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QUANTUM TELEPORTATION OF ENTANGLED STATES IN THE PRESENCE OF NOISE

On the basis of a system of four qubits, the influence of white and colored noises in the states of initially prepared entangled qubit pairs on the final state obtained as a result of the entanglement swapping has been considered. The corresponding density matrices are obtained, and the redistribution of fractions for the pure state and white and colored noises is analyzed. Conditions for the entanglement preservation and destruction in the course of the transition from the initial to the final state are determined. A comparison between the von Neumann entropy for the initial and final states of qubits is carried out.

Keywords: entanglement swapping, teleportation, density matrix, von Neumann entropy

1. Introduction

After von Neumann completed the development of his concept of quantum-mechanical description of processes in nature in 1932, Einstein, Podolsky, and Rosen (EPR) [1], as well as Schrödinger [2], were the firsts who attracted attention to a strange phenomenon in the quantum-mechanical theory. Namely, there may exist quantum-mechanical states of composite systems, the wave functions of which cannot be presented as a product of wave functions of separate subsystems. Later, such states were started to call entangled states, and the phenomenon itself was called entanglement. The very possibility for entangled states to exist follows immediately from the fundamental principle of superposition in the quantum-mechanical theory.

In 1964, Bell [3] adopted the basic principles of work [1] as a working hypothesis and formalized Einstein's global deterministic idea in terms of the so-called Local Hidden-Variable Theory (LHVT). Bell showed that the principles, on which this model is based, give rise unavoidably to certain restrictions on statistical correlations in two-particle systems. The mathematical formulation of such restrictions was presented in the form of Bell's inequalities, which are well-known now. For any LHVT model, Bell's inequalities are strictly satisfied. At the same time, the corresponding calculations of correlations for entangled states following the quantum-mechanical rules violate them. In other words, there are such correlations between the quantities in two subsystems in entangled quantum-mechanical states that cannot be reproduced in the framework of any local hidden-variable model.

Later on, it became clear that the entanglement is one of nature's resources, which can be created, stored, distributed, accumulated, and transferred over a distance. Nowadays, the entanglement is really created and processed in many physical laboratories. The nature of physical carriers for entangled states is very diverse; these can be photons, ions, atoms, molecules, crystals, and so forth. The elucidation of the role of entangled states promoted the appearance of new directions in modern physics, such as quantum calculations, quantum cryptography, and quantum communications.

During a certain period, it was considered that an entangled state of a system composed of two and more quantum-mechanical subsystems (particles, photons) can be created only if the latter emerge either simultaneously from the same source in the course of the same process or as a result of the interaction between them. However, in 1993, Zukowski *et al.* [4] proposed a protocol, in which quantum correlations emerge between particles located at a large distance from one another and possessing no common prehistory. This way to form the entanglement was called the "entanglement swapping", which can be interpreted as the "entanglement exchange", "entanglement switching", or "entanglement transfer". In essence, it is a teleportation of the entanglement over a distance. In what follows, we shall conditionally refer to this phenomenon as the process of entanglement swapping (ES).

The paper by Zukowski *et al.* [4] invoked a wide resonance in the scientific literature. Dur *et al.* [5] developed a protocol of quantum communication over large distances, where ES is its component. Xue *et al.* [6] proposed a scheme of entanglement swapping on the basis of a three-particle state known as the Greenberger–Horn–Zeilenger (GHZ) state,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Along with the theoretical development of ES protocols, real experiments are improved step by step at physical laboratories [7–10].

For the actual capabilities of the quantum-mechanical state transfer over large distances to be correctly estimated, all negative factors that affect the quality of a signal at all stages of its formation and transmission through communication channels must be taken into account. A detailed discussion of this subject can be found in work [11]. In our opinion, in order to study the cumulative influence of all factors, it is necessary that a complete understanding of physical processes in every separate element of the communication channel should be achieved. Entanglement swapping is a mandatory element in the transfer protocols of quantum-mechanical states over a large distance with the use of the so-called quantum repeaters. Therefore, a detailed research of this element with regard for the real process conditions is challenging, in our opinion. This work is aimed at studying theoretically the influence of noise in initially formed entangled states on the final state obtained as a result of the entanglement swapping operation.

2. Some Elements of Quantum Computer Science and the Entanglement Swapping Phenomenon

In order to explain the notation adopted in this work, we recall some elements of quantum computer science and consider the essence of entanglement swapping. Those who have a definite experience in this issue may omit this section.

