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ONE-PHOTON SCATTERING BY *N*-ATOM SYSTEM: APPLICATION TO ONE- AND TWO-MODE RESONATOR

The system of N identical two-level atoms coupled with a quantized electromagnetic field prepared via a one-photon Fock state is investigated. The corresponding N-particle state amplitudes in one- and two-mode resonators are calculated for several space configurations in the cases of closed conservative and open dissipative systems. The nature and the structure of the Weisskopf–Wigner approximation is revealed in the many-body problem. It is shown that the space distribution of atoms, the total number of atoms, and even the available volume for the field modes define the behavior of system's state amplitudes in time. The elaborated theory allows one to analytically describe the time evolution of the system for a quite wide range of the space configurations, if the specific "cyclic" restrictions are applied.

K e y w o r d s: one-photon scattering, one- and two-mode resonator.

1. Introduction

In this work we continue our investigation of the N-atom system coupled with a quantized electromagnetic field initially prepared in the one-photon Fock state. The proposed results represent the application of the theory developed in the recent paper [1]. Here, as a partial case for [1], the distances between atoms are assumed to be quite large, so that the average distances between atoms are larger (or of the same order) than the "resonant transition" wavelength. The ensemble of N two-level atoms is initially in the ground state. The initial one-photon Fock state of the electromagnetic field is specified completely by its wave vectors $\mathbf{k}_0$ with the atomic transition frequency $\omega = c|\mathbf{k}_0|$ and its polarization j ($j = 1, 2$). The main goal of this investigation is to obtain the information about the state of the atomic system and the electromagnetic field in the Weisskopf–Wigner approximation (see [2] Chapt. 6, p. 206 and comments in [1]). The calculations of the state amplitudes are made for several approximations in resonator (cavity) characteristics and for several types of the space configurations of atoms.

A many-body system interacting with a quantized electromagnetic field can possess certain fundamental properties. For example, when the average distances between atoms are much less than the "resonant transition" wavelength of emitted (absorbed) light, the cooperative coupling leads to a substantial radiative shift of the transition energy and a significant change in the decay rate of the ensemble state. The later was revealed through the various theoretical (e.g., some relatively modern researches in [3–6]). The common features in the system behavior can manifests even if initially only one atom or one photon state is excited.

In comparison with other researches in this domain, we will develop a direct consistent solution of the N-particle equations describing the time evolution of the N atomic probability state amplitudes. In addition, in a certain meaning, we will explain the nature of the widely used Weisskopf–Wigner approximation, that was not found in the scientific literature reviewed by us. The question of constructing a correct damping model is not a new one. The discussion of approaches to realistic systems, including damping effects, can be

found, for example, in [10–13]. The later actually introduced the concept of thermalization of the system states. In practice, the thermalization can be combined with coherent effects, as in the case of "superradiance" [6, 14–17], that can in turn be destroyed by the Doppler broadening of the atomic (electronic) resonant frequencies (see the analysis in [18]).

As in work [1], we describe the dynamical evolution of the system, by reviewing, therefore, all the N atomic states. Thus, our work differs from such investigations as [19] and [20], which introduce, generally saying, the evolutional hierarchy when the atoms-field system can be considered as stochastic. For comparison with the results of the mentioned kinetic method, the dynamical system of two atoms coupled with electromagnetic field was analyzed in [21].

The material in the sections of this paper is arranged in the increasing order of generalization, and subsections are in the order of difficulty level.

Let us first provide below some general theoretical premises. More detailed derivations of the corresponding mathematical model can be found in [1].

2. The Equations of Motion for the State Amplitudes

Let therefore consider a collection of N identical atoms located by the corresponding radius-vectors $(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_\alpha, ..., \mathbf{k}r_N)$ and coupled to a one-mode electromagnetic field. Each atom $\alpha = 1, ..., N$ is assumed to have only the two states $|a\rangle_\alpha$ and $|b\rangle_\alpha$ separated by the energy $E_\alpha = E_{a\alpha} - E_{b\alpha} = \hbar\omega$. In the dipole approximation, the closed conservative system of identical atoms with the electromagnetic field in a cavity can be described by the Hamiltonian consisting of the terms representing free atoms and the electromagnetic field with the dipole-field coupling between the atoms and the electromagnetic field modes.

Inasmuch as, at the initial time moment $t = 0$, all the atoms $\alpha = 1, ..., N$ in the ensemble are in the ground state $|b\rangle_\alpha$ and the electromagnetic field is in the Fock state $|1_{\mathbf{k_0}}\rangle$ (that presents one photon with the wave vector $\mathbf{k}_0$), we look for a solution of the corresponding Schrödinger equation in the interacting picture in the following form:

$$\Psi = \sum_{\alpha=1}^{N} \beta_\alpha(t)|b_1 b_2 ... a_\alpha ... b_N\, 0\rangle + \\ + \sum_{\mathbf{k},j} \gamma_{\mathbf{k},j}(t)|b_1 b_2 ... b_N 1_{\mathbf{k},j}\rangle \quad (1)$$

with the initial conditions

$$\beta_\alpha(0) = 0, \quad \gamma_{\mathbf{k},j}(0) = \delta_{\mathbf{k},\mathbf{k_0}}\delta_{j,j_0}, \quad (2)$$

where $\delta_{\mathbf{k},\mathbf{k_0}}$ is the Kronecker delta-symbol ($\delta_{\mathbf{k},\mathbf{k}_0} = 1$ if $\mathbf{k} = \mathbf{k}_0$, and $\delta_{\mathbf{k},\mathbf{k}_0} = 0$ if $\mathbf{k} \neq \mathbf{k}_0$). The same meaning is supposed to have the Kronecker delta-symbol δ_{j,j_0}. The quantities $\beta_\alpha(t)$ ($\alpha = 1, ..., N$) and $\gamma_{\mathbf{k},j}(t)$ ($j = 1, 2$) are the atomic α excited state amplitude with the others in the ground states and the excited Fock field state amplitude of the j-th polarization with the wave vector $\mathbf{k}$, respectively.

Then the corresponding Schrödinger equation in the interacting picture yields the following system of equations:

$$\dot{\beta}_\alpha(t) = i\sum_{\mathbf{k},j} g^*_\alpha(\mathbf{k}, j)\gamma_{\mathbf{k},j}(t) \times \\ \times \exp(-i(\nu_k - \omega)t + i\mathbf{k}\mathbf{r}_\alpha); \quad (3)$$

$$\dot{\gamma}_{\mathbf{k},j}(t) = i\sum_{\delta=1}^{N} g_\delta(\mathbf{k}, j)\beta_\delta(t)\exp(i(\nu_k - \omega)t - i\mathbf{k}\mathbf{r}_\delta), \quad (4)$$

where

$$g_\alpha(\mathbf{k}, j) = \sqrt{\frac{\nu_k}{2\hbar\epsilon_0 V}}\,\wp_\alpha \cdot \mathbf{e}_{\mathbf{k},j}, \quad (5)$$

with

$$\wp_\alpha \cdot \mathbf{e}_{\mathbf{k},j} = e^{i\phi_\alpha}|\wp_\alpha|\cos\theta_{\mathbf{k},j}. \quad (6)$$

Here, ϕ_α denotes some phase, $\theta_{\mathbf{k},j}$ is the angle between the dipole vector $\wp_\alpha = e\langle a|\mathbf{r}_{\alpha\,\mathrm{el}}|b\rangle$ (that represents the non-diagonal transition dipole matrix element of a valence electron for the α-th atom) and the j-th unit polarization vector $\mathbf{e}_{\mathbf{k},j}$ ($j = 1, 2$ and $\mathbf{e}_{\mathbf{k},j} \cdot \mathbf{k} = 0$), and V is the volume available for the atoms-field system.

