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SOLUTIONS OF THE MODEL OF LIQUID AND GAS FILTRATION IN THE ELASTIC MODE WITH DYNAMIC FILTRATION LAW

A filtration model with the generalized Darcy's law making allowance for nonlocal and nonlinear effects has been developed. The expression for the law was derived within the relaxation formalizm of nonequilibrium thermodynamics. The developed model is applied to analyze the influence of relaxation effects on the phase velocity of small wave-like perturbations. The character of nonlinear traveling waves is determined. The properties of polynomial and self-similar solutions are analyzed.

K e y w o r d s: porous medium, generalized Darcy's law, invariant solutions, relaxation.

1. Introduction

A detailed microscopic description of the processes of liquid or gas filtration through porous materials is problematic today, even if we take the capabilities of modern computer facilities into account. Therefore, those processes are usually described in the framework of a continuum approach, by modeling such systems as continuous media characterized by averaged parameters [1–5].

Within the framework of continuum mechanics, the motion of a weakly compressible liquid or gas in an elastically deformable porous medium (the elastic filtration mode) is described by the mass conservation law, equation of state for the moving substance, and filtration law. The latter expresses a relation between the filtration velocity and the pressure gradient. It is often considered in the linear approximation known as Darcy's law. However, today, there exist a significant number of experimental evidence for deviations from Darcy's law [5–9]. This especially concerns nonequilibrium high-intensity processes, when the amplification of nonlocal effects take place [10, 11].

In this work, we propose a generalization of linear Darcy's law by considering the nonlinear and nonequilibrium character of filtration processes. Note that the nonequilibrium behavior is associated with the fact that the time for a local thermodynamic equilibrium to be established in an infinitely small volume of a porous medium is comparable with the characteristic time parameters of the whole system. An additional factor is the presence of field (velocity, stress) nonuniformities, which are characterized by corresponding gradients [12]. In our model, such relaxation processes are taken into account in the dynamic Darcy's equation, whose form is substantiated within the relaxation formalism of nonequilibrium thermodynamics. We also analyze some solutions obtained for the model of elastic filtration mode with regard for the dynamic filtering law.

2. Mathematical Model of Elastic Filtration Mode

Let us briefly recall the basic points of the elastic filtration mode model [2–5]. In its framework, a porous medium is described, by using such an average geometric characteristic as the average porosity $m = V_n/V$, where V_n is the volume of pores in an element of the porous medium, and V the total volume of this element.

As a rule, the total and active porosities are distinguished. The active porosity notion is applicable only to pores that comprise a common network of connected pores and can be filled with a fluid from outside. This is the interpretation of a porosity that is used in the theory of filtration. Then the mass conservation law for a fluid with density ρ_r, which is filtered through a porous element, is expressed by the integral relation

$$\frac{\partial}{\partial t}\int\limits_V m\rho_r dV = -\int\limits_S \rho_r \mathbf{un} dS,$$

where S and V are the surface and volume, respectively, of the porous element; $\mathbf{n}$ is the external normal to the element surface, and $\mathbf{u}$ the filtration velocity. Using the Ostrogradskii–Gauss theorem, we can change to the differential form of the law by assuming that the elementary volume can be arbitrarily small, and the fields are continuous:

$$\frac{\partial\,(m\rho)}{\partial t} + \mathrm{div}\rho\mathbf{u} = 0. \qquad (1)$$

The filtration velocity $\mathbf{u}$ is the main characteristic of the filtration motion. The velocity projection onto the normal $\mathbf{n}$ to the surface element is determined as follows [5]:

$$\mathbf{un} = \lim_{\Delta S\to 0}\frac{\Delta Q}{\rho_r \Delta S},$$

where ΔS is the surface element area, and ΔQ the fluid or gas discharge. The dependence between the filtration velocity vector and the pressure field comprises the essence of the filtration law, which was experimentally established in 1856 by A. Darcy. It looks like

$$\mathbf{u} = -\frac{k}{\mu}\nabla p, \qquad (2)$$

where k is the medium permeability, which does not depend on the fluid properties, but only on the geometric characteristics of the porous medium, and is measured in Darcy units (1 D = 10^{-12} m^2); and μ is the dynamic viscosity of a fluid.