The basic notion in quantum computer science is a quantum bit ("qubit"), which describes a state of a quantum-mechanical two-level system of any physical nature. To describe the state of one qubit in the Hilbert space, it is necessary to set two orthonormalized basis states, which are traditionally denoted by ket-vectors $|0\rangle$ and $|1\rangle$. For a particle with the spin $s = \frac{1}{2}$, the states with the spin projections "up", $|0\rangle \equiv |\uparrow\rangle$, and "down", $|1\rangle \equiv |\downarrow\rangle$, onto a certain axis can be selected as the basis ones. For the photon polarization, the states $|0\rangle$ and $|1\rangle$ may correspond to the horizontal and vertical polarizations, respectively. The basis $\{|0\rangle, |1\rangle\}$ is conventionally referred to as the standard or computational basis. Instead of the computational basis, one may select, e.g., the Hadamard one,

$$|\chi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

or any other one. A change between the bases is carried out with the use of a unitary transformation. The orthonormality of basis states is defined by the scalar products

$$\langle 0|1\rangle = \langle 1|0\rangle = 0, \langle 0|0\rangle = \langle 1|1\rangle = 1,$$

$$\langle \chi_1|\chi_2\rangle = \langle \chi_2|\chi_1\rangle = 0,$$

$$\langle \chi_1|\chi_1\rangle = \langle \chi_2|\chi_2\rangle = 1.$$

Hence, any state of a separate qubit can always be presented in the form

$$|\chi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where the coefficients α and β satisfy the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. The basic difference of a qubit from the classical bit consists in that the state of the former can be a superposition of basis vectors.

The state of a system composed of two qubits is described by a vector in the 4-dimensional Hilbert space, which is a tensor product of Hilbert spaces of separate qubits. The most widespread bases in that space are the computational (standard) basis in the form of a tensor product of corresponding basis states of separate qubits,

$$|0_1\rangle \cdot |0_2\rangle, \quad |0_1\rangle \cdot |1_2\rangle, \quad |1_1\rangle \cdot |0_2\rangle, \quad |1_1\rangle \cdot |1_2\rangle, \tag{1}$$

and the Bell basis,

$$\begin{cases} |\Psi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|0_1\rangle \cdot |1_2\rangle + |1_1\rangle \cdot |0_2\rangle), \\ |\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|0_1\rangle \cdot |1_2\rangle - |1_1\rangle \cdot |0_2\rangle), \\ |\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|0_1\rangle \cdot |0_2\rangle + |1_1\rangle \cdot |1_2\rangle), \\ |\Phi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|0_1\rangle \cdot |0_2\rangle - |1_1\rangle \cdot |1_2\rangle), \end{cases} \tag{2}$$

where the subscripts 1 and 2 denote the qubit number.

The Bell states are also conventionally called EPR-pairs, because just the states of this kind were dealt with in the work by Einstein, Podolsky, and Rosen. All Bell states (2) are entangled, because they cannot be presented as the products of state vectors for separate qubits.

The most interesting and unusual phenomenon that is based on the entanglement is the quantum teleportation, when, by applying Local Operations and Classical Communication (LOCC), an unknown quantum-mechanical state is transmitted over an arbitrary distance. Under local operations, we mean unitary transformations and measurements that are carried out only separately for every of the spatially separated subsystems of a common system.

For the first time, the scenario (protocol) of teleportation of an unknown qubit state was proposed by Bennett *et al.* in [12], where the entanglement of an EPR pair was used for teleportation. Bennett's article stimulated the development of numerous teleportation schemes using entanglement carriers of different origins. Among them, in particular, there are teleportation protocols that use the entanglement of a three-particle GHZ state (Karlsson *et al.* [13]) and a Wigner symmetric state (Shi *et al.* [14]),

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

An improved protocol on the basis of the Wigner asymmetric state (Li *et al.* [15])

$$|W_s\rangle = \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{2}(|010\rangle + |100\rangle)$$

allows the teleportation of the state of a separate photon to be executed with a high probability. More detailed references to the original works on the quantum teleportation can be found, e.g., in works [16, 17].

Now let us proceed to the explanation of the essence of the entanglement swapping phenomenon. Let each of two sources independently form one pair of qubits in one of the entangled EPR states (2). In the experiment [7], short pulses of ultra-violet rays were passed through a nonlinear β-BaB_2O_4 (BBO) crystal. As a result of the parametric down-conversion (PDC) of type II, there emerge photon pairs in the polarization-entangled Bell state $|\Psi^-\rangle$. If two such entangled pairs are created independently, the state vector for the system of four qubits looks like the tensor product of state vectors of separate qubit pairs,

$$|\Psi_{1234}\rangle = |\Psi^-_{12}\rangle \cdot |\Psi^-_{34}\rangle.$$

Afterward, a projective measurement in the Bell basis (2) is carried out on the pair of qubits (photons) 2 and 3. The measuring device is so tuned that the circuit of coincidence of signals from detectors registers an event when the pair of photons 2 and 3 is in the state $|\Psi^-_{23}\rangle$. This operation is called an incomplete projective measurement. Mathematically, this operation is regarded as a projection of the state $|\Psi_{1234}\rangle$ on the state $|\Psi^-_{23}\rangle$.

