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FROM BIALGEBRAS TO OPERADS. QUANTUM LINE AND COOPERAD OF CORRELATION FUNCTIONS

A q-line is a simple example of a braided Hopf algebra. This is just an algebra of polynomials $\Bbbk_q[z]$ with primitive generator and q-deformed statistics.

The (co)action of a q-line on an algebra is a q-derivation. We construct an operad and a cooperad from a bialgebra. In the case of a q-line, this construction is related to the cooperad of correlation functions of I. Kriz et al., which describes vertex algebras.

Modules over the factor-algebra $\Bbbk_q[z]/(z^N)$ are N-complexes. We consider a homotopical category of N-complexes as an example of the q-analog of Maltsiniotis' strongly triangulated category.

The general constructions are considered in the context of iterated monoidal categories with unbiased lax tensor products described in the terms of the Gray tensor products of 2-fold categorical operads of sequential trees Tree.

Keywords: bialgebra, operad, unbiased tensor products, multitensor category, vertex algebra.

1. Introduction

Vertex algebras where discovered by Borcherds and Frenkel–Lepowsky–Meurman in the context of the Monstrous moonshine motivated by the Goddard–Thorn no-ghost theorem from string theory. They realized the Fischer–Griess monster or the Friendly Giant (the largest sporadic finite simple group) as the automorphism group of a monster vertex algebra. Now, a vertex algebra is naturally understand as the holomorphic part of conformal field theory (CFT). The central axiom is that the generalized Jacobi identity comes from the locality principle. In particular, the monster vertex algebra appeared in CFT with 24 free bosons compactified on the torus induced by the Leech lattice and orbifolded by the two-element reflection group.

Today, saying about an "algebra" A of a certain kind, we usually assume that we can realize it as an algebra over an operad $\{O(n)\}_{n\geqslant 0}$. This means that each element of a vector space $O(n)$ determines the n-fold multilinear operation $A^{\times n} \to A$, or, equivalently, for each $n \geqslant 0$, there is a linear map

$$O(n) \otimes A^{\otimes n} \to A. \tag{1}$$

For example, in three most important cases of associative, commutative, or Lie algebras, the component $O(n)$ of the corresponding symmetric operad consists of elements of the free algebra of noncommutative, commutative, or Lie polynomials, with n generators with degree 1 in each variable.

But there is no operad of vertex algebras.

There exist the dual notions of a cooperad and a coalgebra over a cooperad. But it is also possible to consider algebras over cooperads and coalgebras over operads. Suppose that components $O(n)$ of an operad are finite-dimensional vector spaces. Then the collection $\{C(n)\}_{n\geqslant 0}$ of dual spaces $C(n) = O(n)^*$ admits a natural cooperad structure. For an algebra A over O, one can rewrite (1) in a contra-variant form and obtain the maps

$$A^{\otimes n} \to C(n) \otimes A, \tag{2}$$

which give an equivalent description of A as an algebra over a cooperad C. One can imagine $C(n)$ as a space of "functions on operations".

It was recently proved in [1] that the vertex algebras can be described as algebras over a certain cooperad called a cooperad of correlation functions. A component $C(n)$ of this cooperad is a graded algebra of functions on the configuration space of distinct n points, i.e.,

$$C(n) := \Bbbk[z_1, ..., z_n]\big[(z_i - z_j)^{-1}\big]_{i<j} \tag{3}$$

is a localization of the algebra of polynomials. Note that the whole $C(n)$ for $n > 0$ and the graded components of $C(n)$ for $n > 1$ are infinite-dimensional. So, there is no an operad dual to C.

We observe that a localization in (3) can be described via a tensor product, and the whole construction of the above cooperad has bialgebraic nature. We illustrate this in the case of a q-line, a simple example of a braided Hopf algebra, which is just an algebra of polynomials $\Bbbk_q[z]$ with primitive generator and q-deformed statistics. In the case $q = 1$, the above cooperad of correlation functions will be obtained.

The general constructions are considered in the context of iterated monoidal categories with unbiased lax tensor products. We reformulate the unified description of a (symmetric, braided) lax monoidal form [2] in the terms of the 2-fold categorical operads of sequential trees Tree. We consider a version of the Gray tensor product $\boxdot$ for Cat-operads and describe multitensor categories as algebras over Gray tensor products of the operads Tree and Tree$^{\mathrm{op}}$. In particular, an algebra over the operad Tree$\boxdot$Tree$^{\mathrm{op}}$ is a $(1,1)$-tensor category $\mathcal{C}$ with one lax and one colax monoidal structures. A bialgebra in $\mathcal{C}$ is a bilax functor $1 \to \mathcal{C}$. We construct a 2-fold operad and cooperad from such type of bialgebras. Operads in n-fold monoidal categories with strong tensor products were introduced and studied in [3]. In the case of a lax tensor product, the resulting operadic tensor product is also lax. The operadic structure in this case is naturally described in terms of sequential trees.

Finally, we discuss another appearance of a q-line: Modules over a factor-algebra $\Bbbk_q[z]/(z^N)$ are N-complexes. For the usual complexes ($N = 2$), the homotopical and derived categories are examples of triangulated categories (Verdier, 1963). After a long story, Maltsiniotis in 2006 defined a strongly triangulated category, where triangle and octahedron axioms are extended to the list of axioms indexed by hyper-simplexes $\Delta_{n,1}$. We consider analogs of Maltsiniotis' axioms for "higher" hyper-simplexes $\Delta_{n,k}$ depending on fixed k and primitive $(k+1)$-th root of a unit q. We observe that the homotopical category of $(k+1)$-complexes is an example for such axioms.

2. q-Line

Let $\Bbbk$ be a commutative ring.

Example 2.1. *An algebra of polynomials $\Bbbk[z]$ the Hopf algebra structure with primitive generator:*

$$\Delta(z) = z \otimes 1 + 1 \otimes z \quad S(z) = -z.$$

Then, from the bialgebra axiom,

$$\Delta(z^n) = (z \otimes 1 + 1 \otimes z)^n = \sum_{0 \leqslant k \leqslant n} \binom{n}{k} z^k \otimes z^{n-k}.$$

This is just a universal envelope $U(\mathfrak{g})$ for the one-dimensional Lie algebra $\mathfrak{g}$.

One can consider a graded version.

Definition 2.2. *Denote, by $\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})_q$, a category $(\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod}), \otimes, c_q)$ of graded $\Bbbk$-modules $X = \bigoplus_{n\in\mathbb{Z}} X_n$ with tensor product*

$$(X \otimes Y)_n = \bigoplus_{m\in\mathbb{Z}} X_m \otimes Y_{n-m}$$

and braiding (statistics)

$$c_q(x \otimes y) = q^{\deg x \cdot \deg y} \cdot y \otimes x$$

for $q \in \Bbbk^{\times}$.

Example 2.3. *[q-line or Eulerian Hopf algebra of Joni and Rota, 1982] An algebra of polynomials admits the Hopf algebra structure $\Bbbk_q[z]$ in $\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})_q$ with primitive generator z, $\deg z = 1$:*

$$\Delta^{(m)}(z) = \sum_{i+1+j=m} 1^{\otimes i} \otimes z \otimes 1^{\otimes j},$$

$$\Delta^{(m)}(z^n) = \sum_{n_1+\cdots+n_m=n} \binom{n}{n_1 \cdots n_m}_q z^{n_1} \otimes \cdots \otimes z^{n_m}.$$

Here,

$$\binom{n}{n_1 \cdots n_m}_q = \sum_{\sigma\in S_{n_1\cdots n_m}} q^{\ell(\sigma)} \in \mathbb{Z}[q]$$

are Gaussian polynomials*; a sum is over* shuffle *permutations $\sigma \in S_{n_1\cdots n_m}$; $\ell(\sigma)$ is the length of a permutation σ.*

Special cases are q-numbers:

$$[n]_q := \binom{n}{n-1\ 1}_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1},$$

$$[n]_q! := \binom{n}{1 \cdots 1}_q = [1]_q[2]_q \cdots [n]_q,$$

$$\binom{n}{n_1 \cdots n_m}_q = \frac{[n]_q!}{[n_1]_q! \cdots [n_m]_q!}.$$

A graded dual is the Hopf algebra $\Bbbk_q\{z\}$ of divided powers *with basis $z^{(n)}$, $\deg z^{(n)} = n$ for $n \geqslant 0$.*

$$z^{(n)} \cdot z^{(m)} = \binom{n+m}{n}_q z^{(n+m)},$$

$$\Delta\bigl(z^{(n)}\bigr) = \sum_m z^{(m)} \otimes z^{(n-m)}.$$

There is a bialgebra morphism

$$\varphi : \Bbbk_q[z] \to \Bbbk_q\{z\}, \qquad z^n \mapsto [n]_q! z^{(n)}.$$

– *When all $[n]_q \in \Bbbk^\times$, then φ is an isomorphism.*

– *When q is a primitive N-th root of unity, i.e., $[N]_q = 0$ and $[n]_q \in \Bbbk^\times$ for $0 < n < N$, the image $\varphi\big(\Bbbk_q[z]\big)$ is the Hopf algebra*

$$\Bbbk_q[z]/(z^N).$$

Remark 2.4. *q-Line is a special case of a general construction, when the single variable z is replaces by an object $V \in \mathcal{C}$ of a braided monoidal category. The free tensor algebra $TV := \sum_{n\geqslant 0} V^{\otimes n}$ admits a Hopf algebra structure with shuffle coproduct. Another Hopf algebra $T^\vee V$ with the same underlying space is equipped with a shuffle product and cut coproducts.*

The bialgebra morphism $\varphi : TV \to T^\vee V$ is an analog of the (anti-)symmetrizer. The image of this morphism is an analog of the symmetric (external) Hopf algebra SV (respectively, ΛV) (see [4]).

