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# THE CONTRIBUTION OF THE RETARDATION EFFECTS TO THE TOTAL ENERGY SPLITTING OF HYDROGENLIKE ATOMS

*We have shown that the discrepancy between the PSI [1, 2] and CODATA [3] results of high-precision experimental measurements of the proton charge radius can be explained on the basis of the quantum electrodynamics effect, namely through the precise calculation of the contribution of the retardation effects to the total energy splitting. We find that the contribution of these effects is ∼0.3 meV, and this value is very close to the additional term as large as 0.31 meV, which would be required to match the results of PSI measurements with the CODATA value of charge proton radius $r_p$ = 0.8768(69) fm.*

*Keywords*: retardation effects, energy splitting, proton charge radius.

## 1. Introduction

In the latest experimental investigation (performed at Paul Scherrer Institute (PSI)) of the proton structure from the measurement of the Lamb shift ($2S-2P$ transition frequencies) of muonic hydrogen [1, 2] ($\mu p$ atom – a proton orbited by a negative muon), the proton charge radius $r_p = 0.84087(39)$ fm was extracted with the precision higher by an order of magnitude than the CODATA value 0.8768(69) fm [3] (based on H spectroscopy and elastic electron scattering) and at $7\sigma$ variance with the respect to it. If one takes into account that, for the muonic hydrogen Lamb shift, the theoretical uncertainty is equal to 0.004 meV, and the experimental uncertainty is 0.003 meV, it must be admitted that this discrepancy between the CODATA and PSI results, which is at the level 0.3 meV, implies that either the Rydberg constant has to be shifted or the calculation of the quantum electrodynamics effects in atomic hydrogen or muonic hydrogen atoms are insufficient.



As well known, the light muonic atoms have two main features as compared with the electronic hydrogenlike atoms, both of which are connected with the fact that a muon is about 200 times heavier than an electron. First – the contribution of the radiative corrections is greatly enhanced, and second – the leading proton size $r_p$ gives the second largest contribution to the energy shift after the polarization corrections.

The theory relating the Lamb shift to $r_p$ yields [4–6]

$$\Delta E_{\mathrm{L}}^{\mathrm{th}} = (206.0336(15) + \Delta_{\text{finite size}} + \Delta E_{\mathrm{TPE}})\ \mathrm{meV}, \quad (1)$$

where the first term on the right-hand side accounts for radiative, relativistic, and recoil effects, the second term arises from the proton structure and describes the leading finite-size effects, and the third term is determined by two photon exchange effects, including the proton polarizability. According to [7–9], $\Delta E_{\mathrm{TPE}} = 0.0332(20)$ meV. As well known, the contribution of the finite-size proton effect $\Delta_{\text{finite size}}$ to the total energy splitting $\Delta E_{\mathrm{L}} = 2P_{1/2} - 2S_{1/2}$ (Lamb shift) is 1.8 percent of the total $\Delta E_{\mathrm{L}}$ in a "muonic hydrogen" atom, two orders of magnitude more than in a hydrogen atom H. The atomic energy levels of H or $\mu p$ are affected by the finite size of the proton charge distribution by

$$\Delta_{\text{finite size}} = \frac{2\pi Z\alpha}{3} r_p^2 |\Psi(0)|^2, \tag{2}$$

where $|\Psi(0)|$ is the atomic wave function at the origin, $\alpha$ is the fine structure constant, $Z = 1$ – the proton charge, and $r_p$ is the root mean square proton charge radius given in femtometers and defined as $r_p^2 = \int d^3\mathbf{r}\, r^2 \rho_p(\mathbf{r})$, with $\rho_p$ being the normalized proton charge distribution.

The proton size and the proton structure are important, because a lepton (electron or muon) in the $S$-state has a nonzero probability to be inside the proton, which means that the attractive force between the proton and a lepton is reduced, because the electric field inside the charge distribution is smaller than the corresponding field produced by a point charge. For the $S$-states, $|\Psi(0)|^2$ is proportional to $m_r^3$ ($m_r$ is the reduced mass). Since the muon mass $m_\mu$ is 207 times more than the electron mass $m_e$, the $m_r^\mu \approx 186 m_r^e$, which leads to the sharp enhancement of the contribution of the finite-size proton effect $\Delta_{\text{finite size}}$ to the total energy difference $\Delta E_L$ in a "muonic hydrogen" atom in comparison with a hydrogen atom. In addition, the Lamb shift in $\mu p$ differs from H in that the electron vacuum polarization gives the most significant contribution, because the Compton wavelength of the electron (which determines the spatial distribution of the vacuum polarization charge density) is of the order of the muon Bohr radius. This leads to a higher sensitivity to the proton finite size of $\mu p$ in comparison with a hydrogen atom.

For a $\mu p$ atom, the leading finite-size effect $\Delta_{\text{finite size}} = -5.2275(10) r_p^2$ meV is approximately given by Eq. (2) with a correction given in [5]. In this case, the comparison of $\Delta E_{\text{L}}^{\text{th}}$ (Eq. (1)) with that obtained from the measurement of $2S - 2P$ transition frequencies in a "muonic hydrogen" $\Delta E_{\text{L}}^{\text{exp}} = 202.3706(23)$ meV [1, 2] yields $r_p = 0.84087(39)$ fm. If one assume some quantum electrodynamics contributions in $\mu p$ in Eq. (1) were wrong or missing, an additional term as large as 0.31 meV would be required to match the results of above-mentioned measurement with the CODATA value of $r_p = 0.8768(69)$ fm. It should be noted that 0.31 meV is 64 times the claimed uncertainty of Eq. (1).