By regrouping the terms, the state vector $|\Psi_{1234}\rangle$ can be expressed in the form

$$|\Psi_{1234}\rangle = \frac{1}{2}\Big(|\Psi^+_{14}\rangle \cdot |\Psi^+_{23}\rangle - |\Psi^-_{14}\rangle \cdot |\Psi^-_{23}\rangle - $$

$$- |\Phi^+_{14}\rangle \cdot |\Phi^+_{23}\rangle + |\Phi^-_{14}\rangle \cdot |\Phi^-_{23}\rangle\Big).$$

Hence, after the projection operation, the state of qubits 1 and 4 collapses into the Bell state $|\Psi^-_{14}\rangle$,

$$\langle\Psi^-_{23}|\Psi_{1234}\rangle = -\frac{1}{2}|\Psi^-_{14}\rangle,$$

i.e. the pair of qubits with numbers 1 and 4 turns out in the same state $|\Psi^-\rangle$ as the previously prepared states of two pairs, (1, 2) and (3, 4).

The operation of projection on a certain state is an incomplete von Neumann measurement. Therefore, as a result, we obtain a non-normalized vector of state. In this case, if the operation of preparation of the state $|\Psi_{1234}\rangle$ followed by its projection on $|\Psi^-_{23}\rangle$ is multiply repeated, only one of four attempts, on the average, produces the desirable result, $|\Psi^-_{23}\rangle$. However, after the event has been fixed, the state vector is considered to be normalized to 1. Note once more that the previously prepared pairs of qubits were created independently of each other, and they did not interact with each other in any way.

The ES process can be illustrated in the form of a simple diagram shown in Fig. 1 [7]. It should be emphasized that the entanglement of qubits 1 and 4 arises at the time moment of the measurement, i.e. when the projection on the state $|\Psi^-_{23}\rangle$ is executed. Qubits 1 and 4 can be at any distance from each other in this case. In this example, the both qubit pairs are supposed to be prepared in the pure Bell state $|\Psi^-\rangle$, and the ES procedure yields the same pure entangled state $|\Psi^-\rangle$ of qubits 1 and 4.

Now, let us consider the ES phenomenon in the case where each of two initial qubit pairs is formed in the mixed state.

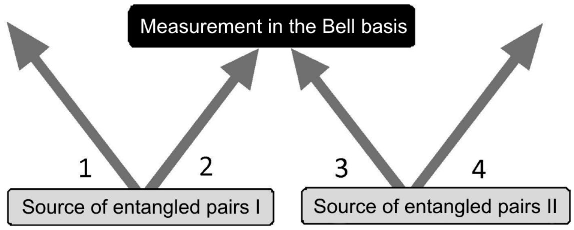

Fig. 1. Diagram of the entanglement swapping process

3. Entanglement Swapping in the Presence of White Noise in Initial States

Let the pure state $|\Psi^-\rangle$ be summed up, in the mixture, with all basis states with the same weight coefficient. The density operator for this state, which is called the Werner state [18], is presented in the form

$$\widehat{\rho}_W = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\hat{I}, \tag{3}$$

where $\hat{I}$ is the unity operator in the four-dimensional space, and p is the state purity parameter, $p \in [0,1]$. At $p = 1$, we obtain the density operator for the pure state $|\Psi^-\rangle$ and, at $p = 0$, the so-called white noise. Noise is called white, because it is formed by all basis states with the same weight coefficient. The corresponding density matrix in the computational basis looks like

$$\rho_W = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & -2p & 0 \\ 0 & -2p & 1+p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}, \tag{4}$$

whereas it is presented in the Bell basis (2) by the diagonal matrix

$$\rho_W^{\mathrm{B}} = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+3p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}. \tag{5}$$

The density operator for this state can be presented in the form of a mixture of Bell states,

$$\widehat{\rho}_W = \frac{1}{4}\Big\{(1-p)|\Psi^+\rangle\langle\Psi^+| + (1+3p)|\Psi^-\rangle\langle\Psi^-| + $$
$$+ (1-p)|\Phi^+\rangle\langle\Phi^+| + (1-p)|\Phi^-\rangle\langle\Phi^-|\Big\}.$$

Since the qubit pairs were prepared independently, the density operator for the system of two qubit pairs can be presented in the form of a tensor product of states (3),

$$\widehat{\rho}_{1234} = \widehat{\rho}_{12} \otimes \widehat{\rho}_{34}. \tag{6}$$

As a result of projection (6) on the Bell state $|\Psi^-_{23}\rangle$, we obtain the density matrix for the pair of qubits 1 and 4,

$$\widehat{\rho}_{14} = \langle\Psi^-_{23}|\widehat{\rho}_{1234}|\Psi^-_{23}\rangle. \tag{7}$$

The matrix representations for $\widehat{\rho}_{14}$ in the computational and Bell bases can be obtained with the use of Eqs. (4) and (5), respectively, and carrying out a simple substitution $p \to p_1 \cdot p_2$ in them, where p_1 and p_2 are the weight coefficients of the pure state in the first (particles 1 and 2) and second (particles 3 and 4), respectively, qubit pairs. Whence, it is evident that, if either of qubit pairs is prepared in the pure state, i.e. $p_1 = 1$ or $p_2 = 1$, the density matrix ρ_{14} exactly reproduces the density matrix of qubit pairs prepared in a mixed initial state. However, if both qubit pairs are prepared in the mixed state, the resulting state of the pair of qubits 1 and 4 is noisier than the state of each of the initially prepared pairs.