Similar universal constructions for categories with a pair of tensor products in [5] cover a wide variety of combinatorial Hopf algebras.

– An action of $\Bbbk_q[z]$ is a (q-)derivation $d(a) := z.a$. The module-algebra axiom turns into the Leibnitz rule:

$$d(ab) = z.(ab) = (z_{(1)}.a)(z_{(2)}.b) = d(a)b + q^{|a|} \cdot a \cdot d(b).$$

– $\Bbbk[z]/(z^N)$-modules are N-complexes

$$\cdots \longrightarrow X_{n-1} \xrightarrow{d} X_n \xrightarrow{d} X_{n+1} \longrightarrow \cdots$$

with $d^N = 0$.

For a fixed primitive N-th root of unity, the category of $\Bbbk[z]/(z^N)$-modules is closed monoidal. This allow one to consider a q-analog of the homological algebra: Kapranov [6], Dubois-Violette [7]: homotopies, homology, q-dg-categories, *etc.*

3. Operads from Bialgebras

A *(plane) collection* is a family $\big\{\mathcal{C}(n)\big\}_{n\geqslant 0}$ of objects of a fixed braided monoidal category (in our case, graded $\Bbbk$-modules).

A tensor product of two collections is a new collection with

$$\mathcal{D}\odot\mathcal{C}(n) := \bigoplus_{m\geqslant 0} \bigoplus_{n_1+\ldots+n_m=n} \mathcal{D}(m)\otimes\mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_m).$$

A *plane operad* (respectively, *cooperad*) is a monoid (respectively, comonoid) in the category of collections. This assumes the associative product μ (respectively, coproduct Δ) with components

$$\mu^{n_1,\ldots,n_m} : \mathcal{C}(m)\otimes\mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_m) \to \mathcal{C}(n),$$
$$\Delta^{n_1,\ldots,n_m} : \mathcal{C}(n) \to \mathcal{C}(m)\otimes\mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_m).$$

Given a graded $\Bbbk$-module X, there is an operad (respectively, cooperad) of endomorphisms

$$\big\{\underline{\mathrm{Hom}}(X^{\otimes n}, X)\big\}_{n\geqslant 0},$$

respectively,

$$\big\{\underline{\mathrm{Hom}}(X, X^{\otimes n})\big\}_{n\geqslant 0}.$$

The structure of an algebra over an operad $\mathcal{E}$ (respectively, that of a coalgebra over a cooperad $\mathcal{C}$) on X is a morphism of operads (respectively, coperads)

$$\mathcal{E}(n) \to \underline{\mathrm{Hom}}(X^{\otimes n}, X),$$
$$\mathcal{C}(n) \to \underline{\mathrm{Hom}}(X, X^{\otimes n}),$$

or, equivalently, morphisms compatible with the product (respectively, coproduct)

$$\mathcal{E}(n)\otimes X^{\otimes n} \to X, \text{ respectively, } \mathcal{C}(n)\otimes X \to X^{\otimes n}.$$

There are dual notions of the algebra over a cooperad and the algebra over an operad described by morphisms

$$X^{\otimes n} \to \mathcal{C}(n)\otimes X, \text{ respectively, } X \to \mathcal{E}(n)\otimes X^{\otimes n}.$$

When all graded components $\mathcal{E}(n)_k$ (respectively, $\mathcal{C}(n)_k$) of an operad (respectively, cooperad) are finite-dimensional, there exists a graded dual cooperad (respectively, operad) with components $\mathcal{E}(n)^*_k$ (respectively, $\mathcal{C}(n)^*_k$). The algebras over a cooperad are algebras over the dual operad, but only when this dual one exists.

The symmetric (braided) operad (cooperad) $\big\{\mathcal{C}(n)\big\}_{n\geqslant 0}$ assumes, for each $n \geqslant 0$, the action of the symmetric group S_n (respectively, braided group B_n) on $\mathcal{C}(n)$ compatible with the product (coproduct).

One can consider a cooperad $\big\{\mathcal{C}(n)\big\}_{n\geqslant 0}$ in the category of graded algebras. This means that each $\mathcal{C}(n)$ is an algebra, and the coproducts

$$\mathcal{C}(n) \to \mathcal{C}(m)\otimes\mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_m)$$

are algebraic morphisms.

Similarly, one can consider an operad in the category of graded coalgebras.

Both conceptions are the examples of bialgebras in the category with a pair of tensor products.

Theorem 3.1. *Let B be a braided bialgebra. Consider the collection $\{B^{\otimes n}\}_{n\geqslant 0}$.*

– It admits the structure of an operad in coalgebras. For $m=\sum_{i\in n} m_i$,

$$B^{\otimes n}\otimes\left(\otimes^{i\in n}B^{\otimes m_i}\right)\xrightarrow[\sim]{c}\otimes^{i\in n}\left(B\otimes B^{\otimes m_i}\right)$$
$$\xrightarrow{\otimes^{i\in n}\left(\Delta^{(m_i)}\otimes 1\right)}\otimes^{i\in n}\left(B^{\otimes m_i}\otimes B^{\otimes m_i}\right)$$
$$\xrightarrow[\sim]{c}(B\otimes B)^{\otimes m}\xrightarrow{\mu^{\otimes m}}B^{\otimes m}.$$

The algebras (respectively, coalgebras) over this operad are module-algebras (respectively, module-coalgebras) over B.

– The collection $\{B^{\otimes n}\}_{n\geqslant 0}$ admits the structure of a cooperad in algebras. We have

$$B^{\otimes m}\xrightarrow{\Delta^{\otimes m}}(B\otimes B)^{\otimes m}\xrightarrow[\sim]{c}$$
$$\otimes^{i\in n}\left(B^{\otimes m_i}\otimes B^{\otimes m_i}\right)\xrightarrow{\otimes^{i\in n}\left(\mu^{(m_i)}\otimes 1\right)}$$
$$\otimes^{i\in n}\left(B\otimes B^{\otimes m_i}\right)\xrightarrow[\sim]{c}B^{\otimes n}\otimes\left(\otimes^{i\in n}B^{\otimes m_i}\right).$$

The algebras (respectively, coalgebras) over this cooperad are comodule-algebras (respectively, comodule-coalgebras) over B.

Example 3.2. *Consider the bialgebra $B=\Bbbk_q[z]$. The components of the corresponding cooperad are the algebras*

$$\Bbbk_q[z]^{\otimes n}\simeq\Bbbk_q[z_1,...,z_n],\quad z_jz_i=qz_iz_j,\ \text{for } i<j.$$

For a partition presented by a monotone map $\varphi: m\to$ $\to n$, the coproduct

$$\Bbbk_q[z_i]_{i\in m}\to\Bbbk_q[t_j]_{j\in n}\otimes\left(\otimes^{j\in n}\Bbbk_q[z_i]_{i\in\varphi^{-1}(j)}\right)$$

is determined by its values on the generators (operadic analog of primitivity)

$$z_i\mapsto z_i+t_{\varphi(i)}.$$

When all q-numbers $[n]_q$ are invertible (in particular, when $q=1$, and $\Bbbk$ is a field of characteristic 0), $\Bbbk_q[x]$ is isomorphic to $\Bbbk_q\{x\}$, and the algebras over the above cooperad are graded algebras equipped with a q-derivative.

For each component consider, a localization

$$\Bbbk_q[z_1,...,z_n]\hookrightarrow\Bbbk_q[z_1,...,z_n]\left[(z_i-z_j)^{-1}\right]_{i<j}.$$

Theorem 3.3. *There exists the unique cooperad structure on the collection of algebras*

$$\left\{\Bbbk_q[z_1,...,z_n]\left[(z_i-z_j)^{-1}\right]_{i<j}\right\}_{n\geqslant 0}$$

such that the above algebra of morphisms determine the morphism of a cooperad of graded algebras.

In the case $q=1$, this is a symmetric cooperad defined in [1] and called a cooperad of *correlation functions*. The algebras over this cooperad are vertex algebras.

A few words about coproducts in the above cooperad:

Consider an algebra $\Bbbk_q[z^{\pm 1}]$ of Laurent polynomials as an object in the category $\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})_q$.

