A.S. SIZHUK, S.M. YEZHOV

Taras Shevchenko National University of Kyiv, Faculty of Physics
(2, Prosp. Academician Glushkov, Kyiv 03022, Ukraine; e-mail: cannabiss@mail.univ.kiev.ua)

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ONE-PHOTON SCATTERING BY AN ATOMIC CHAIN IN ONE- AND TWO-MODE RESONATORS

A chain of N identical two-level atoms coupled with the electromagnetic field, prepared via a single-photon Fock state, is investigated. It is found that, if the interaction between atoms is negligible, than the obtained dynamic equations for the probability amplitudes allow, in a certain sense, an interpretation of the dynamics of states in the classical fashion in terms of a superposition of oscillatory modes of the system under study. The derived equations reveal how a space configuration of the system of atoms affects the dynamics of the atomic states, particularly the "decay" rates of separate atoms and the system as the whole.

K e y w o r d s: one-photon scattering, atomic system, Fock state, Schrödinger equation.

1. Introduction

The collective radiative effects in an atomic system were under intense investigation since the 1950th. Taking the quantization of an electromagnetic field into account, many fundamental and interesting properties of the coupled systems of atoms and a field were revealed in numerous theoretical and experimental works. For example, the collective spontaneous emission from a cloud of N atoms was discovered in [1]. It was shown that if the excited atoms are placed in a volume so that the average distances between atoms are less than the "resonant transition" wavelength of emitted light (the so-called superradiance), than the system can irradiate into the field significantly faster than an isolated atom. The existing correlations between atoms at such distances causes the coherent atomic transitions. The latter is the reason for the emitted light intensity to be proportional to N^2 (the so-called superradiation). As a result, the system radiates "its energy N times faster than a single atom" (see, e.g., [2]). Such property in some sense is the basis for an alternative to lasers' applications. This particular behavior at the spontaneous emission is inherent to the system even if only one atom or one photon state is excited initially in the "atomic cloud-field" system (see, e.g., [3]). Under some conditions, besides the mentioned collective effects, the atomic system can slow down the re-emission as a whole that was explained by the multiphoton exchange mechanism (virtual effects in [4]). A possible application of the specifically prepared atomic states is, for example, the optical quantum-state storage (see [5]).

In the present paper, we study the system of N identical two-level noninteracting atoms coupled with the electromagnetic field initially prepared in a single-photon Fock state. The main goal of the paper is to obtain the information about the temporal behavior of the states of the electromagnetic field and atoms.

The results of our work differ from those in works [6, 7], and [8], where the photon scattering by a large spatially homogeneous system with uniform spatial atomic density coupled to the continuum of modes of a quantized electromagnetic field was discussed. In contrast to those papers, we consider a resonator that limits the number of allowed modes in a defined direction. But, the suggested here model of decay for system's states is also associated with the conventional method of transition to the continuous set of modes of the quantized electromagnetic field in all "other available" (also continuous) directions. At that point, we will demonstrate the some difficulties related to applying the certain approximations. Particularly, we will show that the conventional "relaxational" approximation, which is usually associated with the "thermalization" in the atoms-field system, cannot be directly derived in doing the transition to the continuum. So, the question about the mechanism of state decay remains open. For instance, it can happen that the system has to be discussed as open, which results in the necessity of taking a statistical distribution of the elec-

tromagnetic modes into account. The latter can become the cause of the relaxation exponential in time (see below).

By the aid of the decay parameter D introduced in a certain approximation in Section 4 for the obtained general formalism in Section 2, we made an attempt to reach such form of dynamic equations for the probability amplitudes, that allows us to interpret the system of equations obtained Section 3 in the classical fashion as that describing the oscillatory motion of a multimode system of particles.

We note here that the approaches to realistic systems, including damping effects, were discussed, for example, by M. Lax in [9]. He presented a model with an atoms-reservoir coupling inducing a decrease in the population of excited atomic levels and the corresponding field states. As is cited in [10], the model was studied by several authors, mainly by M. Lax (see [9]). In this model, it is assumed that there is a single important radiation field mode described by an annihilation operator in the reservoir. It is assumed that there are $N \gg 1$ three-level atoms described by their own annihilation operators. These atoms interact with the field, but not with one another. Both the field and atoms are affected by individual "reservoirs" simulating the effects of other radiation modes, phonons in the cavity walls, the pump, *etc.* In this model, a decrease in the population (decay) was introduced by the specific terms corresponding to reservoir-induced transitions. The earlier literature "pertaining to more realistic laser models" (see [10], pp. 236-237) may also be traced from [11].