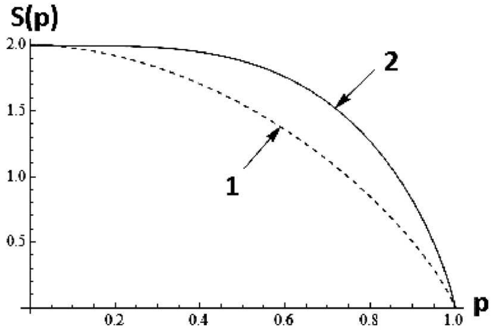

Fig. 2. Dependences of the von Neumann entropy for the state of a qubit pair on the parameter p: (*1*) for the state of each of the initially prepared qubit pairs (the Werner state) and (*2*) for the state of a qubit pair obtained as a result of the ES process

By presenting the density operator in form (3), we assume that the noisiness of a state grows, as the parameter p decreases, because, in this case, the fraction of the required pure state $|\Psi^-\rangle$ in the mixture diminishes, and the noise fraction (the coefficient $1-p$) increases.

Let us examine the case $p_1 = p_2 = p$ in detail. The density operator for the final state of qubits after the ES process termination can be presented in the form similar to Eq. (3), namely,

$$\widehat{\rho}_{14W} = p^2|\Psi^-_{14}\rangle\langle\Psi^-_{14}| + \frac{1-p^2}{4}\hat{I}_{14}. \quad (8)$$

The density matrix of the same state looks like

$$\widetilde{\rho}_{14W} = \frac{1}{4}\begin{pmatrix} 1-p^2 & 0 & 0 & 0 \\ 0 & 1+p^2 & -2p^2 & 0 \\ 0 & -2p^2 & 1+p^2 & 0 \\ 0 & 0 & 0 & 1-p^2 \end{pmatrix}, \quad (9)$$

in the computational basis and like

$$\widetilde{\rho}^{\rm B}_{14W} = \frac{1}{4}\begin{pmatrix} 1-p^2 & 0 & 0 & 0 \\ 0 & 1+3p^2 & 0 & 0 \\ 0 & 0 & 1-p^2 & 0 \\ 0 & 0 & 0 & 1-p^2 \end{pmatrix}, \quad (10)$$

in the Bell one. In other words, the density operator (8) and matrices (9) and (10) can be obtained from Eqs. (3)–(5), respectively, with the use of the simple substitution $p \to p^2$. This means that the states of the initial and final qubit pairs are similar to within the substitution indicated above. Since $p \in [0,1]$, we have that $p^2 \leq p$ and $1-p^2 \geq 1-p$ in this interval, which testifies to an increase of the white noise component fraction.

A comparison between the von Neumann entropy values for the corresponding initial and final states of qubit pairs gives an illustrative demonstration that the weight of the white noise component increased. In Fig. 2, the entropy dependences on the parameter p are plotted for both systems. The solid curve corresponds to the entropy of qubit pair 1 and 4, and the dashed one to the entropy of each initial qubit pair. The figure demonstrates that the entropy of the final qubit pair exceeds the entropy of each qubit pair in the initial state at $0 < p < 1$, which also testifies to the noise increase. The largest entropy difference $\Delta S \approx 0.44$ is observed at $p \approx 0.72$.

The reduced density matrices that correspond to matrices (4) and (9) are proportional to the identical one,

$$\rho_1^{\rm Red} = \rho_2^{\rm Red} = \frac{1}{2}I = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\widetilde{\rho}_1^{\rm Red} = \widetilde{\rho}_4^{\rm Red} = \frac{1}{2}I = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This result is completely clear, because the summation of a pure system state with white noise cannot reduce the entropy in subsystems. Since the reduced density matrices are mixtures of both basis states, the von Neumann entropy for those states does not depend on p and equals 1.

The partially transposed density matrix for the initial state of a qubit pair in the standard basis looks like

$$\rho_W^{\rm PT} = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & -2p \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+p & 0 \\ -2p & 0 & 0 & 1-p \end{pmatrix}. \quad (11)$$

For a two-qubit system, the positive definiteness of the partially transposed density matrix of this system is, according to the Peres–Horodecki criterion [19,20], a necessary and sufficient condition for the entanglement to be absent from the system. The violation of this condition will testify to the presence of an entanglement.