Theorem 3.4. *The algebra $\Bbbk_q[z^{\pm 1}]$ admits a unique $\Bbbk_q[z]$ comodule algebra structure such that the natural embedding $\Bbbk_q[z]\hookrightarrow\Bbbk_q[z^{\pm 1}]$ is a morphism of comodule algebras.*

Proof. The comodule algebra structure on $\Bbbk_q[z]$ is just the coproduct

$$\Delta:\Bbbk_q[z]\to\Bbbk_q[z_1,z_2]\simeq\Bbbk_q[z]\otimes\Bbbk_q[z]\quad z\mapsto z_1+z_2.$$

The coaction is a q-deformed version of the power series extension that maps $(z)^{-(k+1)}$ to the sum

$$\sum_{j\geqslant k}\ \sum_{\substack{p_1,\dots,p_j\in\{0,1\}\\ p_1+\dots+p_j=j-k}} z_2^{-1}z_1^{p_1}z_2^{-1}\cdots z_2^{-1}z_1^{p_j}z_2^{-1}\in\Bbbk_q[z_1,z_2^{\pm 1}]$$

or

$$\sum_{j\geqslant 0}q^{-\binom{j+1}{2}}\binom{j+k}{k}_q z^j\otimes z^{-k-j-1}\in\Bbbk_q[z]\mathbin{\widehat{\otimes}}\Bbbk_q[z^{\pm 1}].$$

How can we put a formal series into the algebraic context?

Formal series are elements of the product of graded components (while polynomials are elements of the direct sum).

Following [1], we hide the formal series in a new tensor product of graded $\Bbbk$-modules:

$$(V\mathbin{\widehat{\otimes}}W)_n:=\operatorname{colim}_{k_0}\prod_{k\geqslant k_0}V_k\otimes W_{n-k}.$$

An interesting feature of this tensor product is that the associativity constrains are not an isomorphism but the embeddings

$$A \mathbin{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}} C) \hookrightarrow (A \mathbin{\widehat{\otimes}} B) \mathbin{\widehat{\otimes}} C$$

in nth graded component:

$$\prod_{\substack{i \geqslant i_0 \\ j \geqslant j_0}} A_i \otimes B_j \otimes C_{n-i-j} \hookrightarrow \prod_{\substack{i \geqslant i_0 \\ i+j \geqslant i_0+j_0}} A_i \otimes B_j \otimes C_{n-i-j}.$$

Now, we need to consider a cooperad $\mathcal{C}(n)$ in a category with two tensor products. Coproducts for this cooperad are the maps

$$\mathcal{C}(n) \to \mathcal{C}(m) \mathbin{\widehat{\otimes}} (\mathcal{C}(n_1) \otimes \dots \otimes \mathcal{C}(n_m)).$$

In our case, for a monotone map $\varphi : m \to n$, the coproduct

$$\begin{aligned}
&\Bbbk_q[z_i]_{i \in m}\big[(z_i - z_j)^{-1}\big]_{i<j} \to \\
&\to \Bbbk_q[t_j]_{j \in n}\big[(t_i - t_j)^{-1}\big]_{i<j} \mathbin{\widehat{\otimes}} \\
&\big(\otimes^{j \in n} \Bbbk_q[z_i]_{i \in \varphi^{-1}(j)}\big[(z_i - z_j)^{-1}\big]_{i<j}\big) \\
&z_i \mapsto z_i + t_{\varphi(i)}, \\
&(z_i - z_j)^{-1} \mapsto \\
&\mapsto \begin{cases} (z_i - z_j)^{-1}, & \varphi(i) = \varphi(j) \\ i_{z_i,z_j}(t_{\varphi(i)} - t_{\varphi(j)} + z_i - z_j)^{-1}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Here, i_{z_i,z_j} means the power series expansion in positive degrees of z_i and z_j.

4. General Constructions

4.1. Finite sets, finite ordinals, and trees

Let $\mathcal{O} \hookrightarrow \mathcal{S}$ be the skeletal categories of a finite ordinal and of finite sets, respectively. Objects in both cases are the natural numbers $n \geqslant 0$ considered as (linear ordered) sets $\mathbf{n} = \{0 < 1 < \dots < n-1\}$; morphisms in $\mathcal{O}$ (respectively, in $\mathcal{S}$) are monotone (respectively, arbitrary) maps $\varphi : \mathbf{m} \to \mathbf{n}$. We can also consider a poset $\mathbf{n}$ as a category and morphisms in $\mathcal{O}$ as functors.

For $j \in \mathbf{n} \in \operatorname{Ob}\mathcal{O}$, the pullback along of $\hat{j} : \mathbf{1} \to \mathbf{n}$, $0 \mapsto j$ determines a functor $\mathcal{O}/\mathbf{n} \to \mathcal{O}/\mathbf{1} \cong \mathcal{O}$, $(\varphi : \mathbf{m} \to \mathbf{n}) \mapsto \varphi^{-1}(j)$. The functor $\iota : \mathcal{O}/\mathbf{n} \to \mathcal{O}^n$, $\varphi \mapsto \big(\varphi^{-1}(j)\big)_{j \in \mathbf{n}}$ is an isomorphism of categories. The category $\mathcal{O}$ becomes a strict monoidal equipped with the products $\sum^{\mathbf{n}} = \big(\mathcal{O}^n \xrightarrow{\iota^{-1}} \mathcal{O}/\mathbf{n} \xrightarrow{\text{dom}} \mathcal{O}\big)$, $(\mathbf{m}_j)_{j \in \mathbf{n}} \mapsto \sum^{j \in \mathbf{n}} \mathbf{m}_j$. This monoidal category is freely generated by the terminal nonsymmetric unital operad $\mathcal{O}(\text{-}, \mathbf{1}) = \{\mathcal{O}(\mathbf{n}, \mathbf{1})\}_{n \geqslant 0}$.

Finite Cartesian product $\prod_{i \in \mathbf{m}} \mathbf{n}_i$ of finite ordinals admits a lexicographic linear order: $(j_i)_{i \in \mathbf{m}} < (j'_i)_{i \in \mathbf{m}}$ iff there exists $i' \in \mathbf{m}$ such that $j_i = j'_i$ for all $i < i'$ and $j_{i'} = j'_{i'}$.

The conventional category Δ is a full subcategory in $\mathcal{O}$ of a nonempty ordinal. The contravariant endofunctor of ideals on the category of partially ordered sets can be restricted to the functor $[\text{-}] : \mathcal{O}^{\text{op}} \to \Delta$. The poset of ideals in $n = \{0 < \dots < n-1\}$ is $[n] = n + 1 = \{0 < \dots < n\}$. For a monotone map $\varphi : n \to m$, the corresponding map $[\varphi] : [m] \to [n]$ is the preimage:

$$[\varphi](k) = \{i \mid \varphi(i) \in k\} = \big\{0 < \dots < \max\{i \mid \varphi(i) \in k\}\big\}.$$

Let $\mathsf{N}\mathcal{J} : \mathcal{O} \to \text{Set}$ be the composition of the functor $[\text{-}] : \mathcal{O} \to \Delta^{\text{op}}$ with the nerve $[n] \mapsto \text{Cat}([n]^{\text{op}}, \mathcal{J})$ of a small category $\mathcal{J}$. Explicitly, n-th component $\mathsf{N}\mathcal{J}_n$ is the set of functors

$$f : \big([n]^{\text{op}} \to \mathcal{J}, \quad (i \leqslant j) \mapsto \big(f_{j \geqslant i} : f(j) \to f(i)\big)$$

that are in the one-to-one correspondence with the sequences of morphisms in $\mathcal{J}$:

$$\begin{aligned} &f(n) \xrightarrow{f_{n-1}} f(n-1) \to \dots \to f(1) \xrightarrow{f_0} f(0), \\ &f_i := f_{i+1 \geqslant i}. \end{aligned}$$

For a monotone map $\varphi : n \to m$, the corresponding map $\varphi_* : \mathsf{N}\mathcal{J}_n \to \mathsf{N}\mathcal{J}_m$ is the pre-composition with $[\varphi] : [m] \to [n]$.

We also consider the category $\operatorname{el}\mathsf{N}\mathcal{J}$ of elements of the co-pre-sheaf $\mathsf{N}\mathcal{J} : \mathcal{O} \to \text{Set}$. The objects of $\operatorname{el}\mathsf{N}\mathcal{J}$ are elements of the nerve $f \in \mathsf{N}\mathcal{J}_n$, $n \geqslant 0$; the morphism from $f \in \mathsf{N}\mathcal{J}_n$ to $g \in \mathsf{N}\mathcal{J}_m$ is a morphism $\varphi : n \to m$ in $\mathcal{O}$ such that $g = \varphi_*(f)$.

Let $\mathcal{J} = \mathcal{O}$ (respectively, $\mathcal{J} = \mathcal{S}$). Then the objects of $\operatorname{el}\mathsf{N}\mathcal{J}$ are called *plane* (respectively, *symmetric*) *sequential forests*. A *plane* (respectively, *symmetric*) *sequential tree* of height n with k leaves is a sequential forest $t : [n]^{\text{op}} \to \mathcal{J}$ with $t(0) = 1$ and $t(n) = k$.

For a forest $f : [n]^{\text{op}} \to \mathcal{J}$, each element $j \in f(0)$ determines a tree $f_{|j} : [n]^{\text{op}} \to \mathcal{J}$ presented by the top row in the following diagram with pullback squares:

$$\begin{tikzcd}
f^{-1}_{n+1\geqslant 0}(j) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \cdots \arrow[r] & f^{-1}_{1\geqslant 0}(j) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \{j\} \arrow[d, hook] \\
f(n+1) \arrow[r, "f_n"] & \cdots \arrow[r] & f(1) \arrow[r, "f_0"] & f(0)
\end{tikzcd}$$

Denote $\mathrm{pTree}(k)$ (respectively, $\mathrm{sTree}(k)$) the full subcategory of $\mathrm{el}\,\mathsf{N}\mathcal{J}$, where the objects are sequential trees with k leaves. The full subcategory of $\mathrm{bTree}(k) \hookrightarrow \mathrm{sTree}(k)$ of *braided sequential trees* is determined by the following condition on an object t:

for each $n \geqslant p > q > r > 0$ and $a, b \in t(p)$, if $a < b$ and $t_{p\geqslant q}(a) > t_{p\geqslant q}(b)$, then $t_{p\geqslant r}(a) \geqslant t_{p\geqslant r}(b)$.