Substituting Eq. (4) into (3), differentiating one more time, and applying the Weisskopf–Wigner approximation (details in [1]), we derive the following dynamical system:

$$\frac{d^2}{dt^2}\beta_\alpha(t) = -\sum_{\delta=1}^{N}\beta_\delta(t)\Phi_{\alpha\delta} - 2D_\alpha\frac{d}{dt}\beta_\alpha(t), \quad (7)$$

where

$$\Phi_{\alpha\delta} = \sum_{j,\,|\mathbf{k}|=k_0} g^*_\alpha(\mathbf{k}, j)g_\delta(\mathbf{k}, j)\exp[i\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_\delta)]. \quad (8)$$

We have also

$$D_\alpha = \frac{1}{2}\frac{8\pi}{3}\frac{2\pi}{2\hbar\varepsilon_0}\left(\frac{1}{2\pi\, c}\right)^3 |\wp_\alpha|^2\omega^3 =$$

$$= \frac{1}{2}\frac{1}{3\pi\,\hbar\varepsilon_0 c^3}\,|\wp_\alpha|^2\omega^3. \tag{9}$$

Here, the coefficients D_α, $\alpha = 1, ..., N$, describe the respective (single atom) rate of decay for the α-th atom excited state. Note that the "non-resonant" terms for the particles with indices different from α were disregarded here by the assumption of quite large interatomic distances (see [1]).

3. Applications

Below, for simplicity, only one polarized mode ($j = 1$) of the resonant field modes is taken into account with the common parameters g_α and $\wp_\alpha$ for $\alpha = 1, ..., N$:

$$g_\alpha(\mathbf{k}, j) = g_\alpha = g > 0 \tag{10}$$

and

$$\wp_\alpha \cdot \mathbf{e}_{\mathbf{k},j} = |\wp| \tag{11}$$

for $|\mathbf{k}| = k_0$. In other words, the space angle distribution for the components $\Phi_{\alpha\delta}$ is disregarded here, assuming that the direction of the transition dipole moment $\wp_\alpha$ for any atom in the system coincides with the photon polarization under the absorption or emission of a resonant photon.

We note also that the cited thereby Rabi frequencies (such like $\Omega_{1,\,2}$ introduced later) and all the quantities that were calculated in all the instances below are present in the SI system of units with the following notation: $\hbar \approx 1.05457 \times 10^{-34}$ J sec/rad; the electric permittivity of free space $\epsilon_0 \approx 8.8542 \times 10^{-12}$ F/m; the speed of light in free space $c = 299792458$ m/sec; resonant wavelength close to the D_2-line of a sodium atom $\lambda_{\rm D} = 589.29 \times 10^{-9}$ m; corresponding circular (in radians per second) resonant frequency $\omega_{\rm res} = \frac{2\pi c}{\lambda_{\rm D}} \approx 0.101747 \times 10^{16}\pi$ rad/sec; non-diagonal so-called "transition" dipole matrix element (in the same order as for the D_2-line transition, i.e., about 1 Debye) $\rho_{\rm ex} \approx 1 \times 3.33564 \times 10^{-30}$ C m. For instance, if the volume available for the atoms-field system has the value equal to $V = 0.001\,{\rm m}^3$, then $g = \rho_{\rm ex}\sqrt{\frac{\omega_{\rm res}}{2\hbar\epsilon_0 V}} \approx 77.8597\sqrt{\pi}$ rad/sec.

3.1. The system in a one-mode resonator

3.1.1. Conservative system: without the damping ($D_\alpha = 0$)

In the resonance approximation, neglecting a frequency detuning, Eqs. (3) and (4) take the following form:

$$\dot\beta_\alpha(t) = ig\,\gamma_k(t)\exp(i\mathbf{k}\mathbf{r}_\alpha), \quad \alpha = 1, ..., N; \tag{12}$$

$$\dot\gamma_k(t) = i\sum_{\delta=1}^{N} g\,\beta_\delta(t)\exp(-i\mathbf{k}\mathbf{r}_\delta). \tag{13}$$

With regard for the above-mentioned initial conditions and requiring the normalized amplitudes for a closed system, we obtain the following solution of the above-introduced system:

$$\gamma_k(t) = \frac{1}{2}\left(\exp(i\Omega_1 t) + \exp(-i\Omega_1 t)\right), \tag{14}$$

$$\beta_\alpha(t) = \frac{1}{2\sqrt{N}}\left(\exp(i\mathbf{k}\mathbf{r}_\alpha + i\Omega_1 t) - \exp(i\mathbf{k}\mathbf{r}_\alpha - i\Omega_1 t)\right), \tag{15}$$

where $\Omega_1 = \sqrt{N}g$.

Therefore, in this simple case, the state amplitudes oscillate with the collective Rabi frequency Ω_1 proportional to the square root of the total particle number. At the same time, the atomic state amplitude is inversely proportional to the the square root of the total particle number, thus decreasing with increasing the number of participating atoms. We see, that, even in such relatively simple resonant approximation, the N-atom system acts as a whole, "distributing" the possibility to absorb the one-photon Fock state energy equally among all the atoms.

Furthermore, it is obvious that the collective Rabi frequency Ω_1 depends on the g parameter and, hence, on the available volume for the atoms-field system. Some interesting features of this dependence will be discussed in the following subsections.

3.1.2. With the damping: $D_\alpha \neq 0$

For $D_\alpha = D$, $\alpha = 1, ..., N$, neglecting the non-diagonal coefficients when $D_{\alpha\delta} \ll D$, the corresponding system of equations (7) reduces to

$$\frac{d^2}{dt^2}\beta_\alpha(t)\exp(-i\mathbf{k}\mathbf{r}_\alpha) = -g^2\sum_{\delta=1}^{N}\beta_\delta(t)\exp(-i\mathbf{k}\mathbf{r}_\delta) -$$

$$-2D\frac{d}{dt}\beta_\alpha(t)\exp(-i\mathbf{k}\mathbf{r}_\alpha). \tag{16}$$

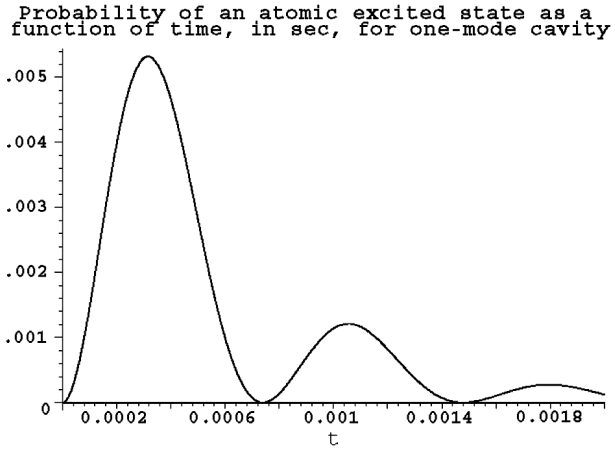

Fig. 1. Time evolution of $|\beta_\alpha(t)|^2$ for $D < \Omega_1$. Here, $V = 0.0001\,\mathrm{m}^3$; total number of particles $N = 100$; single atom decay rate is $D = 1000\,\mathrm{rad/sec}$; $g = 246.2140988\pi^{1/2}\,\mathrm{rad/sec}$; $\Omega_1 = 2462.140988\pi^{1/2}\,\mathrm{rad/sec}$