Some fundamental aspects of the decay processes of a single atom are discussed in [12]. There, the description is performed in terms of the probability density matrix for a rarefied atomic beam on the time scale such that only one atom interacts with the cavity. It was shown that the decay processes (atomic emission) can be due to a certain disbalance between the populations of atoms in the excited (or ground) state and the field Fock states. Especially, the decay conditions arise in the case of a thermal non-equilibrium distribution among the states in the cavity. In addition, the available continuous set of modes can cause frequency shifts.

Within an application, if we consider a superradiating system, the cause of the relaxation (decay) of coherent states can, along with the thermalization of the system states, be the Doppler broadening of the atomic (electronic) resonant frequencies. Some examples of the real systems and the analysis of the application of the theory to the "superradiation" effects are presented in [13]. As a particular case, the cyclotron waves in a plasma are described in the mentioned work to demonstrate a possible application to the physics of a magnetized plasma in tokamak systems.

It is worth to note that the difference between the terminologies used in this paper and in [14] reflects our attention to the dynamics of the system, that is an intermediate step on the way to the kinetics. In more details, the application of the mentioned methodology of the exclusion of boson operators can be found in [15]. For comparison with the results of the kinetic method, the dynamic system of two atoms coupled with electromagnetic field was described in [16].

2. The Equation of Motion for the State Amplitudes

Let us consider a collection of N identical atoms at the positions $\mathbf{r}_1, \mathbf{r}_2, ..., r_\alpha, ..., \mathbf{r}_N$, coupled with a bath of electromagnetic field modes. Each atom $\alpha = 1, ..., N$ is assumed to have only two states $|a\rangle_\alpha$ and $|b\rangle_\alpha$ separated by the energy $E_\alpha = E_{a\alpha} - E_{b\alpha} = \hbar\omega$. Using the dipole approximation, one can write the following Hamiltonian (relative to the ground-point atom energy and field energy):

$$H = H_0 + H_{\rm int} = \hbar\omega \sum_{\alpha=1}^{N} \sigma_\alpha^\dagger \sigma_\alpha + \\ + \sum_{j=1}^{2} \sum_{\mathbf{k}} \hbar\nu_k \hat{a}_{\mathbf{k},j}^\dagger \hat{a}_{\mathbf{k},j} - \sum_{\alpha=1}^{N} \hat{\mathbf{p}}_\alpha \hat{\mathbf{E}}(\mathbf{r}_\alpha), \qquad (1)$$

where H_0 is the Hamiltonian of free atoms and the free electromagnetic field, $H_{...}$ describes the interaction of the atoms with the electromagnetic field; $\sigma_\alpha \equiv |b\rangle_\alpha\langle a|_\alpha$ is the lowering operator for the atom α, and $\hat{\mathbf{E}}(\mathbf{r}_\alpha)$ is the electric field operator evaluated at the position $\mathbf{r}_\alpha$ of the atom α; $\hat{\mathbf{p}}_\alpha$ is the transition dipole moment operator for the atom α; $\hat{a}_{\mathbf{k},j}^\dagger$ and $\hat{a}_{\mathbf{k},j}$ are, respectively, the photon creation and annihilation operators for the mode $\mathbf{k}, j$, where the index $j = 1, 2$ defines the polarization plane.