Darcy's law is obeyed under the following conditions [4, 13]: (i) the fluid moves slowly, so that the inertia effects can be neglected; (ii) there is the mass exchange between the phases, but not the momentum exchange; (iii) the momentum exchange in separate phases associated with the viscous shear is neglected; (iv) the gravitational force is considered to be an external factor and is applied vertically; (v) the viscous displacement obeys Newton's law; (vi) the conditions at the interfaces between the liquid and solid phases correspond to the sticking; (vii) the solid phase is a perfectly rigid body. Note that Darcy's law is written for the excessive pressure.

In order to obtain a closed system of equations for the description of the filtration in a porous medium, Eqs. (1) and (2) should be appended by the equation of state for the fluid or gas, e.g., in the form $\rho = \rho\,(p, T = \text{const})$.

One of the simplest models for nonstationary gas filtration (the model of elastic filtration mode) [5] can be obtained, by assuming that the compressibility of a moving substance very much exceeds the compressibility of the porous medium. Then the change of the medium porosity in time can be neglected, which allows Eq. (1) to be substituted by the equation

$$m\frac{\partial\rho}{\partial t} + \mathrm{div}\rho\mathbf{u} = 0. \qquad (3)$$

As a result, in the one-dimensional case in view of the linear equation of state $\rho = p\rho_0/p_0$ and Darcy's law (2), we obtain the following equation for the pressure:

$$\frac{\partial p}{\partial t} = \frac{k}{2m\mu}\frac{\partial^2\left(p^2\right)}{\partial x^2}.$$

Even in this simple case, the resulting equation is nonlinear. Of course, the assumptions made above about the fluid or gas flow are satisfied rather well in many cases, so that Darcy's law taken in form (2) is enough for a correct description of flow parameters.

3. Generalization of Darcy's Filtration Law

However, experimental data testify that if the filtration velocity is high [2, 8, 14] or, quite the contrary, rather slow, so that anomalous rheological properties of fluids can manifest themselves, there are deviations from relation (2). Let us consider the most well-known generalizations of Darcy's law. First of all, this is the two-component law [14]

$$\nabla p = -\frac{\mu}{k}\mathbf{u} - \beta\frac{\rho u}{\sqrt{k}}\mathbf{u} \equiv f(u)\mathbf{u}, \qquad (4)$$

which was proposed by Forchheimer. Deviations from the linear Darcy's law are observed at Reynolds numbers of an order of 0.1–1.0, whereas the two-component law agrees well with experimental data at Reynolds numbers of an order of 10–100. Another generalization of Eq. (2) was proposed by Brinkman [7, 15]. It looks like

$$\nabla p = -\frac{\mu}{k}\mathbf{u} + \mu'\Delta\mathbf{u},$$

where μ' is the effective viscosity, which can be equal to or even less than the actual fluid viscosity μ. The following nonlinear filtration law also became widespread [5, 14]:

$$\mathbf{u} = -f\left(|\nabla p|\right)\nabla p, \tag{5}$$

where $f(x)$ is a definite smooth function. For two-phase flows, the nonequilibrium Buckley–Leverett model and the Barenblatt model [5] are applied.

The study of the filtration of viscoelastic polymer mixtures, petroleum, and other non-Newtonian fluids showed [14] that relaxation effects can substantially affect the parameters of the nonstationary filtration through a porous medium [9]. To examine the influence of delay effects, we propose to modify the classical Darcy's law within the relaxation formalism of nonequilibrium thermodynamics [10, 11, 16]. Using the operator representation for the permeability, the dynamic Darcy's law can be written in the form

$$u = -\hat{k}\nabla p, \tag{6}$$

where

$$\hat{k} = k^\infty + \frac{k^0 - k^\infty}{1 + \tau D_t}$$

is the dynamic permeability coefficient; k^0 and k^∞ are the equilibrium and frozen permeability coefficients, respectively; D_t the time-differentiation operator, and θ the relaxation time. For the latter, the estimate $\tau \sim k/(m\nu)$, where ν is the kinematic viscosity of a fluid, is known [17]. Formally transforming relation (6), we obtain the following generalized Darcy's equation:

$$\tau\left(u_t + k^\infty\left(\nabla p\right)_t\right) = -u - k^0\nabla p, \tag{7}$$

which can be found, e.g., in works [1, 9, 16].