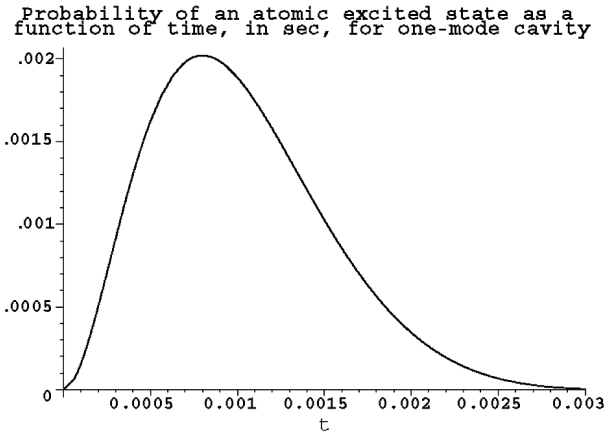

Fig. 2. Time evolution of $|\beta_\alpha(t)|^2$ for $D > \Omega_1$. Here, $V = 0.001\,\mathrm{m}^3$; total number of particles $N = 100$; single atom decay rate is $D = 1000\,\mathrm{rad/sec}$; $g = 77.85973441\pi^{1/2}\,\mathrm{rad/sec}$; $\Omega_1 = 778.5973441\pi^{1/2}\,\mathrm{rad/sec}$

In the case of $D \neq \Omega_1$ and $\beta_\alpha(0) = 0$ for $\alpha = 1, ..., N$, its solutions have the following form:

$$\beta_\alpha(t) = C\left(\exp\left[i\mathbf{kr}_\alpha + \left(-D + \sqrt{D^2 - \Omega_1^2}\right)t\right] - \right.$$
$$\left. - \exp\left[i\mathbf{kr}_\alpha + \left(-D - \sqrt{D^2 - \Omega_1^2}\right)t\right]\right). \qquad (17)$$

Here, it has to be noted that the constant C can be determined, when the initial condition $\frac{d}{dt}\beta_\alpha(t)\,|_{t=0}$ is defined. For example, let us suppose that the initial derivative is defined through the "exactly conservative" initial system of equations (3) with $\gamma_k(0) = 1$ for $k = k_0$ corresponding to the resonant wavelength. Then

$$C = \frac{i\Omega_1}{2\sqrt{N}\sqrt{D^2 - \Omega_1^2}}. \qquad (18)$$

Roughly saying, the found coefficient should yield "right" limitations on the squared modulus of the atomic state amplitude. So that, for a quite wide range of the values of decay coefficient D, the following quite rough restriction can be required:

$$\left|\frac{\sqrt{D^2 - \Omega_1^2}}{\Omega_1}\right| > \frac{1}{2\sqrt{N}}. \qquad (19)$$

The situation $D < \Omega_1$ differs from the case of $D > \Omega_1$ by the character of the evolution in time, as it is shown in Figs. 1 and 2, respectively.

To depict the first case $D < \Omega_1$ in Fig. 1, the following parameters were set up: system's volume $V = 0.0001\,\mathrm{m}^3$; the total number of particles $N = 100$; the space phase $\mathbf{kr} = \pi/6 + 3\pi$; the single atom decay rate $D = 1000\,\mathrm{rad/sec}$. Then $g = 246.2140988\pi^{1/2}\,\mathrm{rad/sec}$; $\Omega_1 = 2462.140988\pi^{1/2}\,\mathrm{rad/sec}$, corresponding on the average to about twenty resonant wavelengths between any two nearest atoms. Note that, here and in the next several statements concerning the provided graphs, the word "average" means that the atoms can actually be located much closer to each other not filling the given volume V uniformly.

Note that the same character of the time evolution as is in Fig. 1, but with the maximum value of the atomic excited state probability about 8.2×10^{-13}, is produced, for example, by the following combination: $V = 10^{-2}\,\mathrm{m}^3$; $N = 5 \times 10^{11}$; the space phase $\mathbf{kr} = \pi/6 + 3\pi$; $D = 10^7\,\mathrm{rad/sec}$. Then $g = 24.62140988 \times \pi^{1/2}\,\mathrm{rad/sec}$ and $\Omega_1 \approx 0.12311 \times 10^8 (2\pi)^{1/2}$. The ratio between the given volume V and the total number of atoms N corresponds, on the average, to about two resonant wavelengths between any two nearest atoms ($|\mathbf{k} \cdot (\mathbf{r}_\alpha - \mathbf{r}_\delta)| \geq 4\pi$).

Figure 2 shows the time evolution for the probability to find an atom in the excited state when $D > \Omega_1$. Such situation corresponds in our model to the volume $V = 0.001\,\mathrm{m}^3$; the total number of particles $N = 100$; the space phase (actually any

choice here is acceptable) is equal, for example, $\mathbf{kr} = \pi/6 + 3\pi$; the single atom decay rate $D = 1000\,\mathrm{rad/sec}$. Then $g = 77.85973441\pi^{1/2}\,\mathrm{rad/sec}$; $\Omega_1 = 778.5973441\pi^{1/2}\,\mathrm{rad/sec}$, corresponding to about two hundreds of resonant wavelengths between any two nearest atoms on the average.

The same character of the time evolution as in Fig. 2, but with the maximum value of the atomic excited state probability about 2.5×10^{-13}, is produced, for example, by the following combination: $V = 10^{-1}\,\mathrm{m}^3$; $N = 5 \times 10^{11}$; the space phase $\mathbf{kr} = \pi/6 + 3\pi$; $D = 10^7\,\mathrm{rad/sec}$. This yields $g = 7.785973441\pi^{1/2}\,\mathrm{rad/sec}$ and $\Omega_1 \approx 0.3893 \times \times 10^7(2\pi)^{1/2}\,\mathrm{rad/sec}$. The values of V and N correspond to about twelve resonant wavelengths between any two nearest atoms on the average ($|\mathbf{k}\cdot(\mathbf{r}_\alpha - \mathbf{r}_\delta)| \geq \geq 24\pi$).

As was supposed in the derivation of the differential equations with the damping terms such like (16) (see the details in work [1]), the available volume V (*assumed here as a rectangular box*!) for the whole atoms-field system defines all the "available" modes for the electromagnetic field. Roughly saying, the restriction on the volume V in this sense determines the lower cutoff frequency for all the available resonant harmonics. In this case, we can discuss the following. Inasmuch as Ω_1 depends on the factor g, the system behavior, generally saying, depends on the "available" volume for the electromagnetic field. The value of volume V can determine one of the inequalities $D < \Omega_1$ and $D > \Omega_1$, therefore defining the character of the relaxation of the system. Besides the above-mentioned parameter V, the collective characteristic frequency Ω_1 depends on the total number of particles N. Therefore, the total number of atoms N defines the character of the relaxation of the system as well. These features are reflected in Figs. 1 and 2 supplied by the corresponding comments.

3.2. The system in a two-mode resonator

3.2.1. Resonance distances
$|\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_{\alpha\pm1})| = 2\pi n$, *where* n *is an integer.* $D_\alpha = 0$

Again, as in the previous subsection, the corresponding system of differential equations can be derived from the initial "closed" conservative equations (3) and (4) in the resonant approximation.