In the interaction representation, the Schrödinger equation (with $\hbar = 1$) is

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{V}_{\rm int} \Psi \qquad (2)$$

with the corresponding Hamiltonian $\hat{V}_{\rm int}$ that describes the interaction of the atoms with photons in the dipole approximation,

$$\hat{V}_{\rm int} = -\hbar \sum_{\mathbf{k},j} \sum_{\alpha=1}^{N} \Big[g_\alpha(\mathbf{k},j) \hat{\sigma}_\alpha \hat{a}^+_{\mathbf{k},j} \times$$
$$\times \exp\Big(i(\nu_{\rm k}-\omega)t - i\mathbf{k}\mathbf{r}_\alpha\Big) + g^*_\alpha(\mathbf{k},j)\hat{\sigma}^+_\alpha \hat{a}_{\mathbf{k},j} \times$$
$$\times \exp\Big(-i(\nu_{\rm k}-\omega)t + i\mathbf{k}\mathbf{r}_\alpha\Big)\Big]. \quad (3)$$

where

$$g_\alpha(\mathbf{k},j) = \sqrt{\frac{\nu_{\rm k}}{2\hbar\epsilon_0 V}}\, \wp_\alpha \cdot \mathbf{e}_{\mathbf{k},j} =$$
$$= e^{i\phi_\alpha}\sqrt{\frac{\nu_{\rm k}}{2\hbar\epsilon_0 V}} |\wp_\alpha| \cos\theta_{\mathbf{k},j}, \quad (4)$$

where ϕ_α denotes some phase, $\cos\theta_{\mathbf{k},j}$ is the angle between the dipole vector $\wp_\alpha = e\langle a|\mathbf{r}_\alpha|b\rangle$ and the j-th unit polarization vector $\mathbf{e}_{\mathbf{k},j}$ ($j=1,2$ and $\mathbf{e}_{\mathbf{k},j}\cdot\mathbf{k}=0$), and V is the normalization volume.

Since, at the initial time moment $t=0$, all atoms $\alpha = 1,...,N$ of the ensemble are in the ground state $|b\rangle_\alpha$ and the electromagnetic field is in the Fock state $|1_{\mathbf{k_0}}\rangle$ (that presents one photon with the wave vector $\mathbf{k}_0$), we look for a solution of the Schrödinger equation in the form

$$\Psi = \sum_{\alpha=1}^{N} \beta_\alpha(t)|b_1 b_2 ... a_\alpha ... b_N; 0\rangle +$$
$$+ \sum_{\mathbf{k},j} \gamma_{\mathbf{k},j}(t)|b_1 b_2 ... b_N; 1_{\mathbf{k},j}\rangle \quad (5)$$

with the initial conditions

$$\beta_\alpha(0) = 0, \quad \gamma_{\mathbf{k},j}(0) = \delta_{\mathbf{k},\mathbf{k_0}}, \quad (6)$$

where $\delta_{\mathbf{k},\mathbf{k_0}}$ is the Kronecker delta, $\delta_{\mathbf{k},\mathbf{k}_0}=1$ if $\mathbf{k}=\mathbf{k}_0$, and $\delta_{\mathbf{k},\mathbf{k}_0}=0$ if $\mathbf{k}\neq\mathbf{k}_0$, $\beta_\alpha(t)$ ($\alpha=1,...,N$) and $\gamma_{\mathbf{k},j}(t)$ ($j=1,2$) are the atomic α excited state amplitude with the others in the ground states and the excited Fock field state amplitude of the j-th polarization with the wave vector $\mathbf{k}$, respectively.

Substituting expression (5) into the corresponding evolution equation (2), we obtain the following equations for the coefficients $\beta_\alpha(t)$ and $\gamma_k(t)$:

$$\dot{\beta}_\alpha(t) = i\sum_{\mathbf{k},j} g^*_\alpha(\mathbf{k},j)\gamma_{\mathbf{k},j}(t)\exp(-i(\nu_k-\omega)t + i\mathbf{k}\mathbf{r}_\alpha), \quad (7)$$

$$\dot{\gamma}_{\mathbf{k},j}(t) = i\sum_{\delta=1}^{N} g_\delta(\mathbf{k},j)\beta_\delta(t)\exp(i(\nu_k-\omega)t - i\mathbf{k}\mathbf{r}_\delta). \quad (8)$$

Integrating Eq. (8) over time yields

$$\gamma_{\mathbf{k},j}(t) = \gamma_{\mathbf{k},j}(0) +$$
$$+ i\int_0^t dt' \sum_{\delta=1}^{N} g_\delta(\mathbf{k},j)\beta_\delta(t')\exp(i(\nu_k-\omega)t' - i\mathbf{k}\mathbf{r}_\delta). \quad (9)$$