A similar result can be obtained differently. In particular, let us write Darcy's law in the nonlocal form,

$$u = -k^\infty\nabla p + \sigma\int\limits_{t_0}^{t}\exp\left[-\frac{t-s}{\tau}\right]\nabla p\left(x,s\right)ds, \tag{8}$$

where σ is a certain parameter. By differentiating this expression and excluding the integral term from the result [also with the help of Eq. (8)], we obtain

$$\tau\left(u_t + k^\infty\left(\nabla p\right)_t\right) = -u - \left(k^\infty - \sigma\tau\right)\nabla p.$$

Making the substitution $k^\infty - \sigma\tau = k^0$, we arrive at Eq. (7). The indicated transformations make it possible to determine the parameter σ in Eq. (8) in terms of the quantities k^∞, k^0, and τ, namely, $\sigma = (k^\infty - k^0)/\tau$. The specific feature of the obtained dynamic Darcy's equation is that the model is reduced to the equation like $u = -k^0\nabla p$ at slow filtration processes and to $u = -k^\infty\nabla p$ at fast processes. In other words, the medium has different permeabilities depending on the characteristic process frequency.

Note that, in this way, a new class of dynamic Darcy's equations can be obtained by extending law (7) onto a nonlinear case. From the analysis of filtration laws (4) and (5), it follows that they can be expressed as the nonlinear relation $\psi(u) = -k^\infty f(\nabla p)$, where ψ and f are nonlinear functions of their arguments. Relaxation corrections to this law are chosen in the integral form, as was done in expression (8):

$$\tau^{-1}\int\limits_{t_0}^{t}\exp\left\{-\frac{t-s}{\tau}\right\}\left(\varphi - \psi\right)ds + \psi =$$
$$= -k^\infty f + \tau^{-1}\int\limits_{t_0}^{t}\exp\left\{-\frac{t-s}{\tau}\right\}\left(k^\infty f - k^0 g\right)ds,$$

where φ is a function depending of the filtration velocity, and g a function depending on the pressure gradient. By differentiating this equality with respect to the time variable t, we obtain a dynamic nonlinear Darcy's equation

$$\tau\left(\psi_t + k^\infty f_t\right) = -\varphi - k^0 g.$$

For instance, if φ, ψ, f, and g are power-law functions, the following filtration law is obtained:

$$\tau\left[u^n + k^\infty\nabla p_2\right]_t = -u^\ell - k^0\nabla p_1. \tag{9}$$

where $p_1 = c_1^2\rho^s$ is the equation of state for a fluid or gas at the equilibrium filtration, and $p_2 = c_2^2\rho$ is its counterpart in the case of frozen relaxation process. Being compared with expression (7), the frozen and equilibrium filtration laws can differ not only by the permeability coefficients, but also by the degree of nonlinearity.

The introduction of terms with time derivatives into Darcy's equation can be considered as making allowance for the time nonlocality in the relations between the generalized forces (the pressure gradient) and the flows (the filtration velocity). In the case where the field of generalized forces rapidly varies in space, spatially nonlocal models are used for the continuum approach to be applicable [10, 12, 18]. The application of nonlocal theory also seems promising in the case where it is not possible to distinguish a sufficiently homogeneous elementary volume in the medium, whose dimensions would justify the applicability of a local Darcy's law, e.g., in the case of fractal porous materials [19]. Hence, let the nonlocal generalization of the filtration law read

$$u(x,t) = -k \int R(x-z)\nabla p(z)dz,$$

where $R(\cdot)$ is a certain relaxation kernel with corresponding properties. Such immanently nonlocal models can be reduced to differential equations, by expanding the pressure gradient in a Taylor series in a vicinity of the point x and by performing the integration. As a result, we obtain

$$u(x,t) = -k\left(I + \sigma^2\nabla^2\right)\nabla p$$

or

$$\left(I + \sigma^2\nabla^2\right)^{-1} u(x,t) = -k\nabla p.$$

The weakly nonlocal case is reduced to the spatially nonlocal Darcy's equation

$$u - \sigma^2\nabla^2 u = -k\nabla p,$$

where σ is the parameter of spatial nonlocality. It is coupled with the correlation radius or with the size of a structural element in the medium.