Without any loss of generality, we set $\mathbf{k}\cdot\mathbf{r}_1 = 0$. Then the condition of the "resonant distances" allows us to refer the problem to the following system of homogeneous linear differential equations:

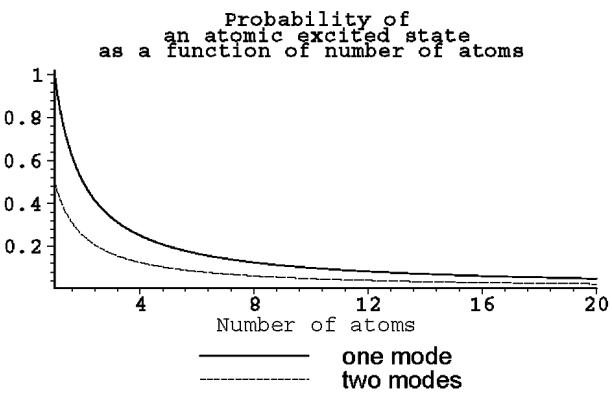

Fig. 3. $|\beta_1|^2(\mathbf{kr}_1, N, t_1)$ for a one-mode cavity, $|\beta_\alpha|^2(N, t_1')$ for a two-mode cavity. N is the total number of atoms in the system, $V = 0.1\,\mathrm{m}^3$, $D = 10^7\,\mathrm{rad/sec}$

$$\frac{d^2}{dt^2}\beta_\alpha(t) = -2g^2\sum_{\delta=1}^{N}\beta_\delta(t), \text{ where } \alpha = 1, ..., N. \quad (20)$$

It is easy to show that

$$\frac{d}{dt}\gamma_{\mathbf{k}}(t) = \frac{d}{dt}\gamma_{-\mathbf{k}}(t) = ig\sum_{\delta=1}^{N}\beta_\delta(t). \quad (21)$$

Therefore, the normalized solution of the above system satisfying the initial conditions $\gamma_{\mathbf{k}}(0) = 1$, $\gamma_{-\mathbf{k}}(0) = 0$, and $\beta_\alpha(0) = 0$ with $\frac{d}{dt}\beta_\alpha(0) = ig$ is

$$\gamma_{\mathbf{k}}(t) = \frac{1}{2}\left[\cos\left(\Omega_1' t\right) + 1\right], \quad (22)$$

$$\gamma_{-\mathbf{k}}(t) = \frac{1}{2}\left[\cos\left(\Omega_1' t\right) - 1\right], \quad (23)$$

$$\beta_\alpha(t) = i\frac{g}{\Omega_1'}\sin\left(\Omega_1' t\right), \quad (24)$$

where $\Omega_1' = g\sqrt{2N}$. As we can see, the frequency Ω_2 for the system in a two-mode resonator differs from that for the system in a one-mode resonator $\Omega_1 = g\sqrt{N}$: $\Omega_1'/\Omega_1 = \sqrt{2}$.

Furthermore, as can be seen from Fig. 3, the probability to find the α-th atom in the excited state for a two-mode resonator is lower in comparison with that for a one-mode resonator. To depict the found dependences of (24) and (17) with (18) on

the total number of atoms N, the parameters were set as follows: $V = 0.1\,\mathrm{m}^3$; $N = 1.20$; $\mathbf{kr}_1 = \frac{2\pi}{3}$; $D = 10^7$ rad/sec; $g = 7.785973441\,\pi^{1/2}$ rad/sec; $\Omega_1(N) = g\sqrt{N}$, $\Omega_1^{'}(N) = g\sqrt{2N}$, $t_1 = \frac{\pi}{(2\Omega_1(N))}$ and $t_1^{'} = \frac{\pi}{(2\Omega_1^{'}(N))}$. Based on the obtained expressions, the same character of the dependence $|\beta_\alpha|(N, t_1^{'})$ on the total number of atoms N is proper for any atom out of the labeled $\alpha = 1, ..., N$, for any value of the volume V, and the single atom decay rate D at the given time moments $t_1^{'}$.

Despite the relative simplicity of the solutions found in the above subsections, the corresponding methodology of calculation is not trivial with respect to the raised physical problems. Some interest therefore invokes the method of solving the N-particle linear differential equations. In the next subsections and sections, we provide the procedure of solving the N-particle linear differential equations in more general cases.

3.2.2. Cyclically distanced atoms $\sum_\alpha^N \sin(\mathbf{kr}_\alpha)\cos(\mathbf{kr}_\alpha) = 0$. $D_\alpha = 0$

From the system of equations (7) with the "resonant" wave vectors $\pm\mathbf{k}$, neglecting the decay terms $D_\alpha = 0$ for $\alpha = 1, ..., N$, we can obtain the following "reduced" system of linear differential equations:

$$\frac{d^2}{dt^2}\beta_\alpha(t) = -2g^2 \sum_{\delta=1}^{N} \beta_\delta(t)\cos(\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_\delta)),\ \alpha = 1, ..., N. \tag{25}$$

Using the notation

$$B_c\,(t) = \sum_{\delta}^{N} \beta_\delta\,(t)\cos\,(\mathbf{kr}_\delta), \tag{26}$$

and the requirement of cyclicity for the atomic space, $\mathbf{kr}_\alpha$ $(\alpha = 1, ..., N)$, we derive from the system of equations (25) that

$$\frac{d^2}{dt^2}B_c = -2g^2 \sum_{\alpha}^{N} \cos^2\,(\mathbf{kr}_\alpha)\,B_c. \tag{27}$$

Therefore, the solution of the above equation is as follows:

$$B_c = \sum_\alpha \beta_\alpha(t)\cos(\mathbf{kr}_\alpha) =$$

$$= C_o\,(\exp\,(i\Omega_2 t) - \exp\,(-i\Omega_2 t)), \tag{28}$$

where $\Omega_2 = g\sqrt{2\sum_\alpha \cos^2(\mathbf{kr}_\alpha)}$.

By analogy,

$$B_s = \sum_\alpha \beta_\alpha(t)\sin(\mathbf{kr}_\alpha) =$$

$$= C_o^{'}\left(\exp\left(i\Omega_2^{'}t\right) - \exp\left(-i\Omega_2^{'}t\right)\right), \tag{29}$$

where $\Omega_2^{'} = g\sqrt{2\sum_\alpha \sin^2(\mathbf{kr}_\alpha)}$.

By definition, the introduced Rabi frequencies $\Omega_2^{'}$ and Ω_2 satisfy the relation $\Omega_2^{'2} + \Omega_2^2 = 2g^2N$.

In the approximation described here, the solution $\beta_\alpha\,(t)$ $(\alpha = 1, ..., N)$ can be found by substituting the above-established expressions for B_c and B_s into the initial second-order differential equation (25) with its following two-time integration over the time variable. The last requires the knowledge of the time derivative for atomic state amplitudes at the initial time moment $t = 0$. In other words, the substitution into the initial system of equations (4) can be performed with regard for the initial conditions $\gamma_{\mathbf{k}}(0) = 1$ and $\gamma_{-\mathbf{k}}(0) = 0$. Substituting the result into the initial system of equations (3) and performing the time integration, we obtain

$$\gamma_{\mathbf{k}}(t) = 2ig\left(i\frac{C_o}{\Omega_2}\,(1 - \cos(\Omega_2 t)) + \right.$$
$$\left. + \frac{C_o^{'}}{\Omega_2^{'}}\left(1 - \cos(\Omega_2^{'}t)\right)\right) + 1, \tag{30}$$

$$\gamma_{-\mathbf{k}}(t) = 2ig\left(i\frac{C_o}{\Omega_2}\,(1 - \cos(\Omega_2 t)) - \right.$$
$$\left. - \frac{C_o^{'}}{\Omega_2^{'}}\left(1 - \cos(\Omega_2^{'}t)\right)\right), \tag{31}$$

$$\beta_\alpha(t) = -2g^2\left(\left[i\frac{C_o}{\Omega_2}\left(t - \frac{1}{\Omega_2}\sin(\Omega_2 t)\right) + \right.\right.$$
$$\left. + \frac{C_o^{'}}{\Omega_2^{'}}\left(t - \frac{1}{\Omega_2^{'}}\sin(\Omega_2^{'}t)\right)\right]\times$$
$$\times\,\exp(i\mathbf{kr}_\alpha) + \frac{1}{2ig}t\,\exp(i\mathbf{kr}_\alpha) +$$
$$+\left[i\frac{C_o}{\Omega_2}\left(t - \frac{1}{\Omega_2}\sin(\Omega_2 t)\right) - \frac{C_o^{'}}{\Omega_2^{'}}\left(t - \frac{1}{\Omega_2^{'}}\sin(\Omega_2^{'}t)\right)\right]\times$$
$$\left.\times\,\exp(-i\mathbf{kr}_\alpha)\right). \tag{32}$$

