the framework of the local realism theory, the correlation function that corresponds to the operator $\hat{A}$ [Eq. (13)] is limited by the values

$$F \le F_{\max} = \begin{cases} 2^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 2^{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

At the same time, the calculation carried out according to the rules of quantum-mechanical theory brings about the value

$$F_{\chi_2} = \langle\chi_2|\hat{A}|\chi_2\rangle = 2^{n-\frac{1}{2}}.$$

Therefore, the quantum-mechanical value of F_{χ_2} exceeds the F-value by a factor of K, where

$$K = \frac{F_{\chi_2}}{F_{\max}} = \begin{cases} 2^{\frac{n-1}{2}} & \text{if } n \text{ is even,} \\ 2^{\frac{n}{2}-1} & \text{if } n \text{ is odd.} \end{cases} \quad (15)$$

We should consistently perform the following operations:

1. Operator (13) is transformed to a form that is similar to the Mermin operator (see the previous section). Below, we will show in a simple way that $F_{\chi_2} = 2^{n-\frac{1}{2}}$.

2. The quantum-mechanical value of the correlation function is calculated for the generalized GHZ state [Eqs. (2) and (7)]:

$$\begin{cases} |\chi\rangle = \alpha|\Uparrow\rangle + \beta|\Downarrow\rangle, \quad |\alpha|^2 + |\beta|^2 = 1 \\ |\chi\rangle = \cos\dfrac{\Theta}{2}|\Uparrow\rangle + e^{i\phi}\sin\dfrac{\Theta}{2}|\Downarrow\rangle. \end{cases}$$

3. The values of the correlation function are determined in arbitrary n-qubit states $|\Psi\rangle$.

4. A set of n-qubit states that violate the Ardehali inequality is found, and the degree of this violation is determined.

Now, the operators $\hat{A}_1$ and $\hat{A}_2$ [Eq. (13)] can be presented in the following form (see Appendix):

$$\hat{A}_1 = -\frac{1}{2}\left(\prod_{j=1}^{n-1}\sigma_+^j + \prod_{j=1}^{n-1}\sigma_-^j\right)\frac{1}{\sqrt{2}}(\sigma_+^n + \sigma_-^n),$$

$$\hat{A}_2 = \frac{1}{2i}\left(\prod_{j=1}^{n-1}\sigma_+^j - \prod_{j=1}^{n-1}\sigma_-^j\right)\frac{1}{i\sqrt{2}}(\sigma_+^n - \sigma_-^n),$$

$$\hat{A} = \hat{A}_1 + \hat{A}_2 = -\frac{1}{2\sqrt{2}}\left\{\left(\prod_{j=1}^{n-1}\sigma_+^j + \prod_{j=1}^{n-1}\sigma_-^j\right)\times\right.$$
$$\left.\times(\sigma_+^n + \sigma_-^n) + \left(\prod_{j=1}^{n-1}\sigma_+^j - \prod_{j=1}^{n-1}\sigma_-^j\right)(\sigma_+^n - \sigma_-^n)\right\} =$$
$$= -\frac{1}{2\sqrt{2}}\left\{\prod_{j=1}^{n}\sigma_+^j + \prod_{j=1}^{n-1}\sigma_-^j\sigma_+^n + \prod_{j=1}^{n-1}\sigma_+^j\sigma_-^n +\right.$$
$$+\prod_{j=1}^{n}\sigma_-^j + \prod_{j=1}^{n}\sigma_+^j - \prod_{j=1}^{n-1}\sigma_-^j\sigma_+^n -$$
$$\left.-\prod_{j=1}^{n-1}\sigma_+^j\sigma_-^n + \prod_{j=1}^{n}\sigma_-^j\right\}.$$

After cancellations, we obtain

$$\hat{A} = -\frac{1}{\sqrt{2}}\left(\prod_{j=1}^{n}\sigma_+^j + \prod_{j=1}^{n}\sigma_-^j\right),$$

or, using the notations given above,

$$\hat{A} = -\frac{1}{\sqrt{2}}(\Sigma_+ + \Sigma_-).$$

The obtained expression for the operator $\hat{A}$ is very simple and makes it possible to easily verify that the state vector $|\chi_2\rangle$ [Eq. (14)] is an eigenvector of this operator:

$$\hat{A}|\chi_2\rangle = -\frac{1}{\sqrt{2}}(\Sigma_+ + \Sigma_-)\frac{1}{\sqrt{2}}(|\Uparrow\rangle - |\Downarrow\rangle) =$$
$$\frac{1}{2}(\Sigma_+|\Downarrow\rangle - \Sigma_-|\Uparrow\rangle) = 2^{n-1}(|\Uparrow\rangle - |\Downarrow\rangle) = 2^{n-\frac{1}{2}}|\chi_2\rangle,$$

Therefore, we obtain $F_{\chi_2} = \langle\chi_2|\hat{A}|\chi_2\rangle = 2^{n-\frac{1}{2}}$. This is no more than the Ardehali result, but calculated in a different way.