For each Tree = pTree, bTree or sTree, we consider the collection $\mathrm{Tree} = \{\mathrm{Tree}(k)\}_{k\geqslant 0}$. We can also consider the collection $\mathrm{Tree}^{\mathrm{op}} = \{\mathrm{Tree}(k)^{\mathrm{op}}\}_{k\geqslant 0}$ of opposite categories. The height of a sequential tree determines an $\mathbb{N}$-grading on objects of the above categories. If $t : [n]^{\mathrm{op}} \to \mathcal{J}$ is a tree with $t(n) = k$, we write $t \in \mathrm{Tree}_n(k)$.

4.2. Cat*-operads of Endofunctors and* $\mathbf{Cat}_{\mathcal{N}}$*-operad of Trees*

Let $\mathrm{Cat}_{\mathcal{N}}$ be the category of small categories $\mathcal{C}$ with an $\mathbb{N}$-grading on objects $\deg : \mathrm{Ob}\,\mathcal{C} \to \mathbb{N}$, and let the functors $F : \mathcal{C} \to \mathcal{C}'$ preserving this grading be $\deg F(X) = \deg X$.

There is another description of this category. Consider Cat as a symmetric monoidal category equipped with Cartesian product $\times$. Then the indiscrete category $\mathcal{N}$, whose objects are natural numbers, admits the structure of a bialgebra in $(\mathrm{Cat}, \times)$ (inherited from additive monoid structure on $\mathbb{N}$):

$$\mathcal{N} \times \mathcal{N} \underset{\Delta : n \mapsto (n,n)}{\overset{\mu : (m,n) \mapsto m+n}{\rightleftarrows}} \mathcal{N} \underset{\varepsilon : n \mapsto 0}{\overset{\eta : 0 \mapsto 0}{\leftrightarrows}} 1.$$

The category $\mathrm{Cat}_{\mathcal{N}}$ can be identified with a category of comodules $\mathcal{C}$ over $\mathcal{J}$ with coaction $\mathcal{C} \to \mathcal{C} \times \mathcal{J}$, $X \mapsto (X, \deg X)$.

There are two monoidal structures on $\mathrm{Cat}_{\mathcal{N}}$:

$\mathcal{N}$-comodule product: this is the Cartesian product of underlying categories $\times^{i\in n}\mathcal{C}_i$ with grading $\deg(X_i)_{i\in n} = \sum_{i\in n} \deg X_i$;

cotensor product over $\mathcal{N}$: this is the categorical product $\times^{i\in n}_{\mathcal{N}}\mathcal{C}_i$ in $\mathrm{Cat}_{\mathcal{N}}$. The forgetful functor $\mathrm{Cat}_{\mathcal{N}} \to \mathrm{Cat}$ admits the structure of a colax monoidal functor. For $n > 0$, this a full subcategory in the Cartesian product $\times^{i\in n}\mathcal{C}_i$ with objects $(X_i)_{i\in n}$ such that all $\deg X_i$ are the same, and $\deg(X_i)_{i\in n} := \deg X_0$. The terminal object in $\mathrm{Cat}_{\mathcal{N}}$ is the regular comodule $\mathcal{N}$.

For a family $(\mathcal{C}_{ij})_{i\in n, j\in m}$ of categories, there is the natural embedding

$$\eta^{nm} : \times^{i\in n} \times^{j\in m}_{\mathcal{N}} \mathcal{C}_{ij} \hookrightarrow \times^{j\in m}_{\mathcal{N}} \times^{i\in n} \mathcal{C}_{ij}.$$

An object $(X_{ij})_{i\in n, j\in m}$, $X_{ij} \in \mathcal{C}_{ij}$ on the left- (respectively, right-)hand side is characterized by the following condition: for each $i \in n$, $\deg X_{ij}$ is independent of j (respectively, $\sum_{i\in n} \deg X_{ij}$ is independent of j). This turns $(\mathrm{Cat}_{\mathcal{N}}, \times, \times_{\mathcal{N}}, \eta)$ into the 2-fold monoidal category.

The category of the collections $\{C(k)\}_{k\geqslant 0}$ of small categories admits the 'operadic' monoidal structure

$$\left(\odot^{i\in n} C_i\right)(k) = \coprod_{t\in \mathrm{Tree}_n(k)} \times^{i\in n} \times^{j\in t(i)} C_i(t_i^{-1}(j)).$$

A monoid in this category is called a Cat-operad. The example of a Cat-operad is the operad $\mathrm{End}_{\mathcal{C}}$ of endomorphisms of a small category $\mathcal{C}$, where $\mathrm{End}_{\mathcal{C}}(k)$ is the category of functors $\mathcal{C}^{\times n} \to \mathcal{C}$.

These monoidal structures on $\mathrm{Cat}_{\mathcal{N}}$ induce an 'operadic' monoidal structure on collections $\{C(k)\}_{k\geqslant 0}$ in $\mathrm{Cat}_{\mathcal{N}}$:

$$\left(\odot^{i\in n} C_i\right)(k) = \coprod_{t\in \mathrm{Tree}_n(k)} \times^{i\in n} \times^{j\in t(i)}_{\mathcal{N}} C_i(t_i^{-1}(j))$$

In particular, the n-th tensor power of a collection C is a collection $C^{\odot n}$, where the category $C^{\odot n}(k)$ is a disjoint union $\coprod_{t\in \mathrm{Tree}_n(k)} C(t)$. An object in $C(t)$ is a coloring c of each vertex $(i \in n, j \in t(i))$ in t by an object $c_{(i,j)} \in C(t_i^{-1}(j))$ such that $\deg c_{(i,j)} = d_i$ is independent of j, and $\deg c = \sum_{i\in n} d_i$. The morphism $c \to c'$ in $C(t)$ is a family of morphisms $c_{(i,j)} \to c'_{(i,j)}$ in $C(t_i^{-1}(j))$.

Definition 4.1. *A* $\mathrm{Cat}_{\mathcal{N}}$*-operad is a monoid* C *in the above monoidal category of collections.*

In particular, this assumes a product $\mu_t : C(t) \to C(k)$ for each $t \in \mathrm{Tree}_n(k)$.

The single tree of height 0 is just a root $\mathbf{1}$. Each natural number k determines a tree $\mathbf{k} \to \mathbf{1}$ of height

1, which is called an *elementary tree* or *corolla.* The (internal) vertex of a tree $t \in \mathrm{Tree}_n(k)$ is an element $(i,j) \in \in \coprod_{i\in n} t(i)$. The vertex (i,j) determines an elementary tree $t_{i|j}$ that is a subtree $t_i^{-1}(j) \to \{j\} \simeq 1$ in the forest $t(i+1) \xrightarrow{t_i} t(i)$.

Given a morphism of trees $\mathrm{Tree}_m(k) \ni s \xrightarrow{\varphi} t \in \in \mathrm{Tree}_n(k)$, for $i \in n$, we denote, by φ_i, the forest obtained by the restriction of $s : [m]^{\mathrm{op}} \to \mathcal{J}$ to the subinterval $[\varphi^{-1}(i)] \simeq [[\varphi](i), [\varphi](i+1)] \hookrightarrow [m]$, i.e., $\varphi_i : p \mapsto \mapsto s(p + [\varphi](i))$. Then, for each $j \in t(i) = s([\varphi](i))$, one can consider the tree $\varphi_{i|j} \in \mathrm{Tree}_{\varphi^{-1}(i)}\, t_i^{-1}(j)$. The assignment $\sqcup^{i\in n} t(i) \ni \varphi \mapsto \varphi_{i|j}$ (coloring a tree t) determines an element of $\mathrm{Tree}(t)$.

Lemma 4.2. *For each tree* $t \in \mathrm{Tree}(k)$, *the category* $\mathrm{Tree}(t)$ *is isomorphic to the slice category* $\mathrm{Tree}(k)/t$.

Proposition 4.3. *The collection* Tree *is a* $\mathrm{Cat}_{\mathcal{N}}$-*operad. For* $t \in \mathrm{Tree}(k)$, *the corresponding product* μ_t *is the domain functor*

$\mathrm{Tree}(t) \simeq \mathrm{Tree}(k)/t \xrightarrow{\mathrm{dom}} Tree(k)$.

4.3. Lax monoidal categories

The monoidal category is a category $\mathcal{C}$ equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $1 \in \mathcal{C}$, and natural isomorphisms (associator and unitors) satisfying coherence identities, which allow one to define the n-fold tensor products $\otimes^n : \mathcal{C}^n \to \mathcal{C}$ unique up to an isomorphism. It is essential that the same notion of unbiased monoidal category (a term of Leinster [8]) involves n-fold tensor products $\otimes^n$ as a part of the definition. Two natural ways to omit the invertibility condition for associators in an unbiased monoidal category produces the dual notions of lax and colax (or oplax) monoidal categories. We choose the term "lax" in the case corresponding to representable multicategories and suitable to consider monoids. One can define the lax (colax, strong, strict) monoidal category as a lax (colax, strong, strict) monoid in Cat .