Finally, by substituting expression (9) in (7), we obtain

$$\dot{\beta}_\alpha(t) = i\sum_{\mathbf{k},j} g_\alpha(\mathbf{k},j)\gamma_{\mathbf{k},j}(0)\exp[-i(\nu_{\rm k}-\omega)t + i\mathbf{k}\mathbf{r}_\alpha] -$$
$$- \sum_{\mathbf{k},j}\sum_{\delta=1}^{N}\int_0^t dt' g^*_\alpha(\mathbf{k},j)g_\delta(\mathbf{k},j)\beta_\delta(t') \times$$
$$\times \exp[i(\nu_{\rm k}-\omega)(t'-t) + i\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_\delta)]. \quad (10)$$

The derived equation (10) will be analyzed in a certain approximation in what follows.

3. Resonance Approximation

Here, we assume that the electromagnetic field can only be in the "resonant" Fock state $|1_{\mathbf{k}_0}\rangle$, in other words:

$$\gamma_{\mathbf{k},j}(t) = 0 \quad \text{if} \quad \nu_k \neq \omega. \quad (11)$$

So, we rewrite here the equation of motion in the case of the resonance approximation $\omega = \nu_k$. Then Eq. (10) takes the form

$$\dot{\beta}_\alpha(t) = i\sum_{j,\,\mathbf{k}} g_\alpha(\mathbf{k},j)\gamma_{\mathbf{k},j}(0)\exp(i\mathbf{k}\mathbf{r}_\alpha) -$$
$$- \sum_{j,\,|\mathbf{k}|=\mathrm{k}_0}\sum_{\delta=1}^{N}\int_0^t dt' g^*_\alpha(\mathbf{k},j)g_\delta(\mathbf{k},j)\beta_\delta(t')\exp[i\mathbf{k}(\mathbf{r}_\alpha - \mathbf{r}_\delta)]. \quad (12)$$

As the next step, we differentiate the above-obtained equation (12) with respect to the time. As a result, we find that the second derivative of an

atomic state amplitude β_α is expressed through a linear combination of the all atomic state amplitudes β_δ, $\delta = 1, ..., N$, as follows:

$$\ddot{\beta}_\alpha(t) = -\sum_{j,\,|\mathbf{k}|=\mathrm{k}_0}\sum_{\delta=1}^{N} g^*_\alpha(\mathbf{k},j)g_\delta(\mathbf{k},j)\beta_\delta(t)\times$$
$$\times\exp[i\mathbf{k}(\mathbf{r}_\alpha-\mathbf{r}_\delta)] = -\sum_{\delta=1}^{N}\Phi_{\alpha\delta}\beta_\delta(t). \quad (13)$$

Thus, we have the system of simple second-order linear differential equations with respect to the time t

$$\frac{d^2}{dt^2}\beta_\alpha(t) = -\sum_{\delta=1}^{N}\beta_\delta(t)\Phi_{\alpha\delta}, \quad (14)$$

where

$$\Phi_{\alpha\delta} = \sum_{j,\,|\mathbf{k}|=k_0} g^*_\alpha(\mathbf{k},j)g_\delta(\mathbf{k},j)\exp[i\mathbf{k}(\mathbf{r}_\alpha-\mathbf{r}_\delta)]. \quad (15)$$

Now, we will analyze the structure of the solutions $\beta_\alpha(t)$ of the system of equations (14). For that, we investigate the properties of the matrix $\Phi_{\alpha\delta}$ and the structure of the parameter ω.

The following evident property for the matrix $\Phi_{\alpha\delta}$ takes place:

$$\Phi_{\delta\alpha} = \sum_{j,\,|\mathbf{k}|=k_0} g^*_\delta(\mathbf{k},j)g_\alpha(\mathbf{k},j)\exp[i\mathbf{k}(\mathbf{r}_\delta-\mathbf{r}_\alpha)] = \Phi^*_{\alpha\delta}. \quad (16)$$

So, the matrix $\Phi_{\delta\alpha}$ is a Hermitian matrix. Thus, all eigenvalues of the matrix are real. Such matrix can be diagonalized by some unitary transformation.