A simple test demonstrates that $|\chi_1\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\rangle + |\Downarrow\rangle)$ is also an eigenvector of the operator $\hat{A}$:

$$\hat{A}|\chi_1\rangle = -2^{n-\frac{1}{2}}|\chi_1\rangle,$$

so that the value of the correlation function is equal to

$$F_{\chi_1} = \langle\chi_1|\hat{A}|\chi_1\rangle = -2^{n-\frac{1}{2}}$$

in this state and to

$$F_\chi = \langle\chi|\hat{A}|\chi\rangle = -2^{n+\frac{1}{2}}\alpha\mathrm{Re}\beta = -2^{n-\frac{1}{2}}\sin\Theta\cos\phi$$

in the generalized GHZ state (2). The vectors $|\chi_1\rangle$ and $|\chi_2\rangle$ are special cases of the state $|\chi\rangle$: at $\Theta = \frac{\pi}{2}$ and $\phi = \pi$,

$$|\chi\rangle \to |\chi_2\rangle,$$
$$F_\chi \to F_{\chi_2} = -2^{n-\frac{1}{2}}\sin\frac{\pi}{2}\cos\pi = 2^{n-\frac{1}{2}};$$

and at $\Theta = \frac{\pi}{2}$ and $\phi = 0$,

$$|\chi\rangle \to |\chi_1\rangle,$$
$$F_\chi \to F_{\chi_1} = -2^{n-\frac{1}{2}}\sin\frac{\pi}{2}\cos 0 = -2^{n-\frac{1}{2}}.$$

The condition that the state $|\chi\rangle$ violates the Ardehali inequality looks like

$$K = \left|\frac{F_\chi}{F_{\max}}\right| > 1 \quad\Rightarrow$$
$$\Rightarrow \begin{cases} 2^{\frac{n-1}{2}}\,|\sin\Theta\cos\phi| > 1 & \text{if } n \text{ is even,} \\ 2^{\frac{n}{2}-1}\,|\sin\Theta\cos\phi| > 1 & \text{if } n \text{ is odd.} \end{cases}$$

The regions on the Bloch sphere corresponding to the states $|\chi\rangle$ that violate the Ardehali inequality are cut off by planes oriented perpendicularly to the axis

Ox, unlike the case of Mermin's inequality, when the corresponding regions are cut off by planes that are oriented perpendicularly to the axis Oy (see Figs. 1 to 3). As the number of particles n grows, the corresponding regions also increase and, as $n \to \infty$, occupy the whole Bloch sphere.

Now, we determine the value $F_\Psi = \langle\Psi|\hat{A}|\Psi\rangle$ for an arbitrary n-qubit state $|\Psi\rangle$. Representing the latter in form (8), taking into account that $\hat{A}|\tilde{\chi}\rangle = 0$, and making allowance for relation (11) between the parameters, we obtain

$$F_\Psi = -2^{n+\frac{1}{2}}|a_1 a_2|\cos\phi$$

or, equivalently,

$$F_\Psi = -2^{n-\frac{1}{2}}(|a_1|^2 + |a_2|^2)\sin\Theta\cos\phi.$$

Then the condition that the given n-qubit state violates the Ardehali inequality looks like

$$\left|\frac{F_\Psi}{F_{\max}}\right| > 1 \Rightarrow \begin{cases} 2^{\frac{n+1}{2}}|a_1 a_2\cos\phi| > 1 & \text{if } n \text{ is even,} \\ 2^{\frac{n}{2}}|a_1 a_2\cos\phi| > 1 & \text{if } n \text{ is odd;} \end{cases}$$

or, equivalently,

$$\begin{cases} 2^{\frac{n-1}{2}}(|a_1|^2 + |a_2|^2)|\sin\Theta\cos\phi| > 1 \text{ if } n \text{ is even,} \\ 2^{\frac{n}{2}-1}(|a_1|^2 + |a_2|^2)|\sin\Theta\cos\phi| > 1 \text{ if } n \text{ is odd.} \end{cases} \quad (16)$$

The conclusions that follow from inequalities (15) and (16) are analogous to those made above for the Mermin inequality. (i) At large enough n-values, all $|\Psi\rangle$-states, for which $Q = (|a_1|^2 + |a_2|^2)\times$ $\times|\sin\Theta\cos\phi| \neq 0$, violate the Ardehali inequality. (ii) In the states, for which $Q = 0$, the Ardehali operator is absolutely incapable of revealing the correlations. (iii) Of all n-qubit states, the states $|\chi_1\rangle$ and $|\chi_2\rangle$ violate the Ardehali inequality most strongly.