The terms proportional to the time variable t can be eliminated by the following choice of the constant coefficients C_o and $C_o^{'}$:

$$C_o = \frac{\Omega_2}{4g}, \quad C_o^{'} = \frac{\Omega_2^{'}}{\Omega_2} i\, C_o. \tag{33}$$

Thus,

$$\gamma_{\mathbf{k}}(t) = \frac{1}{2}\left(\cos(\Omega_2 t) + \cos(\Omega_2^{'} t)\right), \tag{34}$$

$$\gamma_{-\mathbf{k}}(t) = \frac{1}{2}\left(\cos(\Omega_2 t) - \cos(\Omega_2^{'} t)\right), \tag{35}$$

$$\beta_\alpha(t) = ig\left(\frac{1}{\Omega_2}\sin(\Omega_2 t)\cos(\mathbf{kr}_\alpha) + \right.$$
$$\left. + i\frac{1}{\Omega_2^{'}}\sin(\Omega_2^{'} t)\sin(\mathbf{kr}_\alpha)\right), \tag{36}$$

where $\Omega_2 = g\sqrt{2\sum_\alpha \cos^2(\mathbf{kr}_\alpha)}$ and $\Omega_2^{'} = g\sqrt{2\sum_\alpha \sin^2(\mathbf{kr}_\alpha)}$, $\mathbf{k}$ is the "resonant" wave vector for the one-photon Fock state of the electromagnetic field.

It is worth to note that the solution obtained in this section is normalized and satisfies the system of first-order linear differential equations (3), (4). The direct substitution of the found solutions for B_c and B_s into the mentioned initial system of first-order differential equations yields the same expressions for the actual state amplitudes, when the terms linear in time are eliminated by the appropriate choice of the introduced coefficients C_o and $C_o^{'}$. Thus, we can say that the model described in this subsection is a particular case, more carefully a resonant limit, of the conservative system that is inclosed in the initial system of differential equations (3) and (4).

A more deeper role of such constants, like the coefficients C_o and $C_o^{'}$, arising in solving a more general system of differential equations, is revealed in the next section. Here, we can only hint that applying the certain physical limitations on the solutions requires the so-called regularization procedure. As we shall see, the main feature in solving the more general system of N linear differential equations is the possibility to "regularize" the behavior of the system in the limit of long time intervals.

In the next subsection, we investigate a limit of the nonconservative system that is modeled by the system of second-order linear differential equations (7). The principal difference in structures of the solutions for the conservative and nonconservative physical models are shown below.

3.2.3. Cyclically distanced atoms $\sum_\alpha^N \sin(\mathbf{kr}_\alpha)\cos(\mathbf{kr}_\alpha) = 0$ *with the model damping*

From the system of equations (7) in the case of a cavity with the two resonant modes $\mathbf{k} = \pm\mathbf{k}_0$ and identical atoms with $D_\alpha \equiv D$ for $\alpha = 1, ..., N$, one can derive that

$$\frac{d^2}{dt^2}\beta_\alpha(t) = -2g^2\sum_{\delta=1}^{N}\beta_\delta(t)\cos(\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_\delta)) -$$
$$- 2D\frac{d}{dt}\beta_\alpha(t). \tag{37}$$

Using the notation similar to the previous instance,

$$B_c(t) = \sum_{\alpha=1}^{N}\beta_\alpha(t)\cos(\mathbf{kr}_\alpha), \tag{38}$$

and the "cyclic" condition $\sum_\alpha^N \sin(\mathbf{kr}_\alpha)\cos(\mathbf{kr}_\alpha) = 0$ yields the following relatively simple linear differential equation:

$$\frac{d^2}{dt^2}B_c(t) = -2g^2\sum_\alpha \cos^2(\mathbf{kr}_\alpha)B_c(t) -$$
$$- 2D\frac{d}{dt}B_c(t). \tag{39}$$

Therefore, in view of the initial conditions $\beta_\delta(0) = 0$ for $\alpha = 1, ..., N$, the solution of the above equation is as follows:

$$B_c = \sum_\alpha \beta_\alpha(t)\cos(\mathbf{kr}_\alpha) = C(\exp(\Omega_{2+}t) -$$
$$- \exp(\Omega_{2-}t)), \tag{40}$$

where $\Omega_2 = g\sqrt{2\sum_\alpha \cos^2(\mathbf{kr}_\alpha)}$ and

$$\Omega_{2\pm} = -D \pm \sqrt{D^2 - \Omega_2^2}. \tag{41}$$

By analogy,

$$B_s = \sum_\alpha \beta_\alpha(t)\sin(\mathbf{kr}_\alpha) =$$
$$= C^{'}\left(\exp\left(\Omega_{2+}^{'}t\right) - \exp\left(\Omega_{2-}^{'}t\right)\right), \tag{42}$$

where $\Omega_2^{'} = g\sqrt{2\sum_\alpha \sin^2(\mathbf{kr}_\alpha)}$ and

$$\Omega_{2\pm}^{'} = -D \pm \sqrt{D^2 - \Omega^{'2}_2}. \tag{43}$$

It is easy to see that $\Omega^{'2}_2 + \Omega^2_2 = 2g^2N$.

The field probability amplitudes can be obtained using the subsystem of equations (4) of the full "conservative" system of equations (3) and (4). Therefore, substituting (40) and (42) into Eqs. (4) and then considering the restrictions $\beta_\alpha(0) = 0$ for $\alpha = 1, ..., N$ yields

$$\gamma_{\mathbf{k}}(t) = 2ig\left[C\left\{\Omega_{2+}f\left(\Omega_{2+},t\right) - \Omega_{2-}f\left(\Omega_{2-},t\right)\right\} - \right.$$
$$\left. - iC'\left\{\Omega'_{2+}f\left(\Omega'_{2+},t\right) - \Omega'_{2-}f\left(\Omega'_{2-},t\right)\right\}\right] + 1; \quad (44)$$

and

$$\gamma_{-\mathbf{k}}(t) = 2ig\left[C\left\{\Omega_{2+}f\left(\Omega_{2+},t\right) - \Omega_{2-}f\left(\Omega_{2-},t\right)\right\} + \right.$$
$$\left. + iC'\left\{\Omega'_{2+}f\left(\Omega'_{2+},t\right) - \Omega'_{2-}f\left(\Omega'_{2-},t\right)\right\}\right], \quad (45)$$

where

$$f\left(\Omega,t\right) = \frac{\exp\left(\frac{\Omega t}{2}\right)}{\Omega^2}\sinh\left(\frac{\Omega t}{2}\right). \quad (46)$$

Note that we neglected the possible space angle distribution for the direction of the resonant wave vector $\mathbf{k}$.