The unifying way is to describe the (co)lax (symmetric, braided) monoidal category as an algebra over 2-fold (symmetric, braided) $\mathrm{Cat}_{\mathcal{N}}$-operad $\mathrm{Tree}^{\mathrm{op}}$ or Tree . The underlying 2-fold Set-operad of trees is freely generated by elementary trees. For each $\mathrm{Cat}_{\mathcal{N}}$-operad Tree, the generators and the relations are morphisms in the category of elements over the generators and the relations of a monoidal category $\mathcal{O}$. The lax monoidal category structure on a category $\mathcal{C}$ is described in terms of these generators and relations. This assumes a functor $\otimes^n : \mathcal{C}^n \to \mathcal{C}$ for each $n \in \mathbb{N}$ that is a natural transformation $\lambda^\varphi = \lambda^t : \otimes^{i\in n} X_i \to \otimes^{j\in m} \otimes^{i\in\varphi^{-1}(j)} X_i$ for each tree $t = = \big(n \xrightarrow{\varphi} m \to 1\big)$ such that two ways to construct a morphism λ^t for a tree $t = \big(n \xrightarrow{\varphi} m \xrightarrow{\psi} p \to 1\big)$ coincide:

$$\begin{tikzcd}
\otimes^{i\in n} X_i \arrow[r] \arrow[d] & \otimes^{k\in p} \otimes^{i\in(\psi\varphi)^{-1}(k)} X_i \arrow[d] \\
\otimes^{j\in m} \otimes^{i\in\varphi^{-1}(j)} X_i \arrow[r] & \otimes^{k\in p} \otimes^{j\in\psi^{-1}(k)} \otimes^{i\in\varphi^{-1}(j)} X_i.
\end{tikzcd}$$

The lax (symmetric, braided) monoidal functor is a lax morphism of $\mathrm{Tree}^{\mathrm{op}}$-algebras. We can again describe it only on the generators of an operad. This is a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation

$$\phi^n : \otimes^{i\in n} F(X_i) \to F\big(\otimes^{i\in n} X_i\big) \quad \text{for each } n \in \mathbb{N}, \quad (4)$$

such that, for each tree $t = \big(n \xrightarrow{f} m \to 1\big)$, the morphisms $\lambda^{(-)}$ entwine ϕ in $\mathcal{C}$ and $\mathcal{D}$:

$$\begin{tikzcd}
\otimes^{i\in n} F X_i \arrow[r, "\phi^n"] \arrow[d, "\lambda^f"'] & F \otimes^{i\in n} X_i \arrow[d, "F\lambda^f"] \\
\otimes^{j\in m} \otimes^{i\in f^{-1}(j)} F X_i \arrow[r, "\phi^t"] & F \otimes^{j\in m} \otimes^{i\in f^{-1}(j)} X_i.
\end{tikzcd}$$

The natural transformation $(F, \phi) \to (G, \psi)$ between lax (symmetric, braided) monoidal functors is a natural transformation $t : F \to G$ entwining ϕ and ψ.

The terminal category 1 is a lax monoidal in the unique obvious way. An algebra in the lax monoidal category $(\mathcal{C}, \otimes, \lambda)$ is a lax monoidal functor $1 \to \to (\mathcal{C}, \otimes, \lambda)$, $0 \mapsto A$. The natural transformations (4) turn into the algebra multiplications $\mu_A^{(n)} : \otimes^n A \to A$; the compatibility with $\lambda^{(-)}$ turns into the associativity of multiplications.

In the dual case: a coalgebra in the colax monoidal category $(\mathcal{C}, \boxtimes, \rho)$ is a colax monoidal functor $1 \to \to (\mathcal{C}, \boxtimes, \rho)$.

4.4. Multitensor categories and the Gray product of operads

One can consider a lax monoidal category in the 2-category of lax monoidal categories. In such a way,

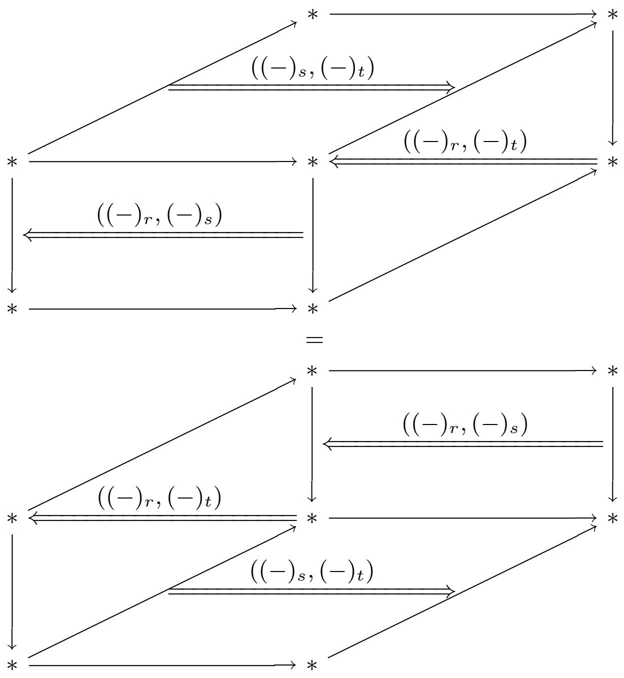

Fig. 1. Cubic relations in the Gray product

the duoidal (or 2-monoidal) categories of [5] are obtained. One can iterate this procedure and obtain a multitensor category with tensor products $\otimes_p$, $p \in N$. For $p < q$, the tensor product $\otimes_q$ is a lax monoidal functor with respect to $\otimes_p$. This assumes the interchange natural transformation

$$\eta^{nm}_{pq} : \otimes^{i\in n}_p \otimes^{j\in m}_q X_{ij} \to \otimes^{j\in m}_q \otimes^{i\in n}_p X_{ij}$$

compatible with λ's. Moreover, for $p < q < r$, they satisfy the braid relation

$$\eta_{qr}\eta_{pr}\eta_{pq} = \eta_{pq}\eta_{pr}\eta_{qr} : \otimes_p \otimes_q \otimes_r \to \otimes_r \otimes_q \otimes_p. \qquad (5)$$

The Gray tensor product turns the category of strict 2-categories and strict 2-functors into a closed monoidal category, where the inner hom is a 2-category of 2-functors, pseudonatural transformations, and modifications. We propose to extend a lax version of the Gray tensor product to the case of multicategories enriched in Cat. The Cat-operad is a Cat-multicategory $\mathcal{E}$ with a single object $*$. As a special case, we obtain the lax Gray tensor product Cat-operad $\boxdot^{k\in\mathbf{p}}\mathcal{E}_k$ of Cat-operads $\mathcal{E}_k$, $k \in \mathbf{p}$. We consider the Cartesian product of underlying sets of the cells of Cat-operads $\mathcal{E}_i$. The dimension of the product cell $(c_i)_{i\in p}$ is $\sum_{i\in p} \dim c_i$. One can compose product cells in each fixed direction $j \in p$, i.e., $(c_i)_{i\in p} \cdot (c'_i)_{i\in p} = (c''_i)_{i\in p}$, when $c_j \cdot c'_j = (c''_i)_{i\in p}$ in the category $\mathcal{E}_j(m)$ for some $m \in \mathbb{N}$, and $c_i = c'_i = c''_i$ for $i \neq j$ and similarly for the operadic composition. The cells of dimension > 2 are assumed to be identical; hence, the cells of dimension 3 describe the relations in the product operad, and the cells of dimension > 3 can be ignored. For each $r \in \mathbf{p}$, there is a morphism of Cat-operads

$$(-)_{(r)} : \mathcal{E}_r \to \boxdot^{k\in\mathbf{p}}\mathcal{E}_k,$$
$$c \mapsto c_{(r)} := \big(\underbrace{*,\dots,*}_{r}, c, \underbrace{*,\dots,*}_{p-r-1}\big);$$

the images are a subject of relations that come from $\mathcal{E}_r$. For each $r < s$ in $\mathbf{p}$ and objects $c \in \mathcal{E}_r(\mathbf{m})$, $d \in \mathcal{E}_s(\mathbf{n})$, there is a morphism

$$(c_{(r)}, d_{(s)}) := \big(\underbrace{*,\dots,*}_{r}, c, \underbrace{*,\dots,*}_{s-r-1}, d, \underbrace{*,\dots,*}_{p-s-1}\big)$$

between the operadic compositions

$$c_{(r)} \circ (d_{(s)})^m \to d_{(s)} \circ (c_{(r)})^n$$

in $\boxdot^{k\in\mathbf{p}}\mathcal{E}_k(\mathbf{mn})$, which can be presented by the diagram

$$\begin{tikzcd} * \arrow[r, "(d_{(s)})^m"] \arrow[d, "(c_{(r)})^n"'] \arrow[dr, phantom, "\Downarrow (c_{(r)}, d_{(s)})"] & * \arrow[d, "c_{(r)}"] \\ * \arrow[r, "d_{(s)}"'] & * \end{tikzcd}$$

These squares respect operadic compositions of two types, horizontal and vertical. For $r < s < t$ in p, such squares satisfy the relation in Fig. 1 corresponding to a cubic 3-cell.