Now, let us try to extend the class of vectors corresponding to the states of n-qubit systems, in which correlations can be revealed. For this purpose, some σ_+^j-multipliers in the first term of the Ardehali operator

$$\hat{A} = -\frac{1}{\sqrt{2}}\left(\prod_{j=1}^{n}\sigma_+^j + \prod_{j=1}^{n}\sigma_-^j\right)$$

should be substituted by the σ_-^j-ones, and, simultaneously, the corresponding multipliers σ_-^j in the second term should be substituted by the σ_+^j-ones:

$$\hat{A} \to \tilde{\hat{A}} = -\frac{1}{\sqrt{2}}(\textstyle\prod_{j=1}^{m}\sigma_-^j \otimes \prod_{j=m+1}^{m}\sigma_+^j +$$
$$+\textstyle\prod_{j=1}^{m}\sigma_+^j \otimes \prod_{j=m+1}^{m}\sigma_-^j).$$

Additionally, the corresponding one-particle states in the state $|\chi_2\rangle$ [Eq. (14)] should be exchanged, $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$:

$$|\chi_2\rangle \to |\tilde{\chi}_2\rangle = \tfrac{1}{\sqrt{2}}(\underbrace{|\downarrow\downarrow ... \downarrow\rangle}_{m} \otimes \underbrace{|\uparrow\uparrow ... \uparrow\rangle}_{n-m} -$$
$$-\underbrace{|\uparrow\uparrow ... \uparrow\rangle}_{m} \otimes \underbrace{|\downarrow\downarrow ... \downarrow\rangle}_{n-m}).$$

It is evident that

$$\langle\tilde{\chi}_2|\tilde{\hat{A}}|\tilde{\chi}_2\rangle = \langle\chi_2|\hat{A}|\chi_2\rangle = 2^{n-\frac{1}{2}}.$$

Let us divide the complete standard basis in the 2^n-dimensional Hilbert space of n-qubit systems into ordered pairs $\{|\eta_k\rangle, |\bar{\eta}_k\rangle\}$, where the vector $|\eta_k\rangle$ is obtained from the vector $|\eta_1\rangle = |\uparrow\uparrow\uparrow ... \uparrow\uparrow\rangle$ by substituting a certain set of $m \leq 2^{n-1}$ one-particle states $|\uparrow\rangle$ with the numbers $j_1, j_2, ..., j_m$ by opposite states $|\downarrow\rangle$, and the vector $|\bar{\eta}_k\rangle$ is obtained from the vector $|\eta_k\rangle$ by substituting all one-particle states in the latter by opposite ones: $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$. For example,

$$|\eta_1\rangle = |\uparrow\uparrow\uparrow ... \uparrow\uparrow\rangle \to |\bar{\eta}_1\rangle = |\downarrow\downarrow\downarrow ... \downarrow\downarrow\rangle,$$
$$|\eta_2\rangle = |\uparrow\uparrow\uparrow ... \uparrow\downarrow\rangle \to |\bar{\eta}_2\rangle = |\downarrow\downarrow\downarrow ... \downarrow\uparrow\rangle,$$

and so on. In terms of those notations, k is a set of the numbers $(j_1, j_2, ..., j_m)$ of those qubits, whose states were changed. It is evident that, in such a way, the complete basis is used to form 2^{n-1} pairs of basis states. Accordingly, the notation $\hat{A}_k$ will mean an operator that is formed from the operator $\hat{A}_1$ by substituting one-particle operators with the numbers $j_1, j_2, ..., j_m$ by opposite ones, $\sigma_+^j \leftrightarrow \sigma_-^j$. The operators $\hat{A}_k$ will be conditionally called the modified Ardehali operators.

