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PHENOMENOLOGICAL RENORMALIZATION GROUP AND CLUSTER APPROXIMATION

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We propose a modification of the Nightingale renormalization group for lattice spin and gauge models by combining it with the cluster decimation approximation. Essential ingredients of our approach are: 1) exact calculation of the partition and correlation functions on a finite lattice strip; 2) preservation of the mass gap or the second moment correlation length, computed in the infinite strip length limit, on each decimation step. The method is applied to studying the general two- and three-dimensional Z(N) models. A perfect agreement with exact results (whenever available) is found. An extension of the method to models with a continuous symmetry is briefly discussed.

K e y w o r d s: phenomenological renormalization group, transfer matrix, Potts model.

1. Introduction

The Migdal–Kadanoff (MK) renormalization group (RG) [1, 2] had a great impact on the development of the real-space RG methods. Its conceptual and technical simplicities have probably been the main reason for the numerous applications of the MK RG ranging from two-dimensional ($2d$) classical spin models to the confinement problem in $4d$ gauge theories. In some special cases like $1d$ and hierarchical lattices, the MK recursion relations (RR) are precise and lead to exact solutions. In other important cases, the MK RG fails to properly predict the phase diagram of a model or gives the string tension, which is too large [3]. Nevertheless, for a number of physically interesting systems, the MK RG does provide qualitatively proper results, which are not very precise, however, on the quantitative level. This fact has impelled researchers to look for a modification of the original MK transformations, which would result in a better quantitative prediction. Indeed, numerous such modifications have been developed in the past.

In the context relevant for this paper, we would like to mention an approach to the RG transformations based on the cluster decimation approximation (CDA) [4]. The CDA utilizes the same restructuring of the lattice as in the conventional MK RG. The main idea is to preserve the free energy after each decimation step. This can be done approximately by computing the exact free energies on small clusters with periodic boundary conditions (BC), e.g., 2×2 or 1×1. The equality of these free energies is used to establish RR for the effective coupling constant on the 1×1 lattice. Then this exact RR is used to approximate the *effective* coupling constant on new $L/2 \times L/2$ lattice. On this way, one gets a considerable improvement of the results over the standard MK RG.

A different class of the RG is represented by a phenomenological renormalization group (PH RG) proposed by Nightingale [5] (for many applications of the PH RG, see [6, 7]). Originally, this approach utilizes the correlation lengths of the partially finite systems. Their scaling relations are then interpreted as the RG equations. When they are expressed *via* eigenvalues of the transfer matrix (TM) on lattices with differ-

ent sizes, one obtains equations, whose solutions approximate the critical temperature and various critical indices.

In the present paper, we develop an approach based on a combination of the PH RG with the CDA. Let $m_r(L)$ be a mass gap of the finite system corresponding to a correlation function in the r-representation, and let b be a rescaling factor. Then, in a vicinity of the critical point, one has a scaling relation of the simple form $bm_r(L) \approx m_r(L/b)$. Basically, our approach can be described as follows.

– The relations $bm_r(L) \approx m_r(L/b)$ are treated as a set of equations for new iterated couplings for the whole range of the bare coupling constants;

– In the spirit of the CDA, we consider the lattice strip $M^{(d-1)} \times L$, $L \to \infty$, as a cluster, where d is the dimension of the system.

The resulting RR when considered as an RG equations can be solved recursively, thus enabling one to get both qualitative and quantitative estimates for the physical observables and critical exponents.

Alternatively, one can employ the second moment correlation length ξ_2 to obtain the phenomenological RG equations. This choice provides a good basis for Monte Carlo simulations on the square clusters of the linear extension L, and the RG transformations are organized in a usual way as transitions $L^d \to (L/2)^d$. In what follows, we use both the mass gap and the second moment correlation length.