If the matrix $\Phi_{\delta\alpha}$ is degenerate, then we have to take the multiple roots into account. The latter means that the general solutions of the system of equations (14) can contain the corresponding powers of the time.

In the case of different eigenvalues of the matrix $(\Phi_{\alpha\delta})$, we find the particular solution of the system of equations (14) in the form

$$\beta_\alpha(t) = A_{\alpha+}e^{-iw_+t} + A_{\alpha-}e^{-iw_-t}. \quad (17)$$

Substituting (17) in the system of equations (14), we obtain the system of linear algebraic equations

$$-w^2_\pm A_{\alpha\pm} + \sum_{\delta=1}^{N}\Phi_{\alpha\delta}\,A_{\delta\pm} = 0; \quad (18)$$

or, in a matrix form,

$$(-w^2\delta_{\alpha\delta} + \Phi_{\alpha\delta})\cdot(A_\alpha) = (S_{\alpha\delta})\cdot(A_\alpha) = 0, \quad (19)$$

where $S_{\alpha\delta} = -w^2\delta_{\alpha\delta} + \Phi_{\alpha\delta}$.

It is known that a homogeneous system of equations, like (18), can has a non-trivial solution $A_{\alpha\pm} \neq 0$ only if its determinant equals zero. In the introduced notation, we have

$$\det(S_{\alpha\delta}) = \det(-w^2\delta_{\alpha\delta} + \Phi_{\alpha\delta}) = 0. \quad (20)$$

As follows from the above notation, we can make a general assumption that, even in the case of a resonator without damping, w has real and imaginary parts $w = \mathrm{Re}(w) + i\mathrm{Im}(w)$.

From the equation of motion (14) for the amplitudes β_α after the substitution of a solution in the form (18), we obtain

$$\sum_\delta(-w^2\delta_{\alpha\delta} + \Phi_{\alpha\delta})A_\delta = 0. \quad (21)$$

It follows that

$$\sum_{\alpha,\delta}(-w^2\delta_{\alpha\delta} + \Phi_{\alpha\delta})A^*_\alpha A_\delta = 0. \quad (22)$$

Therefore,

$$w^2 = \frac{\sum_{\alpha,\delta}\Phi_{\alpha\delta}A^*_\alpha A_\delta}{\sum_{\alpha,\delta}\delta_{\alpha\delta}A^*_\alpha A_\delta}. \quad (23)$$

It is easy to see that the expression

$$\sum_{\alpha,\delta}\delta_{\alpha\delta}A^*_\alpha A_\delta \quad (24)$$

is real. Therefore, since

$$\Phi_{\alpha\delta}A^*_\alpha A_\delta + \Phi_{\delta\alpha}A^*_\delta A_\alpha = \Phi_{\alpha\delta}A^*_\alpha A_\delta + (\Phi_{\alpha\delta}A^*_\alpha A_\delta)^* =$$
$$= 2\mathrm{Re}\,(\Phi_{\alpha\delta}A^*_\alpha A_\delta), \quad (25)$$

the expression

$$\sum_{\alpha,\delta}\Phi_{\alpha\delta}A^*_\alpha A_\delta = \frac{1}{2}\sum_{\alpha,\delta}(\Phi_{\alpha\delta}A^*_\alpha A_\delta + \Phi_{\delta\alpha}A_\alpha A^*_\delta) =$$
$$= \sum_{\alpha,\delta}\mathrm{Re}\,(\Phi_{\alpha\delta}A^*_\alpha A_\delta) \quad (26)$$

is a real quantity and the parameter w squared

$$w^2 = \frac{\sum\limits_{\alpha,\delta} \mathrm{Re}\left(\Phi_{\alpha\delta} A^*_\alpha A_\delta\right)}{\sum\limits_{\alpha} |A_\alpha|^2} \tag{27}$$

is real. Taking into account that

$$w^2 = \mathrm{Re}^2(w) - \mathrm{Im}^2(w) + 2i\mathrm{Re}(w)\mathrm{Im}(w), \tag{28}$$

we conclude that (27) is true (w^2 is real) if $\mathrm{Re}(w) = 0$ or $\mathrm{Im}(w) = 0$, or $\mathrm{Re}(w) = \mathrm{Im}(w) = 0$.