The purpose of this work is to consider the contribution of retardation effects to the total energy splitting $\Delta E_{\text{L}} = 2P_{1/2} - 2S_{1/2}$, which might play the essential role in the above-mentioned problem. Usually, these effects for systems with two charged bodies are investigated in the frame of a nonrelativistic approach or by using the fact that the total relativistic Hamiltonian of the system can be given in the form of an expansion in $\alpha^2$ up to the first correction term

$$V(\mathbf{r}) = \frac{\alpha}{r} - \frac{e^2}{2r}\left[\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 + \frac{(\boldsymbol{\alpha}_1\cdot\mathbf{r})(\boldsymbol{\alpha}_2\cdot\mathbf{r})}{r^2}\right], \tag{3}$$

where $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ are the commuting sets of Dirac matrices, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and the subscripts 1 or 2 distinguish the quantities related to the first and second particles. In this expression, the first term describes the electrostatic Coulomb interaction (exchange by longitudinal photons), and the second part takes the magnetic spin-spin interactions and retardation corrections due to the finite speed of the interaction into account (exchange by transversal photons). However, such approximation for the relativistic interaction between two particles is good only under the assumption that the retardation effects in the spectrum of an atom are small. In the case where these effects are not small, we have to find another way.

We will consider the retardation effects in two ways. The expression for the energy shift in the first-order perturbation theory is as follows:

$$\Delta E_{nlm} = \int \mathrm{d}\mathbf{r}\, \psi_{nlm}^*(\mathbf{r})\, V(\mathbf{r})\, \psi_{nlm}(\mathbf{r}), \tag{4}$$

where $\delta V(\mathbf{r})$ is the correction, which the vacuum polarization inserts to the Coulomb potential $V_{\text{C}}(r) = -\alpha/r$:

$$\delta V(r) = \int \frac{d\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \delta V(\mathbf{q}),$$
$$\delta V(\mathbf{q}) = \frac{4\pi\alpha}{\mathbf{q}^2}\,\Pi(-q^2). \tag{5}$$

Instead of the approximation of the nonrelativistic expression for the scalar term

$$\Pi(\mathbf{q}^2) = \frac{1}{\pi}\int_{t_0}^{\infty} dt \frac{\operatorname{Im}\Pi(t)}{t - \mathbf{q}^2 - i0^+}\frac{\mathbf{q}^2}{t},$$

which is widely used in the works calculating the Lamb shift (see, e.g., [4]), we will use the exact relativistic dispersion relation for the scalar term:

$$\Pi(q^2) = \frac{1}{\pi}\int_{t_0}^{\infty} dt\, \frac{\operatorname{Im}\Pi(t)}{t - q^2 - i0^+}\frac{q^2}{t}. \qquad (6)$$

Here, $q^2 = (q^0)^2 - \mathbf{q}^2$, $\omega = q^0$, and $t_0$ is the lowest particle-production threshold. Second improvement–instead of the second term in (3), we will use the potential describing the retardation of a massive photon propagator or, in other words, we will calculate the effective interaction potential on the frame of the Breit approach [10] using the so-called "dressed" one-photon exchange diagram for the eVP contribution (see Figure).

The paper is organized as follows. First, we consider the massive photon propagator dressed with vacuum polarization in an arbitrary gauge, and then we find it in the Coulomb gauge. We apply the Breit-type approach to calculate a one-photon retardation contribution. For this, we need to find the non-relativistic reduction of the scalar two-particle propagator. Then the effective potential will be built and investigated. In the non-relativistic limit, it can be presented in the form

$$\simeq G_{\rm NR}(\mathbf{p}^2; E)\, V_{\rm NR}(\mathbf{p}, \mathbf{p}')\, G_{\rm NR}(\mathbf{p}'^2; E),$$

where $G_{\rm NR}(\mathbf{p}^2; E)$ is the non-relativistic Green function. This gives us a possibility to calculate the contribution of a retardation of the vacuum polarization to the Lamb shift of hydrogen atoms.

## 2. Photon Propagator

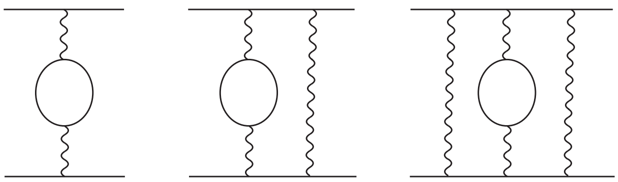
The photon propagator in an arbitrary gauge has the form

$$\Delta^{\mu\nu}(q) = -\frac{1}{q^2}\left(g^{\mu\nu} - \chi^\mu q^\nu - \chi^\nu q^\mu\right), \qquad (7)$$

where $q^\mu = \{q^0, \mathbf{q}\}$ – four-momentum of a photon, $\chi^\nu$ is any four-vector, which will be used for the choice of a gauge. We use the flat Minkowski metric with $\operatorname{diag} g^{\mu\nu} = (1, -1, -1, -1)$.