With small modifications, one can define the Gray product of $\mathrm{Cat}_c\, N$-operads.

Proposition 4.4. *Multitensor (n-oidal) categories are algebras over the lax Gray product* $\mathrm{Cat}_{\mathcal{N}}$*-operad* $\boxdot^n \mathrm{Tree}^{\mathrm{op}}$.

The natural transformations η^{km}_{pq} come from the morphism $\big((t)_p, (s)_q\big)$, where t and s are elementary trees with k and m inputs, respectively. The braid relations (5) come from the cubic relations in Fig. 1.

4.5. 2-Fold bialgebras

The most general 2-fold monoidal category suitable to consider bialgebras can be described in the following equivalent ways:

– colax monoid in the 2-category of lax monoidal categories;

– lax monoid in the 2-category of colax monoidal categories;

– algebra over the lax Gray product $\mathrm{Cat}_{\mathcal{N}}$-operad $\mathrm{Tree} \boxdot \mathrm{Tree}^{\mathrm{op}}$;

– category equipped with the lax monoidal structure $(\otimes, \lambda)$, colax monoidal structure $(\boxtimes, \rho)$, and natural transformations

$$\eta^{nm} : \otimes^{i\in n} \boxtimes^{j\in m} X_{ij} \to \boxtimes^{j\in m} \otimes^{i\in n} X_{ij}$$

compatible with λ and ρ.

We use the name "$(1,1)$-tensor" or "$(1,1)$-oidal" for such a category. A bilax monoidal functor between $(1,1)$-tensor categories is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with natural transformations

$$\phi^n : \otimes^{i\in n} F(X_i) \to F\big(\otimes^{i\in n} X_i\big),$$
$$\psi^n : F\big(\boxtimes^{i\in n} X_i\big) \to \boxtimes^{i\in n} F(X_i)$$

for each $n \in \mathbb{N}$ compatible with λ and ρ, respectively, and satisfying the hexagon identities

$$\begin{tikzcd}
& F \otimes^{i\in n} \boxtimes^{j\in m} X_{ij} \arrow[dr, "\eta^{nm}"] & \\
\otimes^{i\in n} F \boxtimes^{j\in m} X_{ij} \arrow[ur, "\phi^n"] \arrow[d, "\psi^m"'] & & F \boxtimes^{j\in m} \otimes^{i\in n} X_{ij} \arrow[d, "\psi^m"] \\
\otimes^{i\in n} \boxtimes^{j\in m} F X_{ij} \arrow[dr, "\eta^{nm}"'] & & \boxtimes^{j\in m} F \otimes^{i\in n} X_{ij} \\
& \boxtimes^{j\in m} \otimes^{i\in n} F X_{ij} \arrow[ur, "\phi^n"'] &
\end{tikzcd}$$

A bialgebra in the $(1,1)$-tensor category $\mathcal{C}$ is a bilax functor $1 \to \mathcal{C}$, $0 \mapsto B$ with products $\mu^n : \otimes^n B \to B$ and coproducts $\Delta^n : B \to \boxtimes^n B$. The hexagon identities turn into the bialgebra axioms:

$$\begin{tikzcd}
\otimes^n B \arrow[rr, "\mu^n"] \arrow[d, "\otimes^n \Delta^m"'] & & B \arrow[d, "\Delta^m"] \\
\otimes^n \boxtimes^m B \arrow[dr, "\eta^{nm}"'] & & \boxtimes^m B \\
& \boxtimes^m \otimes^n B \arrow[ur, "\boxtimes^m \mu^n"'] &
\end{tikzcd}$$

4.6. Unbiased version of 2-fold (co)operads

We consider two contexts dual each to another and suitable for operads (respectively, cooperads). In both cases, $(\mathcal{C}, \otimes, \boxtimes, \eta)$ is a category with two lax (respectively, colax) monoidal structures and natural transformations $\eta^{nm} : \otimes^n \circ \boxtimes^m \to \boxtimes^m \circ \otimes^n$ compatible with λ's (respectively, with ρ's). We can additionally suppose that the monoidal structure $\boxtimes$ (respectively, $\otimes$) is braided or symmetric.

Suppose that the category $\mathcal{C}$ admits countable categorical coproducts (respectively, products). Then the category of collections in $\mathcal{C}$ admits the new (braided, symmetric) lax monoidal structure

$$\big(\odot^{i\in n} C_i\big)(k) = \coprod_{t\in \mathrm{Tree}_n(k)} \otimes^{i\in n} \boxtimes^{j\in t(i)} C_i(t_i^{-1}(j)) \text{ and,}$$

respectively, the colax monoidal structure

$$\big(\odot^{i\in n} C_i\big)(k) = \prod_{t\in \mathrm{Tree}_n(k)} \boxtimes^{i\in n} \otimes^{j\in t(i)} C_i(t_i^{-1}(j)).$$

In particular, the n-th tensor power $C^{\odot n}(k)$ of a collection C is the coproduct (respectively, product) over $t \in \mathrm{Tree}_n(k)$ of

$$C(t) := \otimes^{i\in n} \boxtimes^{j\in t(i)} C(t_i^{-1}(j)) \text{ and,}$$

respectively,

$$C(t) := \boxtimes^{i\in n} \otimes^{j\in t(i)} C(t_i^{-1}(j)).$$

A (braided, symmetric) operad (respectively, cooperad) is a monoid (respectively, comonoid) in this (braided, symmetric) lax (respectively, colax) monoidal category of collections. More generally, one can explicitly define, without assumptions about categorical (co)products, an operad (respectively, a coperad) as a collection $\{C(n)\}_{n\geqslant 0}$ equipped with products $\mu^t : C(t) \to C(k)$ (respectively, coproducts $\Delta^t : C(k) \to C(t)$) for each tree $t \in \mathrm{Tree}(k)$ such that

$$\mu^t = \mu^s \circ \mu^\varphi, \quad \text{respectively,} \quad \Delta^t = \Delta^\varphi \circ \Delta^s$$

for each morphism $\mathrm{Tree}_n(k) \ni t \xrightarrow{\varphi} s \in \mathrm{Tree}_m(k)$. Here, $\mu^\varphi : C(t) \to C(s)$ (respectively, $\Delta^\varphi : C(s) \to C(t)$) is defined via the composition

$$\begin{aligned}
&C(t) := \otimes^{i\in n} \boxtimes^{j\in t(i)} C(t_i^{-1}(j)) \\
&\xrightarrow{\lambda\lambda} \otimes^{i\in m} \otimes^{i'\in \varphi^{-1}(i)} \boxtimes^{j\in s(i)} \boxtimes^{j'\in s(i')} t^{-1}_{[\varphi](i)+i'}(j') \\
&\xrightarrow{\eta} \otimes^{i\in m} \boxtimes^{j\in s(i)} \otimes^{i'\in \varphi^{-1}(i)} \boxtimes^{j'\in s(i')} t^{-1}_{[\varphi](i)+i'}(j') = \\
&= \otimes^{i\in m} \boxtimes^{j\in s(i)} C(t^\varphi_{i,j}) \\
&\xrightarrow{\otimes^{i\in m}\boxtimes^{j\in s(i)}\mu^{t^\varphi_{i,j}}} \otimes^{i\in m} \boxtimes^{j\in s(i)} C(s_i^{-1}(j)) =: C(s)
\end{aligned}$$

or, respectively,

$$\begin{aligned}
&C(s) := \boxtimes^{i\in m} \otimes^{j\in s(i)} C(s_i^{-1}(j)) \\
&\xrightarrow{\boxtimes^{i\in m}\otimes^{j\in s(i)}\Delta^{t^\varphi_{i,j}}} \boxtimes^{i\in m} \otimes^{j\in s(i)} C(t^\varphi_{i,j}) \\
&= \boxtimes^{i\in m} \otimes^{j\in s(i)} \boxtimes^{i'\in \varphi^{-1}(i)} \otimes^{j'\in s(i')} t^{-1}_{[\varphi](i)+i'}(j') \\
&\xrightarrow{\eta} \boxtimes^{i\in m} \boxtimes^{i'\in \varphi^{-1}(i)} \otimes^{j\in s(i)} \otimes^{j'\in s(i')} t^{-1}_{[\varphi](i)+i'}(j') \\
&\xrightarrow{\rho\rho} \boxtimes^{i\in n} \otimes^{j\in t(i)} C(t_i^{-1}(j)) =: C(t).
\end{aligned}$$

The axioms of a (co)operad imply that we have a functor

$$\mu^{(-)} : \mathrm{Tree}(k) \to \mathcal{C}, \quad \text{respectively,} \quad \Delta^{(-)} : \mathrm{Tree}(k)^{\mathrm{op}} \to \mathcal{C}$$

for each $k \in \mathbb{N}$.