Now, an arbitrary n-qubit state $|\Psi\rangle$ can be written as the expansion

$$|\Psi\rangle = \sum_{k=1}^{2^{n-1}}(a_{1k}|\eta_k\rangle + a_{2k}|\bar{\eta}_k\rangle), \quad \sum_{k=1}^{2^{n-1}}(|a_{1k}|^2 + |a_{2k}|^2) = 1. \quad (17)$$

Introducing the notation

$$|\mu_k\rangle = a_{1k}|\eta_k\rangle + a_{2k}|\bar{\eta}_k\rangle,$$

and taking into account that $\langle\mu_k|\hat{A}_{k'}|\mu_k\rangle = 0$ at $k \neq k'$, it is possible to write

$$\langle\Psi|\hat{A}_k|\Psi\rangle = \langle\mu_k|\hat{A}_k|\mu_k\rangle = F_k = -2^{n+\frac{1}{2}}|a_{1k}a_{2k}|\cos\phi_k, \quad (18)$$

where $a_{1k} = \langle\eta_k|\Psi\rangle = |a_{1k}|e^{i\phi_{1k}}$, $a_{2k} = \langle\bar{\eta}_k|\Psi\rangle = |a_{2k}|e^{i\phi_{2k}}$, and $\phi_k = \phi_{2k} - \phi_{1k}$. Hence, having calculated all projections of the vector $|\Psi\rangle$ onto the basis states of the standard basis, we can determine all F_k-values and, by their comparison, their maximum. Result (18) is classed to the $\hat{A}_k$-type of Ardehali operators.

It is evident that similar vectors $|\mu_k\rangle$ and operators $\hat{A}_k$ could be introduced, when considering the Mermin inequality. The determination of F_k-values is an additional way to reveal the correlations in the state $|\Psi\rangle$.

Thus, if the whole set $\{\hat{A}_k\}$ of modified operators, rather than the Ardehali (Mermin) operator, is applied, the class of vectors describing the states of n-qubit systems, in which the correlations can be revealed on the basis of the violation of corresponding inequalities, can be extended. However, it should be noted that even the whole set of operators $\hat{A}_k$ cannot reveal all kinds of the entanglement in n-qubit states. For instance, if either all coefficients a_{1k} or all coefficients a_{2k} in Eq. (17) equal zero, the whole set of generalized operators $\hat{A}_k$ cannot reveal the correlations in the corresponding states $|\Psi\rangle$. Furthermore, the operators $\hat{A}_k$ cannot reveal the correlations in the Wigner generalized state

$$|\Psi_w\rangle = (\alpha_1|\uparrow\uparrow ... \uparrow\uparrow\downarrow\rangle + \alpha_2|\uparrow\uparrow ... \uparrow\downarrow\uparrow\rangle + ... + \\ + \alpha_n|\downarrow\uparrow ... \uparrow\uparrow\uparrow\rangle).$$

This fact means that the Ardehali and Mermin inequalities cannot be regarded as a generalization of the Bell inequality in the CHSH form to the case of n-qubit quantum systems. Nevertheless, those inequalities are a powerful tool for revealing the correlations in a certain class of n-qubit systems.

4. Conclusions

In this work, a comprehensive analysis is carried out for the capability of the known operators of the Mermin and Ardehali correlation functions to reveal the quantum correlations (entanglement) in generalized two-parameter states of n-qubit GHZ states, as well as in arbitrary n-qubit states. The domains of values for the parameters of state vectors are determined separately for the Mermin and Ardehali inequalities, at which those inequalities become violated. The expressions determining the degree of corresponding violation are derived. A set of state vectors is determined, for which the Mermin and Ardehali operators are absolutely insensitive, i.e. the operators cannot reveal the correlations that are actually present. The proposed generalization of the Mermin and Ardehali operators make it possible to extend the class of n-qubit states, in which the quantum correlations can be revealed. To our opinion, this work completely answered the question about the expediency of the application of the Mermin and Ardehali inequalities to that or another class of multiqubit states.

APPENDIX

Let us demonstrate that the introduced correlation-function operator

$$\hat{A} = \hat{A}_1 + \hat{A}_2,$$

where

$$\hat{A}_1 = -\frac{1}{2}\left(\prod_{j=1}^{n-1}\sigma_+^j + \prod_{j=1}^{n-1}\sigma_-^j\right)\frac{1}{\sqrt{2}}(\sigma_+^n + \sigma_-^n), \\ \hat{A}_2 = \frac{1}{2i}\left(\prod_{j=1}^{n-1}\sigma_+^j - \prod_{j=1}^{n-1}\sigma_-^j\right)\frac{1}{i\sqrt{2}}(\sigma_+^n - \sigma_-^n) \quad (19)$$

completely corresponds to the Ardehali operator (13).