Inasmuch as $\cos\left(\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_\delta)\right) = \cos\left(\mathbf{kr}_\alpha\right)\cos\left(\mathbf{kr}_\delta\right) + \sin\left(\mathbf{kr}_\alpha\right)\sin\left(\mathbf{kr}_\delta\right)$, then, after the substitution of the found superpositions (40) and (42) into the initial equation (37), we derive the following integrable differential equation:

$$\frac{d^2}{dt^2}\beta_\alpha(t) + 2D\frac{d}{dt}\beta_\alpha(t) =$$
$$= -2g^2\left\{\cos\left(\mathbf{kr}_\alpha\right)B_c(t) + \sin\left(\mathbf{kr}_\alpha\right)B_s(t)\right\}. \quad (47)$$

Integrating the left- and right-hand sides of Eq. (47) over the time yields

$$\frac{d}{dt}\beta_\alpha(t) + 2D\beta_\alpha(t) = T_\alpha\left(t\right), \quad (48)$$

where

$$T_\alpha\left(t\right) = -2g^2\left\{\cos\left(\mathbf{kr}_\alpha\right)F_c(t) + \sin\left(\mathbf{kr}_\alpha\right)F_s(t)\right\} +$$
$$+ \frac{d}{dt}\beta_\alpha(0) + 2D\beta_\alpha(0), \quad (49)$$

and

$$F_{c,s}(t) = \int\limits_0^t \left[B_{c,s}(t)\right]dt. \quad (50)$$

According to the definition of the functions $F_{c,s}(t)$,

$$F_c(t) = C\left\{\frac{1}{\Omega_{2+}}\left[\exp\left(\Omega_{2+}t\right) - 1\right] - \right.$$
$$\left. - \frac{1}{\Omega_{2-}}\left[\exp\left(\Omega_{2-}t\right) - 1\right]\right\}; \quad (51)$$

and

$$F_s(t) = C'\left\{\frac{1}{\Omega'_{2+}}\left[\exp\left(\Omega'_{2+}t\right) - 1\right] - \right.$$
$$\left. - \frac{1}{\Omega'_{2-}}\left[\exp\left(\Omega'_{2-}t\right) - 1\right]\right\}. \quad (52)$$

The solution of such linear first-order differential equation, like (48), has the form

$$\beta_\alpha(t) = \frac{1}{\exp\left(2Dt\right)}\int\left[T_\alpha\left(t\right)\exp\left(2Dt\right)\right]dt. \quad (53)$$

The integration in the last expression can be performed and yields

$$\int\left[T_\alpha\left(t\right)e^{(2Dt)}\right]dt =$$
$$= -2g^2\left\{\cos\left(\mathbf{kr}_\alpha\right)C\left\{\frac{1}{\Omega_{2+}}\left[\frac{1}{\Omega_{2+}+2D}\times\right.\right.\right.$$
$$\left.\left.\left.\times\left(e^{((2D+\Omega_{2+})t)} - 1\right) - \right.\right.\right.$$
$$\left.\left.\left. - \frac{1}{2D}\left(e^{(2Dt)} - 1\right)\right] - \frac{1}{\Omega_{2-}}\left[\frac{1}{\Omega_{2-}+2D}\times\right.\right.\right.$$
$$\left.\left.\left.\times\left(e^{((2D+\Omega_{2-})t)} - 1\right) - \right.\right.\right.$$
$$\left.\left.\left. - \frac{1}{2D}\left(e^{(2Dt)} - 1\right)\right]\right\} + \sin\left(\mathbf{kr}_\alpha\right)C'\times\right.$$
$$\times\left\{\frac{1}{\Omega'_{2+}}\left[\frac{1}{\Omega'_{2+}+2D}\left(e^{\left((2D+\Omega'_{2+})t\right)} - 1\right) - \right.\right.$$
$$\left.\left. - \frac{1}{2D}\left(e^{(2Dt)} - 1\right)\right] - \frac{1}{\Omega'_{2-}}\left[\frac{1}{\Omega'_{2-}+2D}\times\right.\right.$$
$$\left.\left.\times\left(e^{\left((2D+\Omega'_{2-})t\right)} - 1\right) - \frac{1}{2D}\left(e^{(2Dt)} - 1\right)\right]\right\}\right\} +$$
$$+ \left(\frac{d}{dt}\beta_\alpha(0) + 2D\beta_\alpha(0)\right)\frac{1}{2D}\left(e^{(2Dt)} - 1\right) + C_0. \quad (54)$$

Therefore,

$$\beta_\alpha(t) =$$

$$= -2g^2 \left\{ \cos(\mathbf{kr}_\alpha) C [H(\Omega_{2+}, D, t) - H(\Omega_{2-}, D, t)] + \right.$$
$$\left. + \sin(\mathbf{kr}_\alpha) C' \left[H\left(\Omega'_{2+}, D, t\right) - H\left(\Omega'_{2-}, D, t\right) \right] \right\} +$$
$$+ \left(\frac{d}{dt}\beta_\alpha(0) + 2D\beta_\alpha(0) \right) \frac{1}{2D} \left(1 - e^{-(2Dt)}\right) + C_0 e^{-(2Dt)}, \quad (55)$$

where

$$H(\Omega, D, t) = \frac{1}{\Omega} \left[\frac{1}{\Omega + 2D} \left(e^{(\Omega t)} - e^{-(2Dt)} \right) - \right.$$
$$\left. - \frac{1}{2D} \left(1 - e^{-(2Dt)}\right) \right]. \quad (56)$$

The initial condition $\beta_\alpha(0) = 0$, for $\alpha = 1/, ..., N$, sets the coefficient C_0 equals 0. The initial time derivative $\frac{d}{dt}\beta_\alpha(0)$ can be determined, for example, if the system of equations (3) is chosen from the initial "conservative" full system of equations (3) and (4) as a basis at the time moment $t = 0$. Then the initial condition for the field state amplitude $\gamma_{\mathbf{k}}(0) = 1$, where $\mathbf{k} = \mathbf{k}_0$, sets the time derivative

$$\frac{d}{dt}\beta_\alpha(0) = ig \left\{ e^{\mathbf{kr}_\alpha} \gamma_{\mathbf{k}}(0) + e^{-\mathbf{kr}_\alpha} \gamma_{-\mathbf{k}}(0) \right\} =$$
$$= ig \left\{ \cos(\mathbf{kr}_\alpha) + i \sin(\mathbf{kr}_\alpha) \right\}. \quad (57)$$

Now, the question arises: how can the coefficients C and C' be chosen correctly? First of all, the choice has to satisfy the limitations on the probability amplitude, yielding the probability limited above by 1 (in more details, it has to be mentioned here the sum of all the squared moduli of the introduced states in the model). Second, the solution with the coefficients have to be consistent with the model decay (damping).