4.7. From bialgebras to operads

Again, we consider two contexts from the previous section for operads (respectively, cooperads) with the additional assumption that the monoidal structure $\boxtimes$ (respectively, $\otimes$) is strong. So in both cases, we can consider a bialgebra (B, μ, Δ) in $(\mathcal{C}, \otimes, \boxtimes, \eta)$. Under these assumptions, we have two dual theorems in the operadic (respectively, cooperadic) context:

Theorem 4.5. *There exists an operad $B^\boxtimes$ with components $B^\boxtimes(n) = \boxtimes^n B$. Hence, for $t \in \mathrm{Tree}_n(k)$,*

$$B^\boxtimes(t) = \otimes^{i\in n} \boxtimes^{j\in t(i)} \boxtimes^{t_i^{-1}(j)} B \simeq \otimes^{i\in n} \boxtimes^{t(i+1)} B.$$

The product μ^t is the composition

$$\begin{aligned}
&\otimes^{i\in n} \boxtimes^{t(i+1)} B \xrightarrow{\otimes^{i\in n}\boxtimes^{j\in t(i+1)}\Delta^{t^{-1}_{n\geqslant i+1}(j)}} \\
&\otimes^{i\in n} \boxtimes^{j\in t(i+1)} \boxtimes^{t^{-1}_{n\geqslant i+1}(j)} B \xrightarrow[\simeq]{\rho} \\
&\otimes^n \boxtimes^k B \xrightarrow{\eta^{nk}} \boxtimes^k \otimes^n B \xrightarrow{\boxtimes^k\mu^n} \boxtimes^k B.
\end{aligned}$$

Theorem 4.6. *There exists a cooperad $B^\otimes$ with components $B^\otimes(n) = \otimes^n B$. Hence, for $t \in \mathrm{Tree}_n(k)$,*

$$B^\otimes(t) = \boxtimes^{i\in n} \otimes^{j\in t(i)} \otimes^{t_i^{-1}(j)} B \simeq \boxtimes^{i\in n} \otimes^{t(i+1)} B.$$

The coproduct Δ^t is the composition

$$\begin{aligned}
&\otimes^k B \xrightarrow{\otimes^k\Delta^n} \otimes^k \boxtimes^n B \xrightarrow{\eta^{kn}} \boxtimes^n \otimes^k B \\
&\xrightarrow[\simeq]{\lambda} \boxtimes^{i\in n} \otimes^{j\in t(i+1)} \otimes^{t^{-1}_{n\geqslant i+1}(j)} B \\
&\xrightarrow{\boxtimes^{i\in n}\otimes^{j\in t(i+1)}\mu^{t^{-1}_{n\geqslant i+1}(j)}} \boxtimes^{i\in n} \otimes^{t(i+1)} B.
\end{aligned}$$

Proof. For a morphism

$$\mathrm{Tree}_n(k) \ni t \xrightarrow{\varphi} s \in \mathrm{Tree}_m(k),$$

$\Delta^\varphi : B^\boxtimes(s) \to B^\boxtimes(t)$ is given by the composition

$$\begin{aligned}
&B^\boxtimes(s) := \boxtimes^{i\in m} \otimes^{s(i+1)} B \xrightarrow{\boxtimes^{i\in m}\otimes^{s(i+1)}\Delta^{\varphi^{-1}(i)}} \\
&\boxtimes^{i\in m} \otimes^{s(i+1)} \boxtimes^{\varphi^{-1}(i)} B \xrightarrow{\boxtimes^{i\in m}\eta^{s(i+1)\,\varphi^{-1}(i)}} \\
&\boxtimes^{i\in m} \boxtimes^{\varphi^{-1}(i)} \otimes^{t([\varphi](i+1))} B \xrightarrow{\boxtimes^{i\in m}\boxtimes^{\varphi^{-1}(i)}\lambda} \\
&\boxtimes^{i\in m} \boxtimes^{j\in\varphi^{-1}(i)} \otimes^{k\in t(i+1)} \otimes^{t^{-1}_{[\varphi](i+1)\geqslant j}(k)} B \\
&\xrightarrow{\boxtimes^{i\in m}\boxtimes^{j\in\varphi^{-1}(i)}\otimes^{k\in t(i+1)}\mu^{t^{-1}_{[\varphi](i+1)\geqslant j}(k)}} \\
&\boxtimes^{i\in m} \boxtimes^{j\in\varphi^{-1}(i)} \otimes^{t(i+1)} B \xrightarrow{\rho^\varphi} \\
&\boxtimes^{i\in n} \otimes^{t(i+1)} B =: B^\boxtimes(t).
\end{aligned}$$

The identity $\Delta^t = \Delta^\varphi \circ \Delta^s$ is verified directly. $\square$

In the cooperadic context, we suppose that the monoidal structure $\otimes$ is braded or symmetric. Then each component $B^\otimes(t) \simeq \boxtimes^{i\in n} \otimes^{t(i+1)} B$ admits a natural algebra structure.

Proposition 4.7. *Under the above assumptions, the operadic coproducts Δ^t are morphisms of this al-*

gebra, i.e., the following diagram commutes:

$$\begin{tikzcd}
\otimes^\ell B^{\otimes}(k) \arrow[r, "\mu^\ell_{B^{\otimes}(k)}"] \arrow[d, "\otimes^\ell \Delta^t_{B^{\otimes}}"'] & B^{\otimes}(k) \arrow[d, "\Delta^t_{B^{\otimes}}"] \\
\otimes^\ell B^{\otimes}(t) \arrow[r, "\mu^\ell_{B^{\otimes}(t)}"] & B^{\otimes}(t).
\end{tikzcd}$$

Proof. Δ^t is a composition of two algebra morphisms, coproduct $\Delta^n_{\otimes^k B}$

$$\otimes^k B \xrightarrow{\otimes^k \Delta^n_B} \otimes^k \boxtimes^n B \xrightarrow{\eta^{kn}} \boxtimes^n \otimes^k B,$$

and

$$\boxtimes^n \otimes^k B \xrightarrow[\simeq]{\lambda} \boxtimes^{i\in n} \otimes^{j\in t(i+1)} \otimes^{t^{-1}_{n\geqslant i+1}(j)} B$$

$$\xrightarrow{\boxtimes^{i\in n} \otimes^{j\in t(i+1)} \mu^{t^{-1}_{n\geqslant i+1}(j)}} \boxtimes^{i\in n} \otimes^{t(i+1)} B. \qquad \square$$

5. *q*-Complexes and Hypersimplices

The (strict) action of a group G on a category $\mathcal{C}$ is a (strict) monoidal functor $G \to \mathrm{End}(\mathcal{C})$.

The *translation structure* is an action of the free group $(\mathbb{Z}, +)$:

$$X \mapsto X[n], \quad n \in \mathbb{Z}.$$

The strict translation structure is just an invertible endofunctor

$$\Sigma : \mathcal{C} \to \mathcal{C}, \quad X[n] = \Sigma^n X.$$

The examples of categories with strict translation structure are:

– the categories of graded modules $\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$ and complexes $C(\mathcal{A})$ with shift

$$X[n]_m := X_{n+m};$$

– partially ordered sets (posets) equipped with automorphism.

The definition of triangulated category by Verdier (1963) inspired by Grothendieck and the close definition of Dold and Puppe (1961) assume the following:

– A category $\mathcal{T}$ with (strict) translation structure. For any such category, one can consider the category $\mathrm{Tri}(\mathcal{T})$ of triangles

$$X \to Y \to Z \to X[1] \qquad \begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
& Z \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

– A replete subcategory of distinguished triangles such that the restriction of a triangle to its base $X \to Y$ is a full functor.

– (Octahedron axiom) Each commutative triangle is a part of octahedra, whose 8 faces are ether commutative or distinguished triangles (in chess order).

For each Abelian category $\mathcal{A}$ (say a category of modules), there are the triangulated categories:

– Category $K(\mathcal{A})$ of complexes up-to-homotopy:

$$f \sim g \;\Leftrightarrow\; f - g = d(h) = [d, h] = dh + hd,$$
$$\deg h = -1,$$

– Derived category $D(\mathcal{A})$ (which is a localization $K(\mathcal{A})\big[\mathrm{qiso}^{-1}\big]$ of the above homotopical category by quasiisomorphisms).

In both cases, a triangle is distinguished if it is isomorphic (in $\mathcal{T}$) to the principal one

$$X \xrightarrow{f} Y \to C(f) \to X[1],$$

where $C(f) := \left(X[1] \oplus Y, \begin{pmatrix} d_{X[1]} & 0 \\ f & d_Y \end{pmatrix}\right)$ is a *cone* of the morphism.

The triangulated categories have a little defect. The cone construction is a functor in the category of complexes, but is not a functor in the homotopical and derived categories. The octahedron axiom improves the non-functoriality of the cone. Is there a better improvement? Omitting a long history, we mention the definition of Maltsiniotis [9].

The *hypersimplex* $\Delta_{n,k}$ is a convex hull of barycenters of $\binom{n+1}{k+1}$ k-dimensional faces of an n-dimensional simplex Δ_n. A triangle and an octahedron are the first members of the sequence $\Delta_{n,1}$ of hypersimplices (with $k = 1$).

For a category $\mathcal{T}$ with translation structure, one can define an n-triangle as a translation preserving the functor $\Bbbk\overline{\Delta}_{n,1} \to \mathcal{T}$, where $\overline{\Delta}_{n,1}$ is a poset with the translation "unfolding" a hypersimplex $\Delta_{n,1} = \overline{\Delta}_{n,1}/\mathbb{Z}$.

The *strongly triangulated category* is a category $\mathcal{T}$ with translation structure equipped with a "coherent family" of the replete subcategories of distinguished n-triangles for each $n \geqslant 0$ such that the restrictions of triangles to the base are full functors.