The electromagnetic gauge invariance constrains the vacuum polarization (VP) to the well-known form

$$\Pi^{\mu\nu}(q) = (g^{\mu\nu} q^2 - q^\mu q^\nu)\,\Pi(q^2), \qquad (8)$$



Photons exchange diagrams of eVP contributions

with the scalar term satisfying the once-subtracted relativistic dispersion relation

$$\Pi(q^2) = \frac{1}{\pi}\int_{t_0}^{\infty} dt\, \frac{\operatorname{Im}\Pi(t)}{t - q^2 - i0^+}\frac{q^2}{t}.$$

The propagator dressed with vacuum polarization is found as

$$\tilde{\Delta}^{\mu\nu}(q) = \Delta^{\mu\alpha}(q)\,\Pi_{\alpha\beta}(q)\,\Delta^{\beta\nu}(q) =$$

$$= \frac{1}{q^4}\left(g^{\mu\alpha} - \chi^\alpha q^\mu - \chi^\mu q^\alpha\right)\left(g_{\alpha\beta}q^2 - q_\alpha q_\beta\right)\times$$

$$\times\left(g^{\beta\nu} - \chi^\beta q^\nu - \chi^\nu q^\beta\right)\Pi(q^2) = \frac{1}{q^2}\Big[g^{\mu\nu} - q^\mu\chi^\nu -$$

$$- q^\nu\chi^\mu + \chi^2 q^\mu q^\nu - \frac{(1 - q\cdot\chi)^2}{q^2} q^\mu q^\nu\Big]\Pi(q^2). \qquad (9)$$

Hence,

$$\tilde{\Delta}^{\mu\nu}(q) = \frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt}{t}\,\frac{\operatorname{Im}\Pi(t)}{t - q^2 - i0^+}\times$$

$$\times\left[g^{\mu\nu} - q^\mu\chi^\nu - q^\nu\chi^\mu + \frac{q^2\chi^2 - (1 - q\cdot\chi)^2}{q^2} q^\mu q^\nu\right]. \qquad (10)$$

In the Coulomb gauge, $\chi$ is chosen as

$$\chi^\mu = \frac{1}{2\mathbf{q}^2}\left(\omega, -\mathbf{q}\right), \qquad (11)$$

where $\omega = q^0$, and $\mathbf{q}^2 = \omega^2 - q^2$. Then the propagator does not mix the spatial and temporal components, $\Delta^{0i} = 0$, and the non-vanishing components are

$$\Delta^{00}(q) = \frac{1}{\mathbf{q}^2}, \quad \Delta^{ij}(q) = \frac{1}{\omega^2 - \mathbf{q}^2}\left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right). \qquad (12)$$

For $\chi$ in the Coulomb gauge, we find

$$q^2\chi^2 - (1 - q\cdot\chi)^2 = 0, \qquad (13)$$

and, hence, the tensor structure remains the same as for the non-dressed propagator. However, the difference consists in that $q^2$ in the denominator is replaced with $(q^2 - t)$, where $t$ is integrated over with some weight.

Let us introduce the massive propagator

$$\Delta^{\mu\nu}(q;t) = -\frac{1}{q^2 - t}\big(g^{\mu\nu} - \chi^\mu q^\nu - \chi^\nu q^\mu\big). \tag{14}$$

Then the dressed propagator is written as

$$\tilde{\Delta}^{\mu\nu}(q) = \frac{1}{\pi}\int\limits_{t_0}^{\infty} dt\, \frac{\operatorname{Im}\Pi(t)}{t}\, \Delta^{\mu\nu}(q;t). \tag{15}$$

In the Coulomb gauge, the non-vanishing components of the massive propagator are

$$\begin{aligned} \Delta^{00}(q;t) &= \frac{1}{q^2 - t}\,\frac{q^2}{\mathbf{q}^2} = \frac{1}{\mathbf{q}^2 + t} + \\ &+ \frac{\omega^2 t}{\mathbf{q}^2(\mathbf{q}^2 + t)(\omega^2 - \mathbf{q}^2 - t)}, \\ \Delta^{ij}(q;t) &= \frac{1}{q^2 - t}\left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right) = \\ &= \frac{1}{\omega^2 - \mathbf{q}^2 - t}\left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right). \end{aligned} \tag{16}$$

In the frame of the Breit approach, only the time-like components $\Delta^{00}$ of the propagator play the essential role. The second term in expression (16) for $\Delta^{00}(q;t)$ describes the contributions of a retardation of the vacuum polarization to the Lamb shift of hydrogen atoms, and it will be used in the future calculations.

## 3. Two-Particle Propagator

To attain the goal, we need to calculate the two-particle (lepton and proton) propagator $G(\boldsymbol{\ell}^2;\, s)$ and to find its non-relativistic reduction.

The lepton (with mass $m_a$) and proton (with mass $m_b$) momenta are written as

$$k = x_a P - \ell, \quad p = x_b P + \ell, \tag{17}$$

where $P = p + k$ and $l$ are is the total and relative momenta, respectively, of two particles. Hence, the momentum fractions are defined as

$$x_a = \frac{s + m_a^2 - m_b^2}{2s}, \quad x_b = \frac{s + m_b^2 - m_a^2}{2s}, \tag{18}$$

and we obtain $x_a + x_b = 1$. Furthermore, $P^2 = s$, and the momentum transfer is

$$q = k' - k = p - p' = \ell - \ell'. \tag{19}$$

In the center-of-mass system, $P = (\sqrt{s}\mathbf{0})$. We obtain obviously $\mathbf{k} = -\mathbf{l} = -\mathbf{p}$.