It is of interest that the choice of $w^2 < 0$ corresponds formally to $\mathrm{Re}(w) = 0$. Hence, we can describe the exponential decay of a "vibration" mode contributing to the state amplitude $\beta_\alpha(t)$. Here, we should say that, from representation (17), the state amplitude $\beta_\alpha(t)$ for each $\alpha = 1, ..., N$ as a solution of the system of dynamic equations (14) is a superposition of the "vibration" modes with different w_m, where w_m is the m-th root of Eq. (20). However, in real applications described in the literature, the decay of a state is still referred to an atomic state for the α-th atom or to some linear combination of the atomic state amplitudes. It is worth to note that the proposed representation for the state amplitude reflects the space structure of the system and, thus, the corresponding correlations between particles.

4. Model of Damping Effect

The decay effect for the state amplitudes of a system can be taken into account in many different ways depending on its nature (see, e.g., [10], [9]). As an example, we describe the system of a resonator and atoms placed into the bath of quantized modes. In other words, the resonator "extracts" only a certain set of modes by amplification among all the modes (of electromagnetic field) allowed in the open system, which are denoted by $\mathbf{k}_{\mathrm{res}}$. Following the idea, we divide the sum over all modes $\sum\limits_{\mathbf{k},j}$ in Eq. (10) into two parts $\sum\limits_{\mathbf{k}=\mathbf{k}_{\mathrm{res}},j}$ and $\sum\limits_{\mathbf{k}\neq\mathbf{k}_{\mathrm{res}},j}$ corresponding to the resonant modes and all non-resonant modes, respectively. Then the total sum over $\mathbf{k}$ can be approximated in the "open" space with a quite large volume V by the integration over some distribution of modes. We have

$$\frac{1}{V}\sum_{\mathbf{k},j} = \frac{1}{V}\sum_{\mathbf{k}=\mathbf{k}_{\mathrm{res}},j} + \frac{1}{V}\sum_{\mathbf{k}\neq\mathbf{k}_{\mathrm{res}},j} \rightarrow$$

$$\rightarrow \frac{1}{V}\sum_{\mathbf{k}=\mathbf{k}_{\mathrm{res}},j} + \lim_{\omega_M\to\infty}\sum_j \left(\frac{1}{2\pi\ c}\right)^3 \int\limits_0^{\omega_M} \omega^2 d\omega \int d\hat{\mathbf{k}}. \tag{29}$$

Then, the second term on the right-hand side of Eq. (10) can be split into two terms, one of which corresponds to the non-resonant modes.

Therefore, let us study the second term on the right-hand side of Eq. (10) after applying the splitting procedure (29). First, we discuss the integral over the non-resonant set of modes in the case of equal atomic indices $\alpha = \delta$:

$$V\sum_j \left(\frac{1}{2\pi c}\right)^3 \int\limits_0^{\omega_M} \omega^2 d\omega \int d\hat{\mathbf{k}}\times$$
$$\times \int\limits_0^t dt' \left|g_\delta(\mathbf{k}, j)\right|^2 \beta_\delta(t') \exp[i(\omega - \omega_{\mathrm{res}})(t' - t)]. \tag{30}$$

In the certain approximation (e.g., in the Weisskopf–Wigner approximation), we rewrite the last expression (30) in the form

$$\sum_{j=1,2} \frac{1}{2\hbar\varepsilon_0}\left(\frac{1}{2\pi c}\right)^3 |\wp|^2 \omega^3_{\mathrm{res}} \int\limits_0^{\omega_M} d\omega \int d\hat{\mathbf{k}}\times$$
$$\times \int\limits_0^t dt' \cos^2\theta_j \beta_\delta(t') \exp[i(\omega - \omega_{\mathrm{res}})(t' - t)]. \tag{31}$$

Here, we used the relation $|\mathbf{k}| = \omega_k/c$ and

$$g^*_\alpha(\mathbf{k}, j) g_\alpha(\mathbf{k}, j) = \frac{\omega}{2\hbar\epsilon_0 V} |\wp|^2 \cos^2\theta_j, \tag{32}$$

where θ_j is the angle between the atomic dipole moment $\wp$ and the electric field polarization vector $\mathbf{e}_j$, $j = 1, 2$. Note that the integration over frequencies includes the resonant frequency in the range of integration because, as is known, a definite integral does not depend on a value of an integrand at one point. The explanations and comments for the approximation used here are given below in the following subsection.