A strongly triangulated category is triangulated in the sense of Verdier. There is the (artificial) example of a triangulated category, which is not strongly triangulated. On the other hand, one can expect that the naturally obtained triangulated categories should be strongly triangulated.

In particular, the cohomological category $H^0(\mathcal{C})$ for a pre-triangulated A_∞-category $\mathcal{C}$ is strongly triangulated [2].

In the rest of the paper, we consider a "q-analog" of the strongly triangulated category, where $q = -1$ is replaced by the N-th primitive root of unity.

5.1. Unfolding of hypersimplices $\boldsymbol{\Delta}_{n,k}$

By $\overline{\Delta}_n$, we denote the set $\mathbb{Z}$ of integers with the usual linear order and translation

$$r[q] = r + q(n+1).$$

One can imagine $\overline{\Delta}_n$ as a linear ordered sum of $\mathbb{Z}$ copies of the interval $[n] = \{0 < 1 < ... < n\}$:

$$... < n[-1] < 0[0] < 1[0] < ... < n[0] < 0[1] < ...$$

Let $\overline{\boldsymbol{\Delta}}$ be a category with objects $\overline{\Delta}_n$, $n \geqslant 0$, and monotone translation preserving maps $\overline{\Delta}_m \to \overline{\Delta}_n$.

The famous category Δ consists of linear ordered sets $[n] = \{0 < 1 < ... < n\}$ and monotone maps.

There is a faithful functor $\Delta \hookrightarrow \overline{\boldsymbol{\Delta}}$

$$\big(\psi : [m] \to [n]\big) \mapsto \big(\overline{\psi} : \overline{\Delta}_m \to \overline{\Delta}_n\big),$$
$$\overline{\psi}(r[q]) = \psi(r)[q], \quad r \in [m].$$

For $n \geqslant 0$, let $\tau : \overline{\Delta}_n \to \overline{\Delta}_n$ be an automorphism $i \mapsto i + 1$.

Proposition 5.1. *Any morphism φ in $\overline{\boldsymbol{\Delta}}$ admits the unique factorization $\overline{\psi} \circ \tau^k$, where ψ is a morphism in Δ, and $k \in \mathbb{Z}$.*

Hence, the category $\overline{\boldsymbol{\Delta}}$ is a realization of Loday's crossed simplicial group $\Delta\mathbb{Z}$ with $\mathrm{Aut}_{\Delta\mathbb{Z}}(\overline{\Delta}_n) \simeq \mathbb{Z}$.

The crossed simplicial group ΔG is an extension of the category Δ by automorphism groups $G_n = \mathrm{Aut}_{\Delta G}([n])$. The famous cyclic category Λ is a factor-category of $\overline{\boldsymbol{\Delta}}$ by relations $\tau^{n+1} = 1$ for $\tau : \overline{\Delta}_n \to \overline{\Delta}_n$

A geometric realization of the corresponding simplicial set is $|\Delta\mathbb{Z}| = \mathbb{R}$. This is related to the Poincare construction of real numbers via slopes.

Let $\overline{\Delta}_{n,m}$ be the set of morphisms (preserving the order and the translation)

$$x : \overline{\Delta}_m \to \overline{\Delta}_n, \quad i \mapsto x_i,$$

equipped with pointwise order and translation structure

$$x[n]_i = x_{n+i}.$$

Such morphism is determined by its segment $(x_0, ..., x_m) \in \mathbb{Z}^{m+1}$. Thus, $\overline{\Delta}_{n,m}$ can be identified with a subposet in $\mathbb{Z}^{m+1}$ determined by the conditions

$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_m \leqslant x_0 + n + 1$$

with the translation

$$(x_0, x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_m, x_0 + n + 1).$$

One can identify $\overline{\Delta}_n$ with $\overline{\Delta}_{n,0}$.

The interior $\overset{\circ}{\overline{\Delta}}_{n,m}$ is a subset of strictly monotone maps $x : \overline{\Delta}_m \to \overline{\Delta}_n$, i.e.,

$$x_0 < x_1 < \cdots < x_m < x_0 + n + 1.$$

A geometric simplex Δ_n is the intersection of a hyperplane $z_0 + ... + z_n = m + 1$ in $\mathbb{R}^{n+1}$ with an orthant $z_i \geqslant 0$, $0 \leqslant i \leqslant n + 1$. We index the basis vectors e_i of $\mathbb{R}^{n+1}$ by elements $i \in \mathbb{Z}_{n+1}$. Denote, by $\overline{k} \in \mathbb{Z}_{n+1}$, the remainder of $k \in \mathbb{Z}$. Then, to each $x \in \overset{\circ}{\overline{\Delta}}_{n,m}$, one can assign a vertex of the hypersimplex $\sum_{i \in [m]} e_{\overline{x}_i} \in \Delta_{n,m}$.

5.2. Distinguished (n, m)-triangles

Let $\mathcal{T}$ be a category with zero object and translation structure. An (n, m)-triangle in $\mathcal{T}$ is a translation preserving the functor

$$F : \overline{\Delta}_{n,m} \to \mathcal{T}$$

such that $F(x) = 0$ for $x \in \partial\overline{\Delta}_{n,m} = \overline{\Delta}_{n,m} \backslash \overset{\circ}{\overline{\Delta}}_{n,m}$. Denote, by $\mathrm{Tri}_{n,m}(\mathcal{T})$, the category of (n, m)-triangles.

Let $\overline{\boldsymbol{\Delta}}_m$ be a category with objects $\overline{\Delta}_{n,m}$ for $n \geqslant 0$, and let the morphisms be generated by

(i) postcompositions with morphism in $\overline{\boldsymbol{\Delta}}$,

(ii) precompositions with $\tau^k : \overline{\Delta}_m \to \overline{\Delta}_m$, $k \in \mathbb{Z}$.

Let q be the primitive $(m + 1)$-th root of unity.

The categories of triangles form Cat-presheave on $\overline{\boldsymbol{\Delta}}_m$:

$$\overline{\boldsymbol{\Delta}}_m^{\mathrm{op}} \to \mathrm{Cat}, \quad \overline{\Delta}_{n,m} \mapsto \mathrm{Tri}_{n,m}(\mathcal{T}).$$

For $h : \overline{\Delta}_{n,m} \to \overline{\Delta}_{n',m}$, the inverse image functor

$$h^* : \mathrm{Tri}_{n',m}(\mathcal{T}) \to \mathrm{Tri}_{n,m}(\mathcal{T})$$

is a precomposition with h and with a translation "twisted" by q.

We consider the following q-analog of Maltsiniotis' axioms for a family of distinguished triangles

$$\mathrm{Tri}^{\checkmark}_{n,m}(\mathcal{T}) \hookrightarrow \mathrm{Tri}_{n,m}(\mathcal{T}), \quad n \geqslant 0:$$

TR0: Every $\mathrm{Tri}^{\checkmark}_{n,m}(\mathcal{T})$ is a replete subcategory in $\mathrm{Tri}_{n,m}(\mathcal{T})$.

TR3 (coherence): For each morphism h in $\overline{\mathbf{\Delta}}_m$, the inverse image of the distinguished triangle is a distinguished triangle. In other words, we have a sub-presheave of distinguished triangles.

TR1-2: The restriction of a triangle to the base is a full functor.

We recall ([6]) that a morphism of N-complexes $f : X \to Y$ is *null-homotopic* if it lies in the image of the operator d^{N-1} in $\underline{\mathrm{Hom}}(X, Y)$.

Theorem 5.2. *The category $K_{N,q}(\mathcal{A})$ of N-complexes up-to-homotopy admits a family of distinguished triangles satisfying the above axioms.*

A q-analog of the cone construction $C = C(A)$, for $x : \overline{\mathbf{N}} \to \Delta_n$ in $\mathbf{\Delta}$ and for a translation preserving $A : \overline{\mathbf{N}} \to \mathcal{T}$, is as follows:

$$C_x = \bigoplus_{i\in\mathbf{N}} A_{x_i}[N-i-1], \quad d_{x,ij} = \begin{cases} 0, & x_i > x_j, \\ q^{x_i} d_{x_i}, & x_i = x_j, \\ a_{x_i x_j}, & x_i < x_j. \end{cases}$$

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ВІД БІАЛГЕБР ДО ОПЕРАД.
КВАНТОВА ПРЯМА ТА КООПЕРАДА
КОРЕЛЯЦІЙНИХ ФУНКЦІЙ

Р е з ю м е

q-пряма – простий приклад заплетеної алгебри Гопфа. Це алгебра поліномів $\Bbbk_q[z]$ з примітивним генератором та q-деформованою статистикою.

(Ко)дія q-прямої на алгебрі – це q-диференціювання. Ми будуємо операду та кооперaду на основі біалгебри. У випадку q-прямої ця конструкція пов'язана з коопeрадою кореляційних функцій І. Кріза та співавторів, яка описує вертексні алгебри.

Модулі над фактор-алгеброю $\Bbbk_q[z]/(z^N)$ – це N-комплекси. Ми розглядаємо гомотопічну категорію N-комплексів як приклад q-аналога сильно триангульованої категорії Мальциніотіса.

Загальні конструкції розглядаються в контексті ітерованих моноідальних категорій з розслабленими тензорними добутками, що описуються в термінах тензорного добутку Грея категорних операд Tree дерев з рівнем.