The non-relativistic reduction of a scalar two-particle propagator should proceed as follows. First, we observe that the scalar two-particle propagator $G(\boldsymbol{\ell}^2;\, s)$ calculated in Appendix is

$$\begin{aligned} G(\boldsymbol{\ell}^2;\, s) &= \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} d\ell_0 \times \\ &\times \frac{1}{\big[(x_a P - \ell)^2 - m_a^2 + i\varepsilon\big]\big[(x_b P + \ell)^2 - m_b^2 + i\varepsilon\big]} = \\ &= \frac{1}{\lambda(s)/s - \boldsymbol{\ell}^2 + i\varepsilon} \times \\ &\times \left(\frac{x_a}{4(m_a^2 + \boldsymbol{\ell}^2)^{1/2}} + \frac{x_b}{4(m_b^2 + \boldsymbol{\ell}^2)^{1/2}}\right), \end{aligned} \tag{20}$$

where

$$\lambda(s) = \frac{1}{4}\big[s - (m_a + m_b)^2\big]\big[s - (m_a - m_b)^2\big]. \tag{21}$$

Everywhere, we substitute $\sqrt{s} = (m_a + m_b) + E$, with $E \ll m_a,\, m_b$. In the non-relativistic limit, we have $(m_{a,b}^2 + \boldsymbol{\ell}^2)^{1/2} \simeq m_{a,b}$:

$$\frac{\lambda(s)}{s} = 2E\frac{m_a m_b}{m_a + m_b} + O(E^2), \tag{22a}$$

$$x_a = \frac{m_a}{m_a + m_b} - E\,\frac{m_a - m_b}{(m_a + m_b)^2} + O(E^2), \tag{22b}$$

$$x_b = \frac{m_b}{m_a + m_b} - E\,\frac{m_b - m_a}{(m_a + m_b)^2} + O(E^2). \tag{22c}$$

Recalling that $\boldsymbol{\ell}^2 = \mathbf{p}^2$, we finally obtain the non-relativistic form:

$$G(\boldsymbol{\ell}^2;\, s) \simeq G_{\rm NR}(\mathbf{p}^2; E) = \frac{1}{4m_a m_b\big(E - \frac{\mathbf{p}^2}{2m_r}\big)}, \tag{23}$$

where $m_r = m_a m_b/(m_a + m_b)$ is the reduced mass. For the fermion-fermion system, we obtain $2m$ from the numerator of each propagator, i.e., the factor $4m_a m_b$ in the denominator is cancelled.

## 4. Effective Potential

Our task is to consider the "dressed" one-photon exchange diagram for the eVP contributions

$$I(\mathbf{p},\mathbf{p}') = \int\limits_{-\infty}^{\infty} \frac{d\ell_0}{2\pi i}\frac{d\ell'_0}{2\pi i} \times$$
$$\times \frac{1}{\left[(x_aP-\ell)^2 - m_a^2 + i\varepsilon\right]\left[(x_bP+\ell)^2 - m_b^2 + i\varepsilon\right]} \times$$
$$\times \frac{1}{\left[(x_aP-\ell')^2 - m_a^2 + i\varepsilon\right]\left[(x_bP+\ell')^2 - m_b^2 + i\varepsilon\right]} \times$$
$$\times V(\ell_0-\ell'_0,\, \mathbf{p},\, \mathbf{p}'). \qquad (24)$$

According to the Breit approach, we now consider the possibility to present it in the non-relativistic limit in the form

$$I(\mathbf{p},\mathbf{p}') \simeq G_{\rm NR}(\mathbf{p}^2;E)V_{\rm NR}(\mathbf{p},\mathbf{p}')G_{\rm NR}(\mathbf{p}'^2;E). \qquad (25)$$

In our approach, we will use the potential

$$V(\ell_0-\ell'_0,\mathbf{p},\mathbf{p}') = (\ell_0-\ell'_0)^2. \qquad (26)$$

Then, using expression (16) for $\Delta^{00}(q;t)$ and integrating (see Appendix), we obtain

$$I(\mathbf{p},\mathbf{p}') = \frac{1}{16}\left(\frac{1}{\lambda(s)/s - \boldsymbol{\ell}^2 + i\varepsilon} + \right.$$
$$\left. + \frac{1}{\lambda(s)/s - \boldsymbol{\ell}'^2 + i\varepsilon}\right)\left(\frac{x_a}{\omega_a} + \frac{x_b}{\omega_b}\right)\left(\frac{x_a}{\omega'_a} + \frac{x_b}{\omega'_b}\right) -$$
$$-\frac{1}{8s}\left(\frac{1}{\omega_a} - \frac{1}{\omega_b}\right)\left(\frac{1}{\omega'_a} - \frac{1}{\omega'_b}\right), \qquad (27)$$

where $\omega_i = (m_i^2 + \boldsymbol{\ell}^2)^{1/2}$, $\omega'_i = (m_i^2 + \boldsymbol{\ell}'^2)^{1/2}$.