We observe that, formally, when the real part of the variable Ω is a negative quantity, i.e., $\mathrm{Re}(\Omega) < 0$, the introduced functions H and f have the following limits for a quite long-time intervals:

$$\lim_{t\to\infty} H(\Omega, D, t) = -\frac{1}{2D\Omega}, \text{ when } \mathrm{Re}(\Omega) < 0; \quad (58)$$

$$\lim_{t\to\infty} f(\Omega, t) = -\frac{1}{2\Omega^2}, \text{ when } \mathrm{Re}(\Omega) < 0. \quad (59)$$

Then

$$\lim_{t\to\infty} \gamma_{\mathbf{k}}(t) = ig \left[C \left\{ \frac{1}{\Omega_{2-}} - \frac{1}{\Omega_{2+}} \right\} - \right.$$
$$\left. - iC' \left\{ \frac{1}{\Omega'_{2-}} - \frac{1}{\Omega'_{2+}} \right\} \right] + 1; \quad (60)$$

$$\lim_{t\to\infty} \gamma_{-\mathbf{k}}(t) = ig \left[C \left\{ \frac{1}{\Omega_{2-}} - \frac{1}{\Omega_{2+}} \right\} + \right.$$
$$\left. + iC' \left\{ \frac{1}{\Omega'_{2-}} - \frac{1}{\Omega'_{2+}} \right\} \right]; \quad (61)$$

$$\lim_{t\to\infty} \beta_\alpha(t) = -\frac{1}{D} g^2 \left\{ C \left\{ \frac{1}{\Omega_{2-}} - \frac{1}{\Omega_{2+}} \right\} \cos(\mathbf{kr}_\alpha) + \right.$$
$$\left. + C' \left\{ \frac{1}{\Omega'_{2-}} - \frac{1}{\Omega'_{2+}} \right\} \sin(\mathbf{kr}_\alpha) \right\} + \frac{1}{2D} \frac{d}{dt}\beta_\alpha(0). \quad (62)$$

As we deal with the open system, it should be expected that, for a quite long time intervals, all the electromagnetic energy of the atoms-field excited states should be emitted into the subsystem causing the state damping. Therefore, let us define the coefficients C and C' in the following manner:

$$C = \frac{i\Omega_{2-}\Omega_{2+}}{2g(\Omega_{2+} - \Omega_{2-})} \quad (63)$$

and

$$C' = -\frac{\Omega'_{2-}\Omega'_{2+}}{2g(\Omega'_{2+} - \Omega'_{2-})}. \quad (64)$$

Then, after the substitution into the expressions for the time limits, one derives the logical final for system's evolution:

$$\lim_{t\to\infty} \gamma_{\mathbf{k}}(t) = 0; \quad (65)$$

$$\lim_{t\to\infty} \gamma_{-\mathbf{k}}(t) = 0; \quad (66)$$

$$\lim_{t\to\infty} \beta_\alpha(t) = 0. \quad (67)$$

The possible space configurations of the atomic system satisfying the condition of "circularity" can be easily found. For example, the set s3a1: $\mathbf{kr_1} \equiv \equiv \mathbf{k} \cdot \mathbf{r}_1 = \frac{\pi}{6}$, $\mathbf{kr_2} = \frac{2\pi}{3}$, and $\mathbf{kr_3} = \pi$. As an instance, it can also be the set s3a2: $\mathbf{kr_1} = \frac{2\pi}{3}$, $\mathbf{kr_2} = \frac{3\pi}{2}$, and $\mathbf{kr_3} = \frac{\pi}{6} + 2\pi$. A space configuration for five atoms can be represented, for example, by such set like s5a1: $\mathbf{kr_1} \equiv \mathbf{k} \cdot \mathbf{r}_1 = \frac{2\pi}{3}$, $\mathbf{kr_2} = \frac{3\pi}{2}$, $\mathbf{kr_3} = \frac{5\pi}{2}$, $\mathbf{kr_4} = \frac{7\pi}{2}$, and $\mathbf{kr_5} = \frac{19\pi}{6}$. Specifically, a combination such like $\mathbf{kr}_i = \pm\frac{\pi}{n} + o_i\left(\frac{\pi}{n}\right) + 2i\pi$ with $i = 1, ..., N$, $n > 2$, and $\left|o_i\left(\frac{\pi}{n}\right)\right| \ll \frac{\pi}{n}$, can generate the necessary magnitudes of the characteristic frequencies of the system Ω_2 and Ω'_2 (that, actually, are the corresponding Rabi

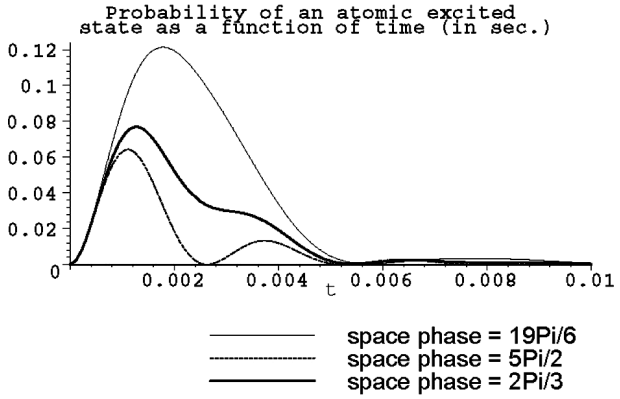

Fig. 4. Time evolution of $|\beta_\alpha(t)|^2$. The volume $V = 10^{-4}\,\mathrm{m}^3$ is filled by the set s5a1 with $D \approx 300\,\mathrm{rad/sec}$ initially coupled with the one-photon Fock state; $g \approx 246.2141\sqrt{\pi}\,\mathrm{rad/sec}$; $\Omega_2 \approx$ $\approx 246.2141\sqrt{2\pi}\,\mathrm{rad/sec}$ and $\Omega_2^{'} \approx 492.4282\sqrt{2\pi}\,\mathrm{rad/sec}$. Solid thin curve is for the space phase $19\pi/6$, dashed curve is for $5\pi/2$, bold curve is built for the space phase $2\pi/3$

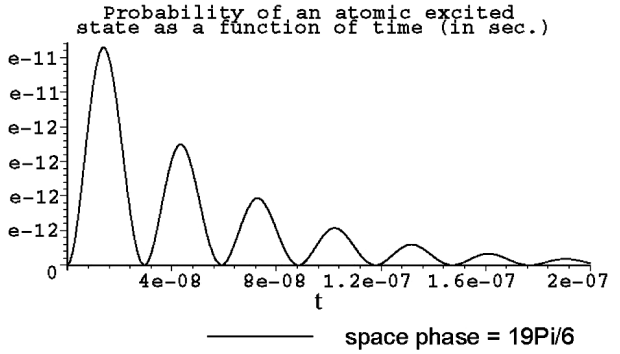

Fig. 5. Time evolution of $|\beta_\alpha(t)|^2$. The volume $V = 10^{-4}\,\mathrm{m}^3$ is filled by totally $N = 10^{12}$ atoms with $D \approx$ $\approx 10^7\,\mathrm{rad/sec}$ initially coupled with the one-photon Fock state; $g \approx 246.2141\sqrt{\pi}\,\mathrm{rad/sec}$; $\Omega_2 = \Omega_2^{'} \approx 0.60421\sqrt{\pi} \times 10^8\,\mathrm{rad/sec}$

frequencies), comparable with the given magnitude of the decay coefficient D.

To illustrate some interesting properties of the described system, we assume that the available volume is $V = 10^{-4}\,\mathrm{m}^3$, somehow filled by the set s5a1 with $D \approx 300\,\mathrm{rad/sec}$ initially coupled with the one-photon Fock state. Then $g = \rho_{\mathrm{ex}}\sqrt{\frac{\omega_{\mathrm{res}}}{2\hbar\epsilon_0 V}} \approx$ $\approx 246.2141\sqrt{\pi}\,\mathrm{rad/sec}$; $\Omega_2 \approx 246.2141\sqrt{2\pi}\,\mathrm{rad/sec}$ and $\Omega_2^{'} \approx 492.4282\sqrt{2\pi}\,\mathrm{rad/sec}$. The corresponding graphs for the excited state probability of each atom are shown in Fig. 4. The local maximum of the probability $|\beta_\alpha(t)|^2$ decreases to the value of the order 10^{-4} in the time interval about 0.015 sec (not shown in the graph).