Let us write down the Ardehali operator in its original form from work [11] and analyze its structure:

$$\hat{A} = \hat{A}_1 + \hat{A}_2,$$

where

$$\hat{A}_1 = (-\sigma_x^1\sigma_x^2\sigma_x^3 ... \sigma_x^{n-1} + \sigma_y^1\sigma_y^2\sigma_x^3 ... \sigma_x^{n-1} + ... - \\ - \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4\sigma_x^5 ... \sigma_x^{n-1} - ... + \\ + \sigma_y^1 ... \sigma_y^6\sigma_x^7 ... \sigma_x^{n-1} + ... - ...)(\sigma_a^n - \sigma_b^n), \quad (20)$$

$$\hat{A}_2 = (\sigma_y^1\sigma_x^2 ... \sigma_x^{n-1} + ... - \sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4 ... \sigma_x^{n-1} - ... + \\ + \sigma_y^1 ... \sigma_y^5\sigma_x^6 ... \sigma_x^{n-1} + ... - \\ - \sigma_y^1 ... \sigma_y^7\sigma_x^8 ... \sigma_x^{n-1} - ... + ...)(\sigma_a^n + \sigma_b^n). \quad (21)$$

Consider the structure of the expression in the first parentheses in $\hat{A}_1$. The first term does not contain σ_y-multipliers. The second term contains two σ_y-multipliers. The notation "+..." means that terms with two multipliers $\sigma_y^i\sigma_y^j$ and all possible superscripts $i < j$ should be included. The number of those terms is equal to the number of 2-combinations from a set of

$n-1$ elements (i.e. the binomial coefficient). The next terms in the considered expression contain all possible combinations with four multipliers, $\sigma_y^i\sigma_y^j\sigma_y^k\sigma_y^l$ $(i<j<k<l)$. The number of those terms is equal to the number of 4-combinations from a set of $n-1$ elements. Note that all terms with the number of σ_y-multipliers multiple of 4 – i.e. $N=4k$, where $k=0$, 1, 2, ... – enter $\hat{A}_1$ with the minus sign, whereas the others with the plus sign.

Analogously, the terms in $\hat{A}_2$ are all possible combinations of σ_y-multipliers, with the number of multipliers in each term being odd and not exceeding $n-1$. The terms with the number of σ_y-multipliers equal to $N=4k+1$ ($k=0$, 1, 2, ...) enter $\hat{A}_2$ with the plus sign, whereas the others with the minus sign.

As an example, we give the complete expressions for $\hat{A}_1$ and $\hat{A}_2$ in the case $n=6$ $(n-1=5)$. To simplify them, we assume that the particle numbers are arranged in the ascending order, so that those numbers are not written down explicitly; for example, $\sigma_y^1\sigma_x^2\sigma_y^3\sigma_x^4\sigma_x^5=\sigma_y\sigma_x\sigma_y\sigma_x\sigma_x$. Then

$$\begin{aligned}\hat{A}_1 &= (-\sigma_x\sigma_x\sigma_x\sigma_x\sigma_x+\sigma_y\sigma_y\sigma_x\sigma_x\sigma_x+\sigma_y\sigma_x\sigma_y\sigma_x\sigma_x+\\&+\sigma_y\sigma_x\sigma_x\sigma_y\sigma_x+\sigma_y\sigma_x\sigma_x\sigma_x\sigma_y+\sigma_x\sigma_y\sigma_y\sigma_x\sigma_x+\\&+\sigma_x\sigma_y\sigma_x\sigma_y\sigma_x+\sigma_x\sigma_y\sigma_x\sigma_x\sigma_y+\sigma_x\sigma_x\sigma_y\sigma_y\sigma_x+\\&+\sigma_x\sigma_x\sigma_y\sigma_x\sigma_y+\sigma_x\sigma_x\sigma_x\sigma_y\sigma_y-\sigma_y\sigma_y\sigma_y\sigma_y\sigma_x-\\&-\sigma_y\sigma_y\sigma_y\sigma_x\sigma_y-\sigma_y\sigma_y\sigma_x\sigma_y\sigma_y-\sigma_y\sigma_x\sigma_y\sigma_y\sigma_y-\\&-\sigma_x\sigma_y\sigma_y\sigma_y\sigma_y)(\sigma_a^n-\sigma_b^n),\\ \hat{A}_2 &= (+\sigma_y\sigma_x\sigma_x\sigma_x\sigma_x+\sigma_x\sigma_y\sigma_x\sigma_x\sigma_x+\sigma_x\sigma_x\sigma_y\sigma_x\sigma_x+\\&+\sigma_x\sigma_x\sigma_x\sigma_y\sigma_x+\sigma_x\sigma_x\sigma_x\sigma_x\sigma_y-\sigma_y\sigma_y\sigma_y\sigma_x\sigma_x-\\&-\sigma_y\sigma_y\sigma_x\sigma_y\sigma_x-\sigma_y\sigma_y\sigma_x\sigma_x\sigma_y-\sigma_y\sigma_x\sigma_y\sigma_y\sigma_x-\\&-\sigma_y\sigma_x\sigma_y\sigma_x\sigma_y-\sigma_y\sigma_x\sigma_x\sigma_y\sigma_y-\sigma_x\sigma_y\sigma_y\sigma_y\sigma_x-\\&-\sigma_x\sigma_y\sigma_y\sigma_x\sigma_y-\sigma_x\sigma_y\sigma_x\sigma_y\sigma_y-\sigma_x\sigma_x\sigma_y\sigma_y\sigma_y+\\&+\sigma_y\sigma_y\sigma_y\sigma_y\sigma_y)(\sigma_a^n+\sigma_b^n).\end{aligned}$$