Therefore, the potential is

$$V_{\rm NR}(\mathbf{p},\mathbf{p}') = G^{-1}(\mathbf{p}^2,s)I(\mathbf{p},\mathbf{p}')G^{-1}(\mathbf{p}'^2,s) =$$
$$= 4m_rE - \mathbf{p}^2 - \mathbf{p}'^2 + \frac{\left(\mathbf{p}^2+\mathbf{p}'^2\right)^2}{2m_am_b} +$$
$$+ \frac{\mathbf{p}^2\mathbf{p}'^2}{2}\left(\frac{1}{m_a^2} + \frac{1}{m_b^2}\right) + O(1/m^4) = 4m_rE - \mathbf{p}^2 - \mathbf{p}'^2 +$$
$$+ \frac{\mathbf{p}^2\mathbf{p}'^2}{2m_r^2} + \frac{\mathbf{p}^4+\mathbf{p}'^4}{2m_am_b} + O(1/m^4). \qquad (28)$$

Omitting the terms suppressed by the masses, we arrive at the following potential describing the retardation of the massive photon propagator:

$$V^{(\rm VPret)}(\mathbf{q};\mathbf{p}^2,\mathbf{p}'^2) = -4\pi\alpha\frac{\omega^2 t}{\mathbf{q}^2(\mathbf{q}^2+t)^2} \overset{\rm NR}{\to}$$
$$\overset{\rm NR}{\to} -4\pi\alpha\frac{t}{\mathbf{q}^2(\mathbf{q}^2+t)^2}\left(4m_rE - \mathbf{p}^2 - \mathbf{p}'^2\right). \qquad (29)$$

## 5. Calculation of the Lamb Shift

The energy shift in the first-order perturbation theory is calculated according to expression (4). In the momentum space, we have

$$\Delta E_{nlm} = \int d\mathbf{r}\int\frac{d\mathbf{p}\,d\mathbf{p}'}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}e^{i(\mathbf{p}-\mathbf{p}'+\mathbf{q})\cdot\mathbf{r}} \times$$
$$\times V(\mathbf{q};\mathbf{p}^2,\mathbf{p}'^2)\,\varphi^*_{nlm}(\mathbf{p}')\varphi_{nlm}(\mathbf{p}), \qquad (30)$$

where the momentum-space wave function is introduced via

$$\psi_{nlm}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\,e^{i\mathbf{p}\cdot\mathbf{r}}\,\varphi_{nlm}(\mathbf{p}). \qquad (31)$$

Substituting the retardation potential

$$V^{(\rm VPret)}(\mathbf{q};\mathbf{p}^2,\mathbf{p}'^2) =$$
$$= -\frac{1}{\pi}\int\limits_{t_0}^{\infty} dt\,{\rm Im}\,\Pi(t)\frac{4\pi Z\alpha}{\mathbf{q}^2(\mathbf{q}^2+t)^2}\frac{\left(\mathbf{p}^2-\mathbf{p}'^2\right)^2}{4m_r^2} \qquad (32)$$

into the above formula for the energy shift, we obtain

$$\Delta E^{(\rm VPret)} = -\frac{Z\alpha}{\pi}\int\limits_{t_0}^{\infty} dt\,{\rm Im}\,\Pi(t)\int d\mathbf{r}W(r;\,t) \times$$
$$\times\left\{\psi^*(\mathbf{r})\frac{\nabla^4}{2m_r^2}\psi(\mathbf{r}) - 2\left[\frac{\nabla^2}{2m_r}\psi^*(\mathbf{r})\right]\frac{\nabla^2}{2m_r}\psi(\mathbf{r})\right\} \qquad (33)$$

with

$$W(r;\,t) = \int\frac{d\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{4\pi}{\mathbf{q}^2(\mathbf{q}^2+t)^2} =$$
$$= \frac{1}{t}\left[\frac{1}{rt}\left(1-e^{-r\sqrt{t}}\right) - \frac{1}{2\sqrt{t}}\,e^{-r\sqrt{t}}\right]. \qquad (34)$$

We focus on the term in the curly brackets. Using the Schrödinger equation

$$\left(\frac{\nabla^2}{2m_r} + E + \frac{Z\alpha}{r}\right)\psi(\mathbf{r}) = 0 \qquad (35)$$

together with the useful identities

$$\boldsymbol{\nabla}\frac{1}{r} = -\frac{\mathbf{r}}{r^3}, \quad \nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r}),$$
$$\mathbf{r}\cdot\boldsymbol{\nabla}\psi(\mathbf{r}) = r\frac{\partial}{\partial r}\psi(\mathbf{r}), \qquad (36)$$

we obtain that the term in the curly brackets reads

$$\{...\} = 2\Big\{ - \psi^*(\mathbf{r}) \left[ \frac{\nabla^2}{2m_r} \left( E + \frac{Z\alpha}{r} \right) \psi(\mathbf{r}) \right] - $$
$$ - \left( E + \frac{Z\alpha}{r} \right)^2 |\psi(\mathbf{r})|^2 \Big\} = $$
$$ = 2 \left\{ \frac{4\pi Z\alpha}{2m_r} \delta(\mathbf{r}) \, |\psi(\mathbf{r})|^2 - \frac{Z\alpha}{m_r} \psi^*(\mathbf{r}) \left( \boldsymbol{\nabla} \frac{1}{r} \right) \cdot \boldsymbol{\nabla} \psi(\mathbf{r}) \right\} = $$
$$ = 2 \left\{ \frac{4\pi Z\alpha}{2m_r} \delta(\mathbf{r}) \, |\psi(\mathbf{r})|^2 + \frac{Z\alpha}{m_r r^2} \psi^*(\mathbf{r}) \frac{\partial}{\partial r} \psi(\mathbf{r}) \right\}. \quad (37)$$