Let us determine the interval of values for the coefficient p, when the initial qubit pair is entangled. First, we diagonalize matrix (11) to obtain

$$\rho_{\rm diag}^{\rm PT} = \frac{1}{4}\begin{pmatrix} 1+p & 0 & 0 & 0 \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 1+p & 0 \\ 0 & 0 & 0 & 1-3p \end{pmatrix}.$$

If $p \in [0,1]$, we have $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1+p}{4} \geq 0$. The unique characteristic number that can be negative is λ_4,

$$\lambda_4 = \frac{1-3p}{4} < 0 \quad \Longrightarrow \quad p > \frac{1}{3}.$$

For the Werner state, this result is well known and was derived for the first time in work [18].

The interval of p-values, for which the state of a qubit pair formed in the course of ES is entangled, can be easily obtained in view of the similarity between the states of the initial and final pairs to within the accuracy of substitution $p \to p^2$. As a result, we have

$$p^2 > \frac{1}{3} \quad \Longrightarrow \quad p > \frac{1}{\sqrt{3}} \approx 0.58.$$

Hence, at $\frac{1}{3} < p < \frac{1}{\sqrt{3}}$, the entanglement resulting from the ES operation gets lost because of the increase of white noise admixture.

4. Entanglement Swapping in the Presence of Colored Noise in Initial States

During some time, it was conventionally considered that, when entangled states are created under laboratory conditions, a real state is a mixed one, which is described by the density operator in the form of a mixture of the required pure, as much as possible, entangled state and a white noise admixture (Werner state (3)). In 2005, Cabello *et al.* [21], having studied the preparation of polarization-entangled states for a pair of photons in the course of parametrical down-conversion (PDC), arrived at a conclusion that really prepared states cannot be colorless. For instance, for the pure Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ to be created in an experimental device, the perfect agreement must be achieved between the photons with respect to both their phases and the time of their arrival at the interference place. Under perfect conditions, photons become undistinguishable at the interference place and, as a result, there emerges an entangled state $|\Psi^-\rangle$ with a certain relative phase between states $|01\rangle$ and $|10\rangle$.

However, the perfect synchronization is impossible in a real device, so that the state $|\Psi^-\rangle$ turns out to be separately mixed with the same states $|01\rangle$ and $|10\rangle$, but without the definite phase correlation. This results in the appearance of colored noise in the real state, and the density operator of this state reads

$$\widehat{\rho}_C = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|), \qquad (12)$$

where p is the parameter characterizing the fraction of the pure state $|\Psi^-\rangle$ in the mixture. Noise in Eq. (12) is called colored, because, in contrast to white noise, it consists of a mixture of basis states with different weighting coefficients.

The authors of work [21] also showed that the violation of the Bell inequality is extremely robust with respect to a variation of the colored noise fraction in state (12). This means that state (12) violates the Bell inequality at any ratio between the pure state $|\Psi^-\rangle$ and colored noise in it (i.e. at every $0 < p \leq 1$). At the same time, the Werner state (3) does not violate the Bell inequality if the white noise fraction exceeds some critical value, i.e. the violation of the Bell inequality turns out unstable with respect to white noise.

On the basis of the experiments on correlations between the polarizations of two photons, Bovino *et al.* [22] convincingly confirmed both the presence of colored noise, the robustness of the Bell inequality violation with respect to colored noise, and its instability with respect to white noise. In work [23], conditions of the Bell inequality violation were studied, provided that noises of both types are simultaneously available in the density matrix.

Now, let us consider and analyze the final state of qubit pair 1 and 4 after the ES operation and in the presence of colored noise in the initial states of two qubit pairs. We assume that the initial state of each qubit pair in the presence of colored noise looks like expression (12). The density matrix of this state looks like

$$\rho_C = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -p & 0 \\ 0 & -p & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (13)$$

in the standard basis and like

$$\rho_C^{\mathrm{B}} = \frac{1}{2}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (14)$$

in the Bell one. Substituting the density operator of the initial qubit pairs (12) with different parameters p_1 and p_2 into expressions (6) and (7), we obtain

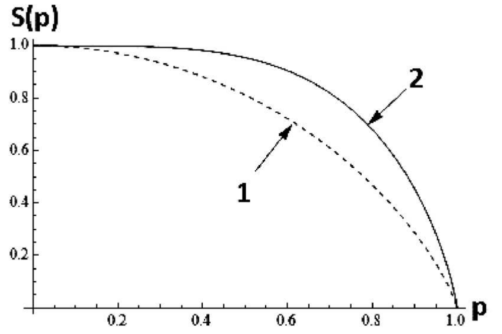

Fig. 3. Dependences of the von Neumann entropy on the parameter p in the presence of colored noise: (1) for the state of each of the initially prepared qubit pairs and (*2*) for the state of a qubit pair obtained as a result of the ES operation

$\widehat{\rho}_{14}$. The matrix representation of $\widehat{\rho}_{14}$ in the computational and Bell bases can be obtained from expressions (13) and (14), respectively, with the use of the simple substitution $p \to p_1 \cdot p_2$ in them. Similarly to the case of white noise, we see that, in the case where one of the initial qubit pairs is prepared in the pure state, i.e. $p_1 = 1$ or $p_2 = 1$, the density matrix ρ_{14} identically reproduces the density matrix of a qubit pair prepared in a mixed initial state.