4. Partial Solutions of the Filtration Model with Dynamic Darcy's Equation

First of all, let us consider the propagation of small perturbations accompanying the filtration of a fluid or gas in the elastic mode. The consideration is carried out in the framework of the model with the dynamic Darcy's equation:

$$m\frac{\partial \rho}{\partial t} + \mathrm{div}\rho\mathbf{u} = 0,$$
$$\tau\left(\frac{d\mathbf{u}}{dt} + k^\infty \frac{d\nabla p}{dt}\right) = -u - k^0\nabla p, \quad p = c^2\rho,$$

where $d/dt = \partial/\partial t + (\mathbf{u}\nabla)$ is the substantial derivative operator, and c the sound velocity in the moving substance. In the one-dimensional case, this system of equations is reduced to the following one:

$$\begin{aligned} &m\rho_t + (\rho u)_x = 0,\\ &\tau\left(u_t + uu_x + k^\infty c^2\left(\rho_{xt} + u\rho_{xx}\right)\right) = -u - k^0c^2\rho_x. \end{aligned} \tag{10}$$

Let us use this model to consider the propagation of small perturbations like

$$\rho = \rho_0 + \varepsilon\rho_1, \quad u = \varepsilon u_1, \quad \varepsilon \ll 1.$$

Then, in view of the smallness of perturbations, the system of equations in the first approximation reads

$$m\frac{\partial \rho_1}{\partial t} + \rho_0\frac{\partial u_1}{\partial x} = 0,$$
$$\tau\left(\frac{\partial u_1}{\partial t} + k^\infty c^2\frac{\partial^2\rho_1}{\partial x\partial t}\right) = -u_1 - k^0c^2\frac{\partial \rho_1}{\partial x}.$$

A solution of this system is sought in the form

$$\rho_1 = r\exp\left(i\left(kx - \omega t\right)\right), \quad u_1 = q\exp\left(i\left(kx - \omega t\right)\right),$$

where k is the wave number, and ω the circular wave frequency. From the consistency condition for the system of equations, a relation between k and ω can be determined in the form

$$\begin{vmatrix} m\omega & -\rho_0 k \\ k^\infty c^2\omega k\tau + k^0c^2ik & 1 - i\omega\tau \end{vmatrix} = 0,$$

which is called the dispersion relation. By calculating the determinant, we obtain

$$k = k' + ik'' = \sqrt{a + ib},$$

where

$$a = \frac{m\omega^2\tau\left(k^0 - k^\infty\right)}{\rho_0c^2\left(\left[k^0\right]^2 + \left[k^\infty\omega\tau\right]^2\right)},$$
$$b = \frac{m\omega\left(k^0 + k^\infty\left[\omega\tau\right]^2\right)}{\rho_0c^2\left(\left[k^0\right]^2 + \left[k^\infty\omega\tau\right]^2\right)},$$

and the damping coefficient k'' is positive ($k'' > 0$). Then

$$k' = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}},$$

and the phase velocity of the perturbation propagation equals

$$v_{\mathrm{ph}} = \frac{\omega}{k'}.$$

Let us evaluate how the relaxation processes affect the phase velocity. For this purpose, let us compare $\upsilon_{\rm ph}\big|_{\tau\neq 0}$ and $\upsilon_{\rm ph}\big|_{\tau=0}$. For definiteness, let us assume that $k^0 < k^\infty$, i.e. the porous medium is more permeable for high-frequency perturbations than for low-frequency ones. Hence, we need to compare the quantities $k'\big|_{\tau\neq 0}$ and $k'\big|_{\tau=0} = \sqrt{m\omega/2\rho_0 c^2}$. The inequality between them is the same as between $\sqrt{a^2+b^2}+a$ and $m\omega/\rho_0 c^2$. Since $a<0$, the inequality sign is determined by the sign of the expression