The two terms in the latter expression will be considered separately, i.e.,

$$\Delta E^{(\mathrm{VPret})} = \Delta E_1^{(\mathrm{VPret})} + \Delta E_2^{(\mathrm{VPret})},$$
$$\Delta E_1^{(\mathrm{VPret})} = -\frac{4(Z\alpha)^2}{m_r} \psi_n^2(0) \int\limits_{t_0}^{\infty} \mathrm{d}t \, \mathrm{Im}\, \Pi(t) W(0;\, t), \quad (38)$$
$$\Delta E_2^{(\mathrm{VPret})} = -\frac{2(Z\alpha)^2}{\pi m_r} \int\limits_{t_0}^{\infty} \mathrm{d}t \, \mathrm{Im}\, \Pi(t) \times$$
$$\times \int\limits_0^{\infty} \mathrm{d}r \, W(r;\, t) \, R_{nl}(r) \, R'_{nl}(r), \quad (39)$$

where $R(r)$ is the radial wave function, and $\psi_n^2(0) = (Z\alpha m_r)^3/\pi n^3$. Furthermore, we easily establish that

$$W(0; t) = \frac{1}{2t^{3/2}}, \quad (40)$$
$$\int\limits_{4m_e^2}^{\infty} \mathrm{d}t \, \frac{\mathrm{Im}\, \Pi^{(1)}(t)}{2t^{3/2}} = -\frac{3\pi\alpha}{64 m_e}. \quad (41)$$

Hence,

$$\Delta E_1^{(\mathrm{VPret})}(2S) = \frac{3\alpha(Z\alpha)^5 m_r^2}{128 m_e} = $$
$$ = \frac{3\alpha(Z\alpha)^4 m_r}{64} \kappa \simeq 200 \ \mu\mathrm{eV}, \quad (42)$$

where $\kappa = Z\alpha m_r/2m_e$.

To calculate the second contribution, we need the derivatives of the radial wave functions

$$R'_{20}(r) = -\frac{1}{\sqrt{2}\, a^{5/2}} \left( 1 - \frac{r}{4a} \right) e^{-r/2a}, \quad (43a)$$
$$R'_{21}(r) = \frac{1}{2\sqrt{6}\, a^{5/2}} \left( 1 - \frac{r}{2a} \right) e^{-r/2a}. \quad (43b)$$

We then find

$$\int\limits_0^{\infty} \mathrm{d}r \, W(r;\, t) \, R_{20}(r) \, R'_{20}(r) = $$
$$ = \frac{6 + 14at^{1/2} + 9a^2 t}{16a^3 t^{3/2} (1 + at^{1/2})^3} - \frac{1}{2a^4 t^2} \log \left( 1 + at^{1/2} \right), \quad (44a)$$
$$\int\limits_0^{\infty} \mathrm{d}r \, W(r;\, t) \, R_{21}(r) \, R'_{21}(r) = $$
$$ = \frac{1}{48 a t^{1/2} (1 + at^{1/2})^3}. \quad (44b)$$

After the integration over $t$, we obtain

$$\Delta E_2^{(\mathrm{VPret})}(2P - 2S) \simeq 100 \ \mu\mathrm{eV}. \quad (45)$$

The total result is

$$\Delta E^{(\mathrm{VPret})}(2P - 2S) = \Delta E_1^{(\mathrm{VPret})}(2S) + $$
$$ + \Delta E_2^{(\mathrm{VPret})}(2P - 2S) \simeq 300 \ \mu\mathrm{eV}. \quad (46)$$

Another way to calculate this result is to use the partial integration:

$$\Delta E^{(\mathrm{VPret})} = -\frac{Z\alpha}{\pi} \int\limits_{t_0}^{\infty} \mathrm{d}t \, \mathrm{Im}\, \Pi(t) \int \mathrm{d}\mathbf{r} \, W(r;\, t) \times$$
$$\times \left\{ \psi^*(\mathbf{r}) \, \frac{\nabla^4}{2m_r^2} \, \psi(\mathbf{r}) - 2 \left[ \frac{\nabla^2}{2m_r} \psi^*(\mathbf{r}) \right] \frac{\nabla^2}{2m_r} \, \psi(\mathbf{r}) \right\},$$
$$ = \frac{(Z\alpha)^2}{\pi m_r} \int\limits_{t_0}^{\infty} \mathrm{d}t \, \mathrm{Im}\, \Pi(t) \int \mathrm{d}\mathbf{r} \, |\psi(\mathbf{r})|^2 \, \frac{1}{r^2} \frac{\partial}{\partial r} W(r;\, t) \quad (47)$$
$$\Delta \tilde{E}_{nlm}^{(\mathrm{VPret})} = -Z\alpha \int \mathrm{d}\mathbf{r} \, |\psi_{nlm}(\mathbf{r})|^2 \, \frac{(-4Z\alpha m_r)}{r} \times$$
$$\times \left[ \frac{1}{rt} \left( 1 - e^{-r\sqrt{t}} \right) - \frac{1}{2\sqrt{t}} \, e^{-r\sqrt{t}} \right]. \quad (48)$$

The term in the square brackets follows from the $q$-integration.