Provided that $p_1 = p_2 = p$, the density operator of the final state of qubits can be presented in a form similar to Eq. (12), namely,

$$\widehat{\rho}_{14C} = p^2|\Psi^-_{14}\rangle\langle\Psi^-_{14}| + \frac{1-p^2}{2}(|0_1 1_4\rangle\langle 0_1 1_4| + |1_1 0_4\rangle\langle 1_1 0_4|).$$

The density matrix of the same state looks like

$$\widetilde{\rho}_{14C} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -p^2 & 0 \\ 0 & -p^2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (15)$$

in the computational basis and like

$$\widetilde{\rho^{\mathrm{B}}}_{14C} = \frac{1}{2}\begin{pmatrix} 1-p^2 & 0 & 0 & 0 \\ 0 & 1+p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in the Bell one. Analogously to the previous case, the states of the initial and final qubit pairs are similar to each other with an accuracy to the substitution $p \to p^2$. Since $p \in [0, 1]$, we have $p^2 \leq p$ and $1 - p^2 \geq 1 - p$ in this interval, which evidences an increase in the fraction of colored noise.

In Fig. 3, the comparison is made between the dependences of the von Neumann entropy on the parameter p for the state of each of the initially prepared qubit pairs (dashed curve) and the final state of qubit pair 1 and 4 (solid curve). One can see that the entropy of final qubit pairs exceeds that of qubit pairs in the initial state if $0 < p < 1$, which also testifies to an increase of the noise fraction. The largest difference between the entropies, $\Delta S \approx 0.21$, is observed at $p \approx 0.76$. Note also that the maximum value of entropy equals $S_{\max} = 1$ (at $p = 0$), whereas $S_{\max} = 2$ in the previous case with white noise. Such a difference is quite clear, because white noise is the most chaotic state.

The reduced density matrices corresponding to matrices (13) and (15) are proportional to the identity matrix, as it was in the previous case. Therefore, the von Neumann entropy for the state of each qubit in the pair does not depend on p and equals 1.

The partially transposed density matrix for the initial state of a qubit pair looks like

$$\rho_C^{\mathrm{PT}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p & 0 & 0 & 0 \end{pmatrix},$$

in the standard basis, and the characteristic values of this matrix are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = p$, and $\lambda_4 = -p$. The value $\lambda_4 < 0$ at $0 < p \leq 1$ testifies that each of the initially prepared qubit pairs is in the entangled state at an arbitrary fraction of colored noise. Using the similarity between the states of initial and final pairs ($p \to p^2$), we arrive at the conclusion that the pair of qubits 1 and 4 will also be entangled after the performance of the ES operation at any values of parameter p.

5. Entanglement Swapping at the Simultaneous Presence of White and Colored Noises in Initial States

Now, let us consider the entanglement swapping operation when both white and colored noises are simultaneously present in the initial states of two qubit pairs. The state of initially prepared pairs is determined in this case by the density operator [23]

$$\widehat{\rho}_{CW} = p|\Psi^-_{12}\rangle\langle\Psi^-| + \frac{r}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|) + \frac{1-p-r}{4}\widehat{I}_{12}, \qquad (16)$$

where p is the fraction of the pure state, $p \in [0,1]$, and r the fraction of the colored state (the weighting factor), $r \in [0, 1-p]$. Let us denote the weighting factor of white noise, which equals $1-p-r$, by q; so that the normalization condition is $p+r+q=1$. The density matrix of state (16) looks like

$$\rho_{CW} = \\ = \frac{1}{4}\begin{pmatrix} 1-p-r & 0 & 0 & 0 \\ 0 & 1+p+r & -2p & 0 \\ 0 & -2p & 1+p+r & 0 \\ 0 & 0 & 0 & 1-p-r \end{pmatrix}, \quad (17)$$

in the standard basis and like

$$\rho^{\mathrm{B}}_{CW} = \\ = \frac{1}{4}\begin{pmatrix} 1-p+r & 0 & 0 & 0 \\ 0 & 1+3p+r & 0 & 0 \\ 0 & 0 & 1-p-r & 0 \\ 0 & 0 & 0 & 1-p-r \end{pmatrix}$$