This paper is organized as follows. In Section 2, we outline a general scheme of RG transformations on the example of the general $Z(N)$ spin models. Here, we also discuss our approach to the computation of the partition and correlation functions on the finite lattice strips. In Section 3, we present some results of our study for two- and three-dimensional $Z(N)$ models. In Section 4, we summarize our results and outline the perspectives.

2. Construction of RG Transformations

We work on a d-dimensional hypercubic lattice $\Lambda_0 \in \in Z^d$ with the lattice spacing $a = 1$ and a linear extension L. By $x = (x_1, x_2, ..., x_d)$, $x_i \in [0, L-1]$, we denote the sites of the lattice, $l = (x, n)$ denotes links, and $e_n, n = (1, ..., d)$ is a unit vector in the n-th direction. For the sake of simplicity, we impose periodic BC. $\Lambda_k \in Z^d$ will denote the lattice with the linear extension $L_k = L/b^k$ obtained after k decimation steps.

Here, we treat models only with interactions between the nearest neighbors. Let us describe how our approximation is built for the general Potts model, i.e., the spin system with discrete $Z(N)$ symmetry (here, N means the number of states a spin s in every site can have, $s = 0, 1, ..., N-1$). For the details of the model, see, e.g., review [8] and references therein. The partition function (PF) of the general $Z(N)$ model can be written down as

$$Z(\Lambda_0, \{t_k\}) = \prod_{x \in \Lambda_0} \left(\frac{1}{N} \sum_{s(x)=0}^{N-1} \right) \times$$

$$\times \prod_{l \in \Lambda_0} Q\left[\{t_k\}; s(x) - s(x + e_n)\right]. \quad (1)$$

The most general $Z(N)$-symmetric Boltzmann weight is defined as

$$Q\left[\{t_k\}; s\right] = \sum_{k=0}^{N-1} t_k \exp\left[\frac{2\pi i}{N} ks\right]. \quad (2)$$

The set of coupling constants $\{t_k\}$ can be chosen to satisfy

$$t_0 = 1, \quad 0 \le t_k \le 1, \quad t_k = t_{-k} = t_{k+N}. \quad (3)$$

The two-point correlation function in the representation $r = 1, ..., N-1$ reads

$$\Gamma_r(\Lambda_0, \{t_k\}; R) = \left\langle \exp\left[\frac{2\pi i}{N} r(s(x) - s(x+R))\right] \right\rangle. \quad (4)$$

If all couplings are set to be equal, i.e. $t_1 = t_2 = = ... = t_{N-1}$, one gets the standard Potts model. For the vector Potts model, the coupling constants are given by

$$t_k = C_k/C_0, \; C_k = \sum_{m=0}^{N-1} \exp\left[\beta \cos\frac{2\pi}{N} m + \frac{2\pi i}{N} mk\right]. \quad (5)$$

Our goal is to present both partition and correlation functions on a decimated lattice in the form

$$Z(\Lambda_0, \{t_k\}) = A(\{t_k\}) Z(\Lambda_1, \{t_k^{(1)}\}),$$

$$\Gamma_r(\Lambda_0, \{t_k\}; R) = \quad (6)$$

$$= \; G_r(\{t_k\}, R) \Gamma_r(\Lambda_1, \{t_k^{(1)}\}; R/2)$$

with unchanged Boltzmann weight and a new set of couplings $\{t_k^{(1)}\}$. Actually, any real-space RG amounts to a prescription of how to (approximately) compute, *e.g.*, the constant $A(\{t_k\})$ and new couplings $\{t_k^{(1)}\}$. We propose here to use the CDA of a certain type. Let us explain the main idea on the example of the two-dimensional lattice taking the rescaling factor $b = 2$. Extensions to higher dimensions and to other values of b are straightforward.