Recalling the external dispersive integral over $t$, we conclude that the retardation of VP results in the following central and local potential

$$V^{(\mathrm{VPret})}(r) = 4(Z\alpha)^2 m_r \frac{1}{\pi} \int\limits_{t_0}^{\infty} \mathrm{d}t \, \mathrm{Im}\, \Pi(t) \frac{1}{rt} \times$$
$$\times \left[ \frac{1}{rt} \left( 1 - e^{-r\sqrt{t}} \right) - \frac{1}{2\sqrt{t}} \, e^{-r\sqrt{t}} \right]. \quad (49)$$

To compute the $2P-2S$ splitting, we consider

$$\int d\mathbf{r}\Big(|\psi_{210}|^2-|\psi_{200}|^2\Big)\Big[\frac{1}{M^2r^2}\left(1-e^{-Mr}\right)-$$

$$-\frac{1}{2Mr}\,e^{-Mr}\Big]=\frac{aM}{12}\frac{1-2aM}{(1+aM)^4},\qquad(50)$$

where $a=(Z\alpha m_r)^{-1}$ is the Bohr radius.

Thus, we have again the same result for the Lamb shift

$$\Delta E^{(\mathrm{VPret})}(2P-2S)=$$

$$=Z\alpha\frac{1}{3\pi}\int\limits_{t_0}^{\infty}dt\,\frac{\mathrm{Im}\,\Pi(t)}{\sqrt{t}}\,\frac{1-2a\sqrt{t}}{(1+a\sqrt{t})^4}\simeq 300\ \mu\mathrm{eV}.\qquad(51)$$

## 6. Conclusion

We have shown that the discrepancy between the PSI [2] and CODATA [3] results of the high-precision experimental determination of the proton charge radius, which has triggered a lively discussion adressing not only the accuracy of these experiments, but also the bound-state quantum electrodynamics, proton structure, Rydberg constant, and even a possibility of a new physics, can be explained on the basis of the quantum electrodynamics effect, namely the precise calculation of the contribution of the retardation effects to the total energy splitting. We find that the contribution of these effects is ~0.3 meV. This value is very close to 0.31 meV, which would be required to match the results of PSI measurements with the CODATA value of charge proton radius $r_p=0.8768(69)$ fm.

We have to note that our results, which are based on the one-photon exchange, is not gauge invariant. One should consider the two-photon diagrams, which are indeed gauge invariant, to obtain the complete contribution, $(Z\alpha)^4(m^2/M)$ and $(Z\alpha)^4 m(m/M)^2$. This will be done in our future work, but obviously that the account for these terms does not change drastically the results obtained in the present work.

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**APPENDIX. Integrals**

The integral

$$I(\mathbf{p},\mathbf{p}')=$$

$$=\int\limits_{-\infty}^{\infty}\frac{d\ell_0}{2\pi i}\frac{d\ell_0'}{2\pi i}\,\frac{1}{\big[(x_aP-\ell)^2-m_a^2+i\varepsilon\big]\big[(x_bP+\ell)^2-m_b^2+i\varepsilon\big]}\times$$

$$\times\frac{1}{\big[(x_aP-\ell')^2-m_a^2+i\varepsilon\big]\big[(x_bP+\ell')^2-m_b^2+i\varepsilon\big]}\times$$

$$\times(\ell_0-\ell_0')^2\qquad(\mathrm{A1})$$

can be easy rewritten as

$$I(\mathbf{p},\mathbf{p}')=G_2(\boldsymbol{\ell}^2;\,s)G(\boldsymbol{\ell}'^2;\,s)-2G_1(\boldsymbol{\ell}^2;\,s)G_1(\boldsymbol{\ell}'^2;\,s)+$$

$$+G(\boldsymbol{\ell}^2;\,s)G_2(\boldsymbol{\ell}'^2;\,s),$$

where

$$G(\boldsymbol{\ell}^2;\,s)=\frac{1}{2\pi i}\times$$

$$\times\int\limits_{-\infty}^{\infty}d\ell_0\,\frac{1}{\big[(x_aP-\ell)^2-m_a^2+i\varepsilon\big]\,\big[(x_bP+\ell)^2-m_b^2+i\varepsilon\big]}=$$

$$=\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}d\ell_0\,\frac{1}{\big[(x_aP_0-\ell_0-\omega_a+i\varepsilon)(x_aP_0-\ell_0+\omega_a-i\varepsilon)\big]}\times$$

$$\times\,\frac{1}{\big[(x_bP_0+\ell_0-\omega_b+i\varepsilon)(x_bP_0+\ell_0+\omega_b-i\varepsilon)\big]},\qquad(\mathrm{A2})$$

$$\omega_i=(m_i^2+\boldsymbol{\ell}^2)^{1/2}.$$

The poles in the lower half-plane are

$$\ell_0^{(a)}=x_aP_0+\omega_a-i\varepsilon,\qquad(\mathrm{A3})$$

$$\ell_0^{(b)}=-x_bP_0+\omega_b-i\varepsilon.\qquad(\mathrm{A4})$$

Noting the extra minus sign from going over the lower half-plane contour, we obtain

$$G(\boldsymbol{\ell}^2;\,s)=-\Big\{\frac{1}{2\omega_a[(P_0+\omega_a)^2-\omega_b^2+i\varepsilon]}+$$

$$+\frac{1}{2\omega_b[(P_0-\omega_b)^2-\omega_a^2+i\varepsilon]}\Big\}=-\Big\{\frac{1}{4\omega_a(x_as+\omega_a\sqrt{s}+i\varepsilon)}+$$

$$+\frac{1}{4\omega_b(x_bs-\omega_b\sqrt{s}+i\varepsilon)}\Big\}.\qquad(\mathrm{A5})$$