4.1. Some remarks about the Weisskopf–Wigner approximation

In the Weisskopf–Wigner approximation (see [8], pp. 206–209), we replace ω^3 in the integrand of expression (30) by ω^3_{res} and the lower limit of the integration over

the frequency ω by $-\infty$. Then, after the integration over the spatial angle $d\mathbf{k}$ in the equation, we represent the integral over frequency by the delta-function to express the approximation for a quite long time intervals:

$$\int\limits_{-\infty}^{\infty} d\omega \exp[i(\omega-\omega_{\rm res})(t'-t)] = 2\pi\delta\left(t'-t\right). \qquad (33)$$

Usually, this approach is argued by the the fact that (see [8], p. 207), "in the emission spectrum, the intensity of light associated with the emitted radiation is going to be centered about the atomic transition frequency $\omega_{\rm res}$." Respectively, the quantity ω^3 varies slightly around the resonance frequency (here, $\omega^3_{\rm res}$) during the quite long time interval.

At the same time, the integration in expression (31) over frequencies (and over spatial angles in the case where $\mathbf{r}_\alpha = \mathbf{r}_\delta$ in Eq. (10)) without imposing any additional conditions such like that mentioned above does not reveal any delta-functional properties for the following time integration:

$$\lim_{\omega_M\to\infty}\int\limits_0^{\omega_M}\omega^3\exp[i\omega(t'-t)]d\omega = \lim_{\omega_M\to\infty}\frac{1}{i(t'-t)}\times$$

$$\times\left[\omega_M^3\exp[i\omega_M(t'-t)] - 3\frac{1}{i(t'-t)}\times\right.$$

$$\times\left\{\omega_M^2\exp[i\omega_M(t'-t)] - \frac{2}{i(t'-t)}\omega_M\exp[i\omega_M(t'-t)] + \right.$$

$$\left.\left.+2\left(\frac{1-\cos\left(\omega_M(t'-t)\right)}{(t'-t)^2} - i\frac{\sin\left(\omega_M(t'-t)\right)}{(t'-t)^2}\right)\right\}\right]. \qquad (34)$$

We see that the integral also yields terms that can be responsible for a shift of the resonant frequency ω_0, at which the atomic-field transitions can occur (see, for comparison, [12]).

Some redemption from the analytic difficulties could be made, by assuming that the coefficients $g_\alpha(\mathbf{k}, j)$ are distributed in space over a volume corresponding to an atomic diameter a. This characteristic size a can impose some restrictions on the minimum wavelength and, thus, the frequency ω_M. In addition, a specific assumption about the atomic susceptibility and the time scale can change the character of the frequency integration yielding the delta-functional limits (see, for comparison, [17] and [18]).

Let us continue the calculation after the digression.

Following the Weisskopf–Wigner approximation, the integration over the spatial angle is done by using the relation $\sum\limits_{j=1,2}\cos^2\theta_j = 1-\cos^2\theta_3$, where θ_3 is the angle between the wave vector $\mathbf{k}$ and the dipole moment $\wp$, so that

$$\int\limits_0^{\pi}\int\limits_0^{2\pi}\left(1-\cos^2\theta\right)\sin\theta\, d\varphi d\theta = \frac{8\pi}{3}. \qquad (35)$$

Having the delta-function dependent on the time difference, we can easily perform the integration over the time. As a result, we have

$$\dot\beta_\alpha(t) = i\sum_{\mathbf{k},j} g_\alpha(\mathbf{k},j)\gamma_{\mathbf{k},j}(0)\exp[-i(\nu_k-\omega)t + i\mathbf{k}\mathbf{r}_\alpha] -$$

$$-\frac{8\pi}{3}\frac{2\pi}{2\hbar\varepsilon_0}\left(\frac{1}{2\pi\, c}\right)^3|\wp|^2\omega^3_{\rm res}\beta_\alpha(t) -$$

$$-\sum_{|\mathbf{k}|=\omega_{\rm res}/c,j}\int\limits_0^t dt'\left|g_\alpha(\mathbf{k},j)\right|^2\beta_\alpha(t') -$$

$$-\sum_{\mathbf{k},j}\sum_{\delta=1,\delta\neq\alpha}^{N}\int\limits_0^t dt' g^*_\alpha(\mathbf{k},j)g_\delta(\mathbf{k},j)\beta_\delta(t')\times$$

$$\times\exp[i(\omega_k-\omega_{\rm res})(t'-t) + i\mathbf{k}(\mathbf{r}_\alpha-\mathbf{r}_\delta)]. \qquad (36)$$