Let $\Lambda = (M \times L)$ be a strip of a $2d$ lattice with a width $M \ll L$ fixed and periodic BC in both directions. Define the free energy and the correlation function in the thermodynamic $L \to \infty$ limit as

$$F(M, \{t_k\}) = \log \lambda_0(M, \{t_k\}) =$$

$$= \lim_{L\to\infty} \frac{1}{L} \log Z(M \times L, \{t_k\}), \qquad (7)$$

$$\Gamma_r(M, \{t_k\}; R) =$$

$$= \lim_{L\to\infty} \Gamma_r(M \times L, \{t_k\}; R). \qquad (8)$$

Suppose that $\Gamma_r(M, \{t_k\}; R)$ has the following very general form:

$$\Gamma_r(M, \{t_k\}; R) = D_r(M, \{t_k\}, R) \left[B_r(M, \{t_k\})\right]^R. \quad (9)$$

Here, the function $B_r(M, \{t_k\})$ describing the exponential decay is parametrized as

$$B_r(M, \{t_k\}) = \frac{\lambda_r(M, \{t_k\})}{\lambda_0(M, \{t_k\})}, \qquad (10)$$

and the function $D_r(M, \{t_k\}, R)$ has utmost a power-like decay. Functions λ_r can be considered, e.g., as the eigenvalues of the corresponding transfer matrices. Our basic idea is to present the correlation function $\Gamma_r(M, \{t_k\}; R)$ *via* the correlation function $\Gamma_r(M/2, \{t_k^{(1)}\}; R/2)$, calculated on the strip of the width $M/2$ with new couplings $\{t_k^{(1)}\}$, in the form

$$\Gamma_r(M, \{t_k\}; R) =$$

$$= \frac{D_r(M, \{t_k\}, R)}{D_r(M/2, \{t_k^{(1)}\}, R/2)} \Gamma_r(M/2, \{t_k^{(1)}\}; R/2). \qquad (11)$$

Comparing (9) with the last equation, one concludes that (11) holds if

$$B_r^2(M, \{t_k\}) \;=\; B_r(M/2, \{t_k^{(1)}\}) \qquad (12)$$

for all $r = 1, ..., N-1$. This system of $N-1$ equations determines new couplings $t_k^{(1)}$ on the lattice strip $(M/2, L/2)$. We use these exact relations to approximate the partition and correlation functions on Λ_0 as

$$Z(\Lambda_0, \{t_k\}) =$$

$$= \left[\frac{\lambda_0(M, \{t_k\})}{\lambda_0^{(1/2)}\left(M/2, \{t_k^{(1)}\}\right)}\right]^{L^2/M} Z\left(\Lambda_1, \{t_k^{(1)}\}\right), \qquad (13)$$

$$\Gamma_r(\Lambda_0, \{t_k\}; R) =$$

$$= \frac{D_r(M, \{t_k\}, R)}{D_r\left(M/2, \{t_k^{(1)}\}, R/2\right)} \Gamma_r\left(\Lambda_1, \{t_k^{(1)}\}; R/2\right). \qquad (14)$$

Equations (7)–(8) and (13)–(14) define our RG transformations. Renormalized coupling constants are computed from Eqs. (12), which are, of course, the equations of the PH RG. In this form, the PH RG becomes a much more flexible tool, for it can be treated as an iterative procedure. This is essential for the systems with many coupling constants, be it original bare coupling constants or the constants generated during iterations. The second important gain is that, within this approach, one can compute not only critical characteristics of the system but also many thermodynamical functions in the whole range of the coupling constants. In addition, instead of the mass gap preservation, one can take any appropriate quantity, whose scaling is known. For example, one could work with the second moment correlation length which is more suitable for the numerical Monte Carlo computations.

Similarly to any RG of this type, the main technical difficulty arises from the desire to obtain as precise results as possible. It is well known that the precision of the results systematically improves when the lattice strip width M becomes larger and larger [5]. However, the computation of the mass gap on a large strip requires the diagonalization of big transfer matrices, and this appears to be an important obstacle in the applications of this method. In what follows, we use an approach that allows us to go to larger strip widths. Here, we briefly outline two main ingredients of our approach.