Finally, using the fact that

$$x_a^2s-\omega_a^2=x_b^2s-\omega_b^2=\lambda(s)/s-\boldsymbol{\ell}^2,\qquad(\mathrm{A6})$$

we find

$$G(\boldsymbol{\ell}^2;\,s)=-\frac{1}{\lambda(s)/s-\boldsymbol{\ell}^2+i\varepsilon}\times$$

$$\times\left(\frac{x_as-\omega_a\sqrt{s}}{4\omega_as}+\frac{x_bs+\omega_b\sqrt{s}}{4\omega_bs}\right)=$$

$$=-\frac{1}{\lambda(s)/s-\boldsymbol{\ell}^2+i\varepsilon}\left(\frac{x_a}{4\omega_a}+\frac{x_b}{4\omega_b}\right).\qquad(\mathrm{A7})$$

The next step is as follows:

$$G_1(\boldsymbol{\ell}^2;\, s) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} d\ell_0\, \ell_0 \times$$

$$\times \frac{1}{\left[(x_a P - \ell)^2 - m_a^2 + i\varepsilon\right]\left[(x_b P + \ell)^2 - m_b^2 + i\varepsilon\right]} =$$

$$= \frac{1}{2\pi i}\frac{1}{2\sqrt{s}}\int\limits_{-\infty}^{\infty} d\ell_0 \frac{1}{\left[(x_a P_0 - \ell_0 - \omega_a + i\varepsilon)(x_a P_0 - \ell_0 + \omega_a - i\varepsilon)\right]} -$$

$$- \frac{1}{2\pi i}\frac{1}{2\sqrt{s}}\int\limits_{-\infty}^{\infty} d\ell_0 \frac{1}{\left[(x_b P_0 + \ell_0 - \omega_b + i\varepsilon)(x_b P_0 + \ell_0 + \omega_b - i\varepsilon)\right]},$$

$$\omega_i = (m_i^2 + \boldsymbol{\ell}^2)^{1/2}. \tag{A8}$$

The poles in the lower half-plane are

$$\ell_0^{(a)} = x_a P_0 + \omega_a - i\varepsilon \tag{A9}$$

in the first integral and

$$\ell_0^{(b)} = -x_b P_0 + \omega_b - i\varepsilon \tag{A10}$$

in the second integral. Noting the extra minus sign from going over the lower half-plane contour, we obtain

$$G_1(\boldsymbol{\ell}^2;\, s) = \frac{1}{4\sqrt{s}}\left\{\frac{1}{\omega_a} - \frac{1}{\omega_b}\right\}. \tag{A11}$$

Then

$$G_2(\boldsymbol{\ell}^2;\, s) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} d\ell_0\, {\ell_0}^2 \times$$

$$\times \frac{1}{\left[(x_a P - \ell)^2 - m_a^2 + i\varepsilon\right]\left[(x_b P + \ell)^2 - m_b^2 + i\varepsilon\right]} =$$

$$= \frac{1}{2\pi i}\frac{1}{2\sqrt{s}}\int\limits_{-\infty}^{\infty} d\ell_0 \ell_0 \frac{1}{\left[(x_a P_0 - \ell_0 - \omega_a + i\varepsilon)(x_a P_0 - \ell_0 + \omega_a - i\varepsilon)\right]} -$$

$$- \frac{1}{2\pi i}\frac{1}{2\sqrt{s}}\int\limits_{-\infty}^{\infty} d\ell_0 \ell_0 \frac{1}{\left[(x_b P_0 + \ell_0 - \omega_b + i\varepsilon)(x_b P_0 + \ell_0 + \omega_b - i\varepsilon)\right]},$$

$$\omega_i = (m_i^2 + \boldsymbol{\ell}^2)^{1/2}. \tag{A12}$$

The poles in the lower half-plane are

$$\ell_0^{(a)} = x_a P_0 + \omega_a - i\varepsilon \tag{A13}$$

in the first integral and

$$\ell_0^{(b)} = -x_b P_0 + \omega_b - i\varepsilon \tag{A14}$$

in the second integral. Noting the extra minus sign from going over the lower half-plane contour, we obtain

$$G_2(\boldsymbol{\ell}^2;\, s) = \frac{1}{4}\left\{\frac{x_b}{\omega_b} + \frac{x_a}{\omega_a}\right\}. \tag{A15}$$


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ВНЕСОК ЕФЕКТІВ ЗАПІЗНЮВАННЯ ДО ПОВНОЇ ЕНЕРГІЇ РОЗЩЕПЛЕННЯ ВОДНЕПОДІБНИХ АТОМІВ

Р е з ю м е

Ми показали, що розбіжність результатів PSI та CODATA з високоточного експериментального виміру зарядового радіуса протона може бути пояснена в рамках квантової електродинаміки, а саме завдяки врахуванню внеску ефектів запізнювання до повної енергії розщеплення. Розрахований нами внесок становить ∼0,3 меВ, що є дуже близьким до значення 0,31 меВ, на яке відрізняються результати вимірювання зарядового радіуса протона PSI та CODATA.