Now, consider the expression for $\hat{A}_1$. It can be rewritten in the form

$$\hat{A}_1=-\frac{1}{2}\left(\prod_{j=1}^{n-1}\sigma_+^j+\prod_{j=1}^{n-1}\sigma_-^j\right)\frac{1}{\sqrt{2}}(\sigma_+^n+\sigma_-^n). \qquad (22)$$

The expression in the first parentheses equals

$$\begin{aligned}\hat{B}&=\left(\prod_{j=1}^{n-1}\sigma_+^j+\prod_{j=1}^{n-1}\sigma_-^j\right)=\\&=(\sigma_x^1+i\sigma_y^1)(\sigma_x^2+i\sigma_y^2)\,(\sigma_x^3+i\sigma_y^3)...\,(\sigma_x^{n-1}+i\sigma_y^{n-1})+\\&+(\sigma_x^1-i\sigma_y^1)(\sigma_x^2-i\sigma_y^2)(\sigma_x^3-i\sigma_y^3)\,...\,(\sigma_x^{n-1}-i\sigma_y^{n-1}).\end{aligned}$$

Let us forget, for the present, the true meaning of σ_x and σ_y, and let us consider them to be certain real-valued parameters. Then

$$\begin{aligned}\hat{B}&=2\mathrm{Re}\{(\sigma_x^1+i\sigma_y^1)(\sigma_x^2+i\sigma_y^2)\times\\&\times(\sigma_x^3+i\sigma_y^3)\,...\,(\sigma_x^{n-1}+i\sigma_y^{n-1})\}.\end{aligned}$$

Removing the parentheses and calculating the real part of the obtained expression, as well as taking into account that $i^{4k}=(-i)^{4k}=1$ for $k=0,1,2,...$, whereas $i^{2(2k+1)}=-1$, we obtain that Eq. (22) accurately coincides with Eq. (20). A similar consideration demonstrates that $\hat{A}_2$ in form (19) accurately coincides with $\hat{A}_2$ introduced by M. Ardehali [Eq. (21)].

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ДЕТЕКТУВАННЯ ЗАПЛУТАНОСТІ БАГАТОКУБІТОВИХ КВАНТОВИХ СИСТЕМ ЗА КРИТЕРІЯМИ МЕРМІНА І АРДЕХАЛІ

Р е з ю м е

У роботі досліджується можливість виявлення заплутаності в n-кубітових узагальнених двопараметричних GHZ-станах, а також в довільних n-кубітових станах за допомогою нерівності Мерміна і нерівності Ардехалі з числа отримавших узагальнену назву нерівностей Мерміна–Ардехалі–Белінського–Клишко. Отримано формули для розрахунку значень кореляційних функцій Мерміна і Ардехалі в довільних квантових n-кубітових станах та критерій порушення відповідних нерівностей конкретними станами. Виявлено сукупність станів, абсолютно нечутливих до операторів Мерміна і Ардехалі. Запропоновано (модифіковані) оператори Мерміна і Ардехалі, сукупність яких дозволяє розширити клас n-кубітових станів, в яких можна виявити наявність квантових кореляцій.