1. The partition and correlation functions are computed in a dual formulation. For example, for $2d$ mod-

els, one finds

$$Z(\Lambda_0,\{t_k\}) = \sum_{r_n=0}^{N-1} \prod_{x\in\Lambda_0} \left(\frac{1}{N}\sum_{s(x)=0}^{N-1}\right)\times$$
$$\times \prod_{l\in\Lambda_0} t_{s(x)-s(x+e_n)+\eta_n}, \qquad (15)$$

where $\eta_n = r_n$, $n = 1, 2$ on a set of dual links, which forms a closed path due to periodic BC, and $\eta_n = 0$, otherwise. The transfer matrix is also constructed in the dual formulation. The largest eigenvalue of the transfer matrix with $r_1 = r_2 = 0$ gives the dominant contribution to the partition function, while the largest eigenvalue of the transfer matrix with $r_1 = r \neq 0$, $r_2 = 0$ gives the dominant contribution to the correlation function in the r representation in the first direction.

2. The transfer matrix itself is constructed not on the spin configurations, but on coefficients describing the evolution of all *independent* couplings. Such couplings are extracted after the first direct summation over all spins from a given column in a strip. This approach automatically involves all lattice symmetries and significantly reduces the size of the matrix.

Let us demonstrate how this approach works on an example of the $2d$ standard $Z(N)$ Potts model on the lattice strip $\Lambda = 2 \times L$. The direct computation of the TM leads to a matrix of the size $N^2 \times N^2$. In the dual representation described in entry 1, the partition function becomes

$$Z(2\times L,\{t_k\}) = \frac{1}{N}\times$$
$$\times \sum_{r=0}^{N-1}\sum_{\{k(x)\}=0}^{N-1}\prod_{x=1}^{L}\left[t_{k(x)}t_{k(x+e_n)+r}\alpha_{k(x)+k(x+e_n)+r}\right], \qquad (16)$$

where

$$\alpha_k = \frac{\sum_{m=0}^{N-1} t_m t_{k-m}}{\sum_{m=0}^{N-1} t_m^2}$$

appears due to periodic BC in the transverse direction. It follows from Eq. (16) that

$$Z(2\times L,\{t_k\}) = \frac{1}{N}\sum_{r=0}^{N-1}\sum_{k=0}^{N-1}[\lambda_{k,r}(\{t_k\})]^L. \qquad (17)$$

Here, $\lambda_{k,r}(\{t_k\})$ are N eigenvalues of the $N\times N$ transfer matrix

$$W_{k_1k_2}(r) = t_{k_1}t_{k_2+r}\alpha_{k_1+k_2+r}. \qquad (18)$$

For the correlation function, a general answer is

$$\Gamma_j(2\times L,\{t_k\};R) = \frac{Z_j(2\times L,\{t_k\};R)}{Z(2\times L,\{t_k\})},$$
$$Z_j(2\times L,\{t_k\};R) = \qquad (19)$$
$$= \frac{1}{N}\sum_{r=0}^{N-1}\mathrm{Tr}[W_{k_1k_2}(r)]^{L-R}[W_{k_1k_2}(r+j)]^R.$$

One can see that the size of the TM in the dual representation is already reduced from $N^2 \times N^2$ to $N \times N$. A further reduction can be achieved by using the prescription described in entry 2 above. In our case, the transfer matrix for a $2 \times L$ strip with periodic BC takes the form

$$W = \begin{pmatrix} 1 & \frac{(N-1)t((N-2)t+2)}{(N-1)t^2+1} \\ \frac{t^3((N-2)t+2)}{(N-1)t^2+1} & \frac{t^2(t(3t+N((N-3)t+2)-4)+1)}{(N-1)t^2+1} \end{pmatrix}, \qquad (20)$$

and the corresponding matrix, from which the correlation function is built, reads

$$W_1 = t\begin{pmatrix} \frac{t((2N-3)t+2)+1}{(N-1)t^2+1} & \frac{(N-2)\left((N-2)t^2+2t\right)}{(N-1)t^2+1} \\ \frac{2t\left((N-2)t^2+2t\right)}{(N-1)t^2+1} & \frac{t(t(5t+N((N-4)t+2)-6)+1)}{(N-1)t^2+1} \end{pmatrix}. \qquad (21)$$