Again, splitting the third term on the right-hand side of the last equation into "resonant" and "non-resonant" parts, we can introduce the "non-resonant" terms for each $\delta\neq\alpha$ through the multiplication of the corresponding coefficient $D_{\alpha\,\delta}$ and the δ-th state amplitude, having the frequency and time integrations approximated by the time integration with a delta-functional core. The more detailed procedure of calculation in this case can be found, as an example, in the recent work [18]. In the related "resonant" part, only the time integration is retained. Therefore, taking this and the initial condition (6) into account and differentiating Eq. (36) one more time, we derive the equations

$$\frac{d^2}{dt^2}\beta_\alpha(t) = -\sum_{\delta=1}^{N}\beta_\delta(t)\Phi_{\alpha\delta} - 2\sum_{\delta=1}^{N}D_{\alpha\delta}\frac{d}{dt}\beta_\delta(t), \quad (37)$$

where

$$D_{\alpha\,\alpha} = \frac{1}{2}\frac{8\pi}{3}\frac{2\pi}{2\hbar\varepsilon_0}\left(\frac{1}{2\pi c}\right)^3 |\wp|^2\omega_{\rm res}^3 =$$

$$= \frac{1}{2}\frac{1}{3\pi\hbar\varepsilon_0 c^3}|\wp|^2\omega_{\rm res}^3, \qquad (38)$$

and $\Phi_{\alpha\delta}$ are defined in the previous section. Here, the coefficients $D_{\alpha\alpha}$, $\alpha = 1, ..., N$, describe the respective rate of single-atom decay for the α-th atom excited state. Since the atomic transition dipole moment is scaled in debyes, such single-atom decay usually occurs in the optical region through a spontaneous emission during 10^{-7}–10^{-9} s. The coefficients $D_{\alpha\delta}$, $\alpha = 1, ..., N$ and $\delta = 1, ..., N$, with $\alpha \neq \delta$ describe the collective decay rates for each α-th atom in the excited state and can be represented in the fashion given in Appendix in work [18].

We can see from (37) that the equations of motion for the state amplitudes $\beta_\alpha(t)$, $\alpha = 1, ..., N$ have form similar to that describing the motion of coupled classical oscillators.

5. Conclusion

Thus, we have investigated the system of N identical two-level noninteracting atoms coupled with the electromagnetic field initially prepared via a single-photon Fock state. The obtained form of dynamic equations for the probability amplitudes allows, in a certain sense, an interpretation of the dynamics of states in the classical fashion as the motion of a multimode vibrating system of particles. The "multimode vibrating system" arises, because an atomic spatial configuration defines the excitation ability of an atom in respect to the whole system. Respectively, the space configuration of an atomic chain preset the "decay" rates of the states of the system.

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А.С. Сіжук, С.М. Єжов

ОДНОФОТОННЕ
РОЗСІЮВАННЯ АТОМНИМ ЛАНЦЮГОМ
В ОДНО- ТА ДВОМОДОВОМУ РЕЗОНАТОРАХ

Р е з ю м е

Досліджено систему N ідентичних дворівневих невзаємодіючих атомів, що приготовлена в однофотонному фоківському стані електромагнітного поля. Показано, що при нехтуванні взаємодією між атомами отримані динамічні рівняння для амплітуд ймовірностей дозволяють інтерпретацію динаміки станів у класичному стилі, а саме, в термінах суперпозиції коливальних мод досліджуваної системи. Отримані рівняння розкривають вплив відносного просторового розташування атомів на динаміку станів, зокрема розпад як окремих атомів, так і системи в цілому.