in the Bell one. Substituting the density operator of the initial qubit pairs (16) with different parameters p_1, p_2, r_1, and r_2 into expressions (6) and (7), we obtain the following matrix representations for the state of qubit pair 1 and 4 in the final state after executing the ES operation: in the computational basis,

$$\rho_{14CW} = \frac{1}{4}\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 \\ 0 & a_3 & a_2 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix},$$

where $a_1 = 1-(p_1+r_1)(p_2+r_2)$, $a_2 = 1+(p_1+r_1)(p_2+r_2)$, $a_3 = -2p_1p_2$, and, in the Bell basis,

$$\widetilde{\rho}^{\mathrm{B}}_{14CW} = \frac{1}{2}\operatorname{diag}\big(1-2p_1p_2+(p_1+r_1)(p_2+r_2);$$
$$1+2p_1p_2+(p_1+r_1)(p_2+r_2);$$
$$1-(p_1+r_1)(p_2+r_2);\ 1-(p_1+r_1)(p_2+r_2)\big).$$

At $p_1 = p_2 = p$ and $r_1 = r_2 = r$, the density operator of the final state of qubits can be presented in the form similar to Eq. (16), namely,

$$\widehat{\rho}_{14} = \widetilde{p}\,|\Psi^-_{14}\rangle\langle\Psi^-_{14}| + \frac{\widetilde{r}}{2}(|0_1 1_4\rangle\langle 0_1 1_4| + \\ + |1_1 0_4\rangle\langle 1_1 0_4|) + \frac{\widetilde{q}}{4}\widehat{I}_{14}, \quad (18)$$

where

$$\widetilde{p} = p^2, \quad \widetilde{r} = r^2 + 2pr, \\ \widetilde{q} = 1-(\widetilde{p}+\widetilde{r}) = 1-(p+r)^2. \quad (19)$$

Let us analyze the fraction redistribution among the pure state and colored and white noises owing to the ES process. If $p=1$ (so that $r=0$ and $q=0$), we have $\widetilde{p}=1$, $\widetilde{r}=0$, and $\widetilde{q}=0$. Hence, the pure initial state transforms into a pure one, i.e. the noise components are absent from the final state.

If $p=0$ and $r \neq 0$ (then, $q = 1-r$), we have $\widetilde{p}=0$, $\widetilde{r}=r^2$, and $\widetilde{q}=1-r^2$. Since $r^2 < r$ $(0<r<1)$ and $1-r^2 > 1-r$, the fraction of colored noise decreases and that of white noise increases after the ES process.

If $p=0$ and $r=0$ (then, $q=1$), we have $\widetilde{p}=0$, $\widetilde{r}=0$, and $\widetilde{q}=q=1$, i.e. white noise does not vary.

If $0<p<1$, we have $p^2<p$ and $(p+r)^2 < p+r$, so that $\widetilde{p}<p$, $\widetilde{q}>q$, and, therefore, the fraction of the pure state decreases and that of white noise increases.

If the fraction of the pure state in the initial state (16) is equal to the fraction of white noise, i.e. $p = 1-(p+r)$ (whence, $p=\frac{1-r}{2}$), the fraction of colored noise does not change after the ES process. Really,

$$\widetilde{r} = r^2 + 2pr = r^2 + 2\cdot\frac{1-r}{2}\cdot r = r.$$

Then it is obvious that, if $p<q$ (i.e. if $p<\frac{1-r}{2}$), we obtain $\widetilde{r}<r$, which means that the colored noise fraction decreases, i.e. the growth of white noise occurs owing to a reduction of the pure state and colored noise fractions.

If $p>q$ (i.e. if $p>\frac{1-r}{2}$), the colored and white noises grow owing to a reduction of the pure state fraction.

The density matrix of state (18) looks like

$$\widetilde{\rho}_{14CW} = \frac{1}{4}\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 \\ 0 & a_3 & a_2 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix}, \quad (20)$$

in the standard basis (here, $a_1 = 1-(p+r)^2$, $a_2 = 1+(p+r)^2$, and $a_3 = -2p^2$) and like

$$\widetilde{\rho}^{\mathrm{B}}_{14CW} = \frac{1}{2}\operatorname{diag}\big(1-2p^2+(p+r)^2; 1+2p^2+(p+r)^2; \\ 1-(p+r)^2; 1-(p+r)^2\big)$$

in the Bell one.