We see that the size of the TM is finally reduced from $N^2 \times N^2$ to 2×2. The difference between the sizes of the original matrix and the reduced one increases drastically with N and M.

Our results for eigenvalues, from which the partition function is built, coincide with those of [9], while the corresponding correlation functions are calculated for the first time.

3. Application to $Z(N)$ Models

Below, we apply our procedure to some $2d$ $Z(N)$ spin models and to $3d$ $Z(N)$ lattice gauge theories.

Our first example concerns the $Z(2)$ and $Z(3)$ models. Here, the standard and vector models coincide. For $N = 2$, i.e. for the Ising model, one has $t_0 = 1$ and $t_1 = t$. While $t_0 = 1$ and $t_1 = t_2 = t$ for $N = 3$. We take

$$t \equiv t^{(0)} = \tanh\beta$$

as an initial value for $N = 2$ and

$$t \equiv t^{(0)} = (\exp(3\beta/2) - 1)/(\exp(3\beta/2) + 2)$$

for $N = 3$. In the case of the Ising model, the RR reads

$$t^{(1)} = \left[\frac{2t(1+t)^2}{1+2t^2+t^4+\sqrt{1+14t^4+t^8}}\right]^2. \qquad (22)$$

A non-trivial fixed point of this RR gives the critical value β_c. Expanding the RR around β_c, one computes the critical index ν. For $N > 2$, the algorithm is the same, although the equation itself is much more cumbersome. The results for several N are summarized in Table 1.

Another application is the calculation of the free energy. To this end, one needs to solve an RG equations for several decimation steps. For example, for an $M \times L$ strip, the free energy of the $2d$ Ising model is calculated as a finite sum

$$F = \frac{2\cosh(\beta)}{M} \sum_{i \in \mathrm{Iter}} \frac{1}{4^{i+1}} \log \frac{\lambda_0(\{t_i\}, M)}{\sqrt{\lambda_0(\{t_{i+1}\}, M/2)}}. \qquad (23)$$

The following figures, Fig. 1–3, demonstrate the results for different strip widths M and different ranges of the coupling constant.

It is seen clearly from Fig. 1 even for the smallest width that the exact and approximate curves already after the 5th iteration (i.e., the sum in (23) contains 5 terms) are practically indistinguishable in this range.

On a smaller scale around the critical point (Fig. 2, 3), one can see easily how fast the approximate functions converge with iterations to the exact free energy of the Ising model. If, for $M = 2$, the approximate curve still differ after 5 iterations from the exact one (Fig. 2) then, e.g., for $M = 16$, the approximate and exact free energies once again become practically indistinguishable (Fig. 3).

In the next example, let us consider the general $Z(4)$ spin model on the lattice strips of different

Table 1. **Critical coupling and critical exponents obtained from CDA for $Z(N)$ ($N = 2, 3, 4, 5$) Potts models; subscript e denotes an exact value (see, e.g., [8])**

N	β_c	ν	β_{ce}	ν_e
2	0.435666	0.987299	0.440687	1
3	0.327572	0.874834	0.335018	5/6
4	0.266501	0.807699	0.251263	2/3
5	0.246802	–	0.234872	–