Suppose now that there are $N \sim 10^{12}$ atoms with the single atom decay rate by five orders higher than in the previous case, i.e., $D \approx 10^7\,\mathrm{rad/sec}$, in the same value $V = 10^{-4}\,\mathrm{m}^3$. It can correspond to the quite realistic situation, where the average distance between any nearest atoms is equal to about twenty resonant wavelengths: $|\mathbf{r}_\alpha - \mathbf{r}_\delta| \geq 20\lambda_D$, $\alpha, \delta = 1, ..., N$. Then $g \approx 246.2141\sqrt{\pi}\,\mathrm{rad/sec}$. If we approximate the characteristic Rabi frequencies Ω_2 and $\Omega_2^{'}$ by the value $g\sqrt{(N)}$, i.e., $\Omega_2 = \Omega_2^{'} \approx 0.60421\sqrt{\pi} \times$ $\times 10^8\,\mathrm{rad/sec}$, then the corresponding graph of the atomic excited state probability is shown in Fig. 5.

Note that if the volume $V = 10^{-4}\,\mathrm{m}^3$ is filled by totally $N = 10^{12}$ atoms with the single atom decay rate $D = 300\,\mathrm{rad/sec}$, as in the first situation with the space configuration s5a1, then the initial probability local maximum $|\beta_\alpha(t)|^2_{\mathrm{loc.\,max}}$ decreases from the value 1.6×10^{-11} to 5×10^{-14}, i.e., lower by three orders, in the time interval about 0.01 sec.

3.2.4. Some comments about the "physical" properties of the obtained solutions

Note that the solution obtained in this section (55) does not satisfy the initial system of differential equations (3) and (4) for a closed conservative system of N atoms and the electromagnetic field. The direct substitution of the above-found solution (40) and (42) for B_c and B_s in the mentioned initial system of linear first-order differential equations (3) and (4) yields such state amplitudes like

$$\beta_\alpha^{\mathrm{untrue}}(t) = -2g^2\left[\exp(i\mathbf{kr}_\alpha)\left\{C(f(\Omega_{2+}, t) - \right.\right.$$
$$\left. - f(\Omega_{2-}, t)) - iC^{'}\left(f\left(\Omega_{2+}^{'}, t\right) - f\left(\Omega_{2-}^{'}, t\right)\right)\right\} +$$
$$+ \exp(-i\mathbf{kr}_\alpha)\left\{C\left(f(\Omega_{2+}, t) - f(\Omega_{2-}, t)\right) + \right.$$
$$\left. \left. + iC^{'}\left(f\left(\Omega_{2+}^{'}, t\right) - f\left(\Omega_{2-}^{'}, t\right)\right)\right\}\right], \qquad (68)$$

where $\alpha = 1, ..., N$, C, $C^{'}$, and the function $f(\Omega, t)$ were already introduced above. The alternative uncorrect solutions (68), $\alpha = 1, ..., N$, are not limited above, when the parameters of the system (for example, such like V and g) run through a quite wide range of values. Therefore, at first look, this fact does not permit any physical application of the above "untrue" solutions (68). But the condition of absence of

any terms linear in time for the solutions generates the same expressions (63) and (64) for the coefficients C and $C^{'}$ in the direct substitution, giving somehow a similar "vanishing" behavior of the "untrue" solution (68) for quite long-time intervals. It is of interest that the following backward substitution of the provided above "untrue" expressions (68) for $\alpha = 1, ..., N$ into the definitions for B_c and B_s does not give the "true" forms (40) and (42). Hence, the found "correct" solution of the form (55) cannot be counted as a particular one (or as a limit of such) for the system of equations (3) and (4) that represents only a closed conservative system of atoms with an electromagnetic field. Note that the method proposed in the previous subsection of calculating the initial derivatives can be argued by a relatively slow decay of the excited state of the system. So that, for a quite small interval of time, the initial conditions can approximately satisfy the "conservative" equations (3) and (4). Thus, we can say that the model described in this section, besides the atoms and the electromagnetic field, implicitly contains a third participant guaranteeing a relaxation of the total system. The last property was strongly imposed by the choice (63) and (64) for the coefficients C and $C^{'}$.

4. Conclusion

We have investigated the system of N identical two-level long distanced atoms. The functional dependence of the N atomic state amplitudes on the space configuration and the time is derived in the Weisskopf–Wigner approximation.

It is shown that, with increasing the volume V of the system, the maximum value of probability to find an atom in its excited state decreases.

The nature of Weisskopf–Wigner approximation was revealed through the provided application to the many-body system. The found solution (55) cannot be counted as a particular one, or as a limit of such, for the initial system of equations (3) and (4) that represents only a closed conservative system of atoms and an electromagnetic field. Thus, we can say that the model described in this work, besides the atoms and the electromagnetic field, implicitly contains a third participant guaranteeing a relaxation of the total system in time. It is worth to note that the "complete" decay of the excitations of the system was strongly imposed by the choice of the coefficients C (63) and $C^{'}$ (64).

The described methods of solving the system of linear differential equations can be applied even to more general situations where the boundary "circular" conditions are not satisfied. In certain cases, the problem allows one to extend the system by adding a subsystem of relatively non-sufficient number of atoms (or an atom when the expression $\left|\sum_{\alpha}^{N} \sin(\mathbf{kr}_\alpha)\cos(\mathbf{kr}_\alpha)\right| < 1$ is actual) without influencing the main characteristics under our interest. In addition, it is of interest to investigate the dependence of the value of the mentioned construction $\left|\sum_{\alpha}^{N} \sin(\mathbf{kr}_\alpha)\cos(\mathbf{kr}_\alpha)\right|$ on the location of the origin of a coordinate system.

The possibility of extension of the described method of "cyclic bonds" requires an additional investigation in the case of elements $\Phi_{\alpha\delta}$ (see Eq. 8) distributed over the space angle. It can happen that the appropriate description will become possible in the terms of spherical harmonics. Then the introduced "cyclic" conditions should be reformulated.

The theoretical material proposed by us requires an experimental verification. The described model system can be realized on atomic or molecular systems (developing the "visible" or "infrared" modifications of the method proposed in [9], for example), chains of trapped ions (like in [22]), *etc.*

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ОДНОФОТОННЕ РОЗСІЯННЯ
СИСТЕМОЮ N АТОМІВ: ЗАСТОСУВАННЯ
ДО ОДНО- ТА ДВОМОДОВОГО РЕЗОНАТОРА

Р е з ю м е

Розглянуто систему N однакових дворівневих атомів, взаємодіючих з квантованим електромагнітним полем у резонансному наближенні. Відповідні N-частинкові амплітуди стану в одно- та двомодовому резонаторах пораховані для декількох просторових конфігурацій у випадках замкнутої консервативної та відкритої дисипативної систем. Природу і структуру наближення Вайскопфа–Вігнера було розкрито для даного типу багаточастинкових задач. Було показано, що просторове розташування атомів та "доступний" об'єм для мод поля визначають поведінку амплітуд станів системи з часом. Розроблена теорія дозволяє аналітично описувати еволюцію системи у часі для достатньо широкого спектра просторових конфігурацій, якщо запроваджені специфічні "циклічні" обмеження.