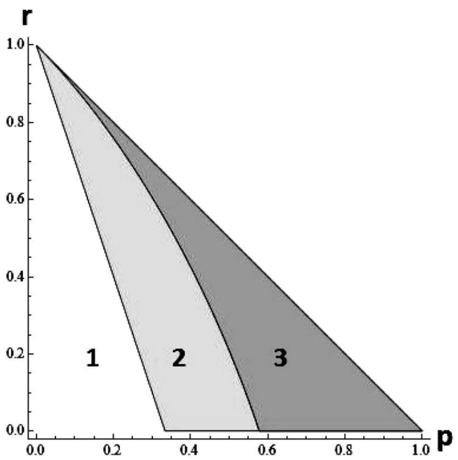

Fig. 4. Diagram for the values of parameters p and r, at which (*1*) the system in the initial and final states is non-entangled, (*2*) the system is entangled in the initial state and non-entangled in the final one, and (*3*) the initial and final states of the systems are entangled

For any $p \in [0, 1]$ and any $r \in [0, 1-p]$, the von Neumann entropy for the state of the final qubit pair, $S_{fin}(p, r)$, exceeds the entropy $S_{in}(p, r)$ for the state of each of the initial qubit pairs. The largest difference between those entropies, $\Delta S \approx 0.4$, is observed at $p \approx 0.72$ and $r \approx 0.15$.

The reduced density matrices that correspond to matrices (17) and (20) are proportional to the identity matrix, as it was in the previous cases, and, accordingly, the von Neumann entropy for the state of each qubit in the pair does not depend on p and r, being equal to 1.

The partially transposed density matrix for the state of each of the initially prepared pairs looks like

$$\rho_{CW}^{\mathrm{PT}} = \\ = \frac{1}{4}\begin{pmatrix} 1-p-r & 0 & 0 & -2p \\ 0 & 1+p+r & 0 & 0 \\ 0 & 0 & 1+p+r & 0 \\ -2p & 0 & 0 & 1-p-r \end{pmatrix} \qquad (21)$$

in the standard basis. To find the range of values for coefficients p and r, in which the state of each of the initially prepared pairs is entangled, let us firstly diagonalize matrix (21),

$$\rho_{\mathrm{diag}}^{\mathrm{PT}} = \\ = \frac{1}{4}\begin{pmatrix} 1+p+r & 0 & 0 & 0 \\ 0 & 1+p+r & 0 & 0 \\ 0 & 0 & 1+p-r & 0 \\ 0 & 0 & 0 & 1-3p-r \end{pmatrix}.$$

Owing to the conditions $p \in [0, 1]$ and $r \in [0, 1-p]$ imposed on the weight coefficients, only one characteristic value of the partially transposed matrix can be negative, namely,

$$1 - 3p - r \leq 0 \quad \Longrightarrow \quad 3p + r \geq 1. \qquad (22)$$

This result was obtained earlier in work [23].

Now, let us determine the range of p and r values, in which the state of a qubit pair formed by the ES operation is entangled. For this purpose, we use relations (19) and (22) to obtain

$$3\widetilde{p} + \widetilde{r} \geq 1 \quad \Longrightarrow \quad 3p^2 + r^2 + 2pr \geq 1 \quad \Longrightarrow \\ \Longrightarrow \quad 2p^2 + (p + r)^2 \geq 1.$$

Figure 4 exhibits the range of values for the parameters p and r, in which the initial and final systems of qubits are either entangled or not. One can see that there is a range of parameters (it is marked by number 2 in Fig. 4), for which the state of each of the initially prepared pairs is entangled, whereas the state of a pair obtained as a result of the ES operation is non-entangled. In other words, there may arise a situation after the entanglement swapping process, when the final state of qubits 1 and 4 is non-entangled owing to the influence of noise, whereas the initial qubit pairs are entangled.

6. Conclusions

1. If either of qubit pairs is prepared in the pure state ($p_1 = 1$ or $p_2 = 1$), the density matrix ρ_{14} of a qubit pair obtained as a result of the entanglement swapping process identically reproduces the density matrix of the second qubit pair prepared in the mixed state. If the both qubit pairs are prepared in the mixed state, the final state of qubit pair 1 and 4 turns out noisier than the state of each of the initially prepared pairs, which results in an increase of the von Neumann entropy for the qubit pair.

2. Depending on the initial distribution among the fractions of the pure state and colored and white noise, different scenarios for the redistribution of those fractions are implemented in the course of the ES process.

3. The range of values for the parameters p and r was determined, at which an entangled state transforms into the non-entangled one in the course of entanglement swapping.

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КВАНТОВА ТЕЛЕПОРТАЦІЯ ЗАШУМЛЕНИХ ЗАПЛУТАНИХ СТАНІВ

Р е з ю м е

У системі чотирьох кубітів розглянуто вплив наявності білого та кольорового шуму в станах початково приготованих заплутаних пар кубітів на кінцевий стан пари кубітів, що отримується в результаті операції безконтактного заплутування. Побудовано відповідні матриці густини, проаналізовано перерозподіл часток чистого стану та білого і кольорового шуму. З'ясовано умови збереження та руйнування заплутаності при переході від початкового до кінцевого стану. Проведено також порівняння ентропії фон Неймана початкового і кінцевого станів пари кубітів.