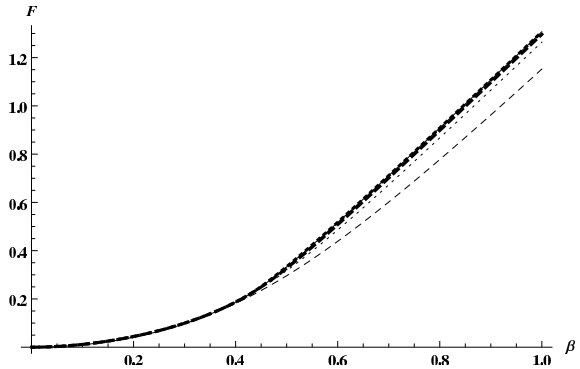

Fig. 1. Free energy of the Ising model calculated for the transition $2 \times L \to 1 \times L$. The solid line stands for the exact free energy; the dashed line is the free energy after one iteration, while the dotted line is a free energy after 5 iterations

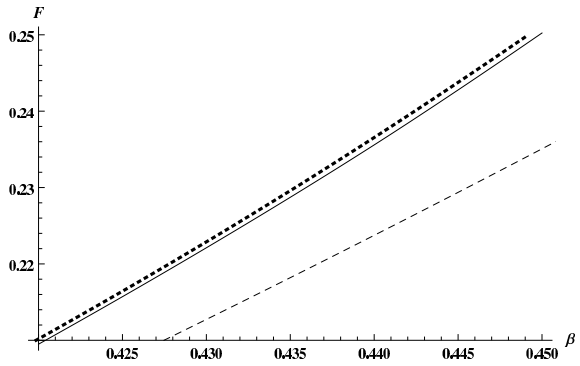

Fig. 2. Free energy of the Ising model in a vicinity of the critical point calculated for the transition $2 \times L \to 1 \times L$. The solid line stands for the exact free energy; the dashed line is a free energy after one iteration, while the dotted line is the free energy after 5 iterations

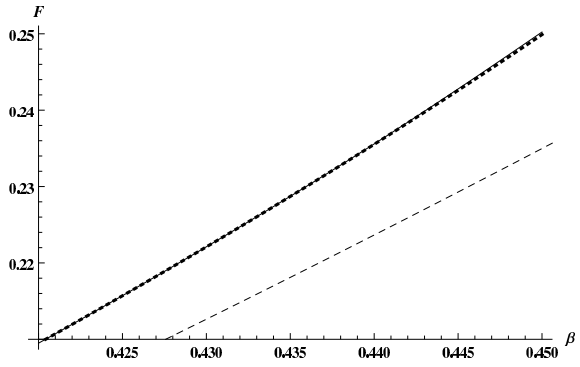

Fig. 3. Free energy of the Ising model in a vicinity of the critical point calculated for the transition $16 \times L \to 8 \times L$. The solid line stands for the exact free energy; the dashed line is the free energy after one iteration, while the dotted line is the free energy after 5 iterations

Table 2. **Critical coupling and critical index ν for the $Z(4)$ model obtained from iterations $M_1 \times L \to M_2 \times L$. Here, the subscripts "s" and "v" stand for the standard and vector models**

$M_1 \to M_2$	t_s	ν_s	t_v	ν_v	$t_2(t_1 = 0)$	ν
Exact	1/3	2/3	0.414214	1	0.414214	1
$4 \to 2$	0.321807	0.730859	0.402491	0.934196	0.402491	0.934217
$6 \to 2$	0.326477	0.734682	0.406834	0.94718	0.406835	0.947154
$6 \to 4$	0.330436	0.741442	0.411239	0.970341	0.411239	0.970634
$8 \to 2$	0.328659	0.736383	0.408962	0.95484	0.408962	0.954708
$8 \to 4$	0.331406	0.743497	0.412259	0.97711	0.41226	0.977542
$8 \to 6$	0.332276	0.746328	0.41329	0.98673	0.413294	0.987379

Table 3. **Critical coupling and critical index ν obtained from CDA using the second moment correlation length compared to those obtained from CDA using the correlation function described in Section 2 for the transitions $2 \times L \to 1 \times L$ in the standard Potts $Z(N)$ models**

N	$\beta_{c\xi_2}$	ν_{ξ_2}	$\beta_{c\Gamma}$	ν_Γ
2	0.427479	1.04218	0.435666	0.987299
3	0.320582	0.932411	0.327572	0.874834
4	0.260379	0.866739	0.266501	0.807699
5	0.240463	–	0.246802	–

Table 4. **Critical coupling β_c and critical exponent ν for $Z(N), N = 2, 3, 5, 13$ calculated *via* ξ_2 preservation requirement. Subscript "e" stands for the exact value, when available**

N	L	β_c	$\beta_{c(e)}$	ν	$\nu_{(e)}$
2	16	0.441905		1.04733	
	32	0.440965		1.01295	
	64	0.440664	0.440687	0.998986	1.0
3	8	0.33703		0.971028	
	16	0.33531		0.887692	
	32	0.33505		0.857852	
	64	0.3350186	0.335018	0.849067	5/6
5	16	0.234663		–	–
	32	0.234726		–	–
	64	0.2348156	0.234872	–	–
13	8	0.1165		–	–
	16	0.117		–	–
	32	0.1175	0.117482	–	–

widths M. The phase structure of this model is well known, so that we can check both the validity and the precision of our approach. Fixed points are extracted from the solution of the recursion relations, and the critical lines can be obtained after just several transformations, since the RG iterations converge rather fast. The critical index ν is derived in a usual way by linearising the RG equations around fixed points. In this case, we find 4 fixed points corresponding to: 1) the standard Potts model ($t_1 = t_2$), 2) the vector model ($t_2 = t_1^2$), 3) and 4) models defined on the lines $t_1 = 0$ and $t_2 = 1$. Our results are summarized in Table 2 (results for the line $t_2 = 1$ reproduce ones for $t_1 = 0$).

In else another example, we require the preservation of the second moment correlation length ξ_2 during RG steps instead of the mass gap preservation. To this end, we first re-formulate ξ_2,

$$\xi_2 = \frac{\sqrt{\frac{\chi}{F} - 1}}{2\sin(\pi/L)}, \tag{24}$$

in terms of the transfer matrix in the dual representation. For the transitions $2 \times L \to 1 \times L$, this expressions was obtained analytically for standard $Z(N)$ models for arbitrary N. The resulting ξ_2 is expressed *via* eigenvalues of the transfer matrix and corresponding weights, but is too cumbersome to be presented here. The numerical calculations for the transitions $2 \times L \to 1 \times L$ based on this exact expression for ξ_2 lead to the results given in Table 3.

For larger lattices, we have to employ the numerical methods. In this case, we consider a square cluster of size $L \times L$ and the RG of the form $L \to L/2$. Table 4 summarizes the results of our Monte Carlo simulations for standard $Z(N)$ Potts models for several values of N. Subscript e refers to exact values. The Monte Carlo simulations were performed using the cluster algorithm with 10^6–10^7 measurements per point depending on the value of N.

Table 5. **$3d$ $Z(N)$ standard Potts models: β_c from the PH RG (column two) and from the Monte Carlo simulations of Ref. [10] (column three). $3d$ $Z(N)$ vector models: β_c from the PH RG (column 4) and from the Monte Carlo simulations (column five); critical index ν from the PH RG (column six) and from the Monte Carlo simulations (column seven). Numbers for columns five and seven are taken from Ref. [11]**

N	Potts model		Vector model			
	β_c	β_c^{MC}	β_c	β_c^{MC}	ν	ν^{MC}
2	0.77706	0.761414(2)	0.77706	0.761395(4)	0.616656	0.6306
3	1.17186	1.084314(8)	1.17186	1.0844(2)	–	–
4	1.34363	1.288239(5)	1.55411	1.52276(4)	0.616657	0.62933
5	1.50331	1.438361(4)	2.17896	2.17961(10)	0.692226	0.6681
6	1.62881	1.557385(4)	2.99296	3.00683(7)	0.699208	0.6756
8	1.81941	1.740360(6)	5.09472	5.12829(13)	0.699583	0.6748

We see that, in all these examples, our results are in perfect agreement with known results, whenever they are available.

We have also studied the general $3d$ lattice $Z(N)$ gauge models for various N. In this case, there are no reference results except from the case $N = 2$ (see,*e.g.*, [12]), and our results are original and new. In calculations, we also used duality transformations and performed RG iterations with the smallest cluster $2 \times 2 \times L$. In Table 5, we present the estimates of the critical points β_c both in standard Potts gauge models and in vector gauge models and compare them with the results of Monte Carlo numerical simulations.

4. Summary and Perspectives

In this paper, we have considered a modification of the PH RG by combining it with the CDA. Within these frameworks, we have derived exact representations for the partition and correlation functions on the decimated lattice. Recursion relations are derived from the requirement of the preservation of the mass gap (or the second moment correlation length) of the system for each representation of the correlation function on each RG iteration. We have also proposed a new method of calculation of the mass gap on a finite lattice strip based on the duality transformations and on the construction of the transfer matrices for the coefficients describing the evolution of all *independent* couplings. As a simple example, which allows analytical calculations, we employ this approach in details for the $2d$ Potts model on the lattice strip $2 \times L$.

As an application of this method, we have studied various two-dimensional $Z(N)$ spin models on the lattice strips of different widths M and three-dimensional $Z(N)$ lattice gauge theories. In all cases, we have computed the critical points of phase transitions and the critical index ν, which governs the exponential decay of the correlation function in a vicinity of the critical point. As is seen from the tables given in Section 3, our RG equations, obtained even on small lattice clusters, give very reasonable approximations to exact or numerically computed values. Furthermore, we calculated the free energy of the Ising model after several iterations steps. Since the Ising model is exactly solvable, this gave us an opportunity to check how good is our RG in describing the thermodynamical quantities. Figures 1–3 show that the free energy obtained just after 5 iterations reproduces the exact free energy very well. We can conclude from this example that the convergence to the thermodynamic limit is rather fast. Finally, we have described RG based on the preservation of the second moment correlation length. In this case, we have been able to perform numerical MC computations on square lattice clusters. Table 4 demonstrates that the RG constructed on clusters of a moderate size gives very precise values for both critical points and index ν.

The technics presented here provides a unified approach to the analysis of general Potts models and is, from authors' point of view, conceptually somewhat simpler, as compared to more conventional approaches. On the technical side, it provides a comparable (or better) precision of calculations, while requiring quite less computational resources.

The above technics can be straightforwardly extended to any Abelian spin model with continuous

symmetry like the XY model or to lattice $U(1)$ gauge theory. However, the more work is needed to extend the present approach to non-Abelian models. Investigations in these directions are in progress and will be given elsewhere.

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ФЕНОМЕНОЛОГІЧНА РЕНОРМАЛІЗАЦІЙНА ГРУПА ТА КЛАСТЕРНЕ НАБЛИЖЕННЯ

Р е з ю м е

Запропоновано модифікацію феноменологічної ренормалізаційної групи для спінових та калібрувальних моделей на ґратці, що базується на поєднанні її з кластерним децимаційним наближенням. Суттєвими інгредієнтами нашого підходу є: 1) точне обчислення функції розподілу та кореляційної функції на скінченій смузі; 2) вимога збереження масової щілини або другого моменту кореляційної довжини, обчислених у ліміті нескінченної довжини смуги, на кожному децимаційному кроці. Даний метод застосовано для вивчення загальних дво- та тривимірних $Z(N)$ моделей. Продемонстровано дуже добре відтворення точних результатів у всіх випадках, коли ці останні відомі. Коротко обговорено можливе розширення метода на моделі з неперервними симетріями.