$$b^2 - \left(\frac{m\omega}{\rho_0 c^2}\right)^2 + \frac{2am\omega}{\rho_0 c^2} = \frac{m^2\omega^2 q(k^0-k^\infty)(k^0)^{-2}}{c^4\rho_0^2([k^0]^2+[k^\infty q]^2)^2}\times$$
$$\times\,(k^{\infty\,2}(k^0+k^\infty)q^3 + 2k^0 k^{\infty\,2}q^2 + 2k^0k^\infty q + 2k^{0\,3}),$$

where $q=\omega\tau>0$. This expression is evidently negative if $k^0<k^\infty$, so that $k'\big|_{\tau\neq 0} < k'\big|_{\tau=0}$. Using a similar consideration, we can show that, in the case $k^0>k^\infty$, we obtain $k'\big|_{\tau\neq 0} > k'\big|_{\tau=0}$.

Hence, the following **statement** is valid:

If the permeability of a porous medium is higher for high-frequency perturbations than for low-frequency ones, i.e. $k^0<k^\infty$, then the phase velocity of the filtration with relaxation is higher than the phase velocity without relaxation, $\upsilon_{\rm ph}\big|_{\tau\neq 0} > \upsilon_{\rm ph}\big|_{\tau=0}$. Vice versa, provided that $k^0>k^\infty$, the phase velocities in the nonequilibrium process exceed their counterparts in the equilibrium case, $\upsilon_{\rm ph}\big|_{\tau\neq 0} < \upsilon_{\rm ph}\big|_{\tau=0}$.

4.1. Wave-like solutions of the nonequilibrium filtration model

The wave propagation in porous media in the case of nonlinear filtration law (5) was considered in work [5], where the behavior of the pressure and the filtration velocity at the front of a stationary wave was determined. Let us analyze those modes in the case of dynamic Darcy's equation.

Wave-like solutions of model (10) have the form

$$\rho = R(\xi),\quad u = U(\xi),\quad \xi = x - st, \tag{11}$$

where s is the constant velocity of the wave front. Substituting Eq. (11) into Eq. (10), integrating the resulting relation from the background $\xi=\infty$ to the current value ξ, and taking into account that $U(\infty)=0$ and $R(\infty)=R_0$, we obtain a quadrature

$$RU = ms\,(R-R_0)$$

and a system of ordinary differential equations

$$\frac{dR}{d\xi} = W,$$
$$\frac{dW}{d\xi} = \frac{ms\left(1-\frac{R_0}{R}\right) + k^0c^2W - \tau s\,(s-U)\,m\frac{W}{R^2}R_0}{\tau k^\infty c^2\,(s-U)}. \tag{12}$$

This is an autonomous dynamic system, which has a nontrivial stationary point Q in the phase plane (R,W) with the coordinates $Q\,(R_0,0)$. In a vicinity of this point, system (12) is described by a linearized system with the matrix

$$A = \begin{pmatrix} 0 & 1\\ \alpha & \beta\end{pmatrix},$$

where

$$\alpha = \frac{m}{\tau G_0} > 0,\quad \beta = \frac{\theta}{\tau s} - \frac{sm}{G_0},$$
$$G_0 = k^\infty c^2 R_0,\quad \theta = k^0/k^\infty.$$

It is evident that the eigenvalues $\lambda_{1,2}$ of this matrix should be calculated from the condition $\lambda^2-\beta\lambda-\alpha=0$. They equal $\lambda_{1,2}=\beta\pm\sqrt{D}$, where $D=\beta^2+4\alpha>0$. Therefore, they are real-valued and have different signs at all parameter values. From whence, it follows that the stationary point is a saddle one. The only trajectory entering it is a stable separatrix (see Fig. 1).

Let us analyze the behavior of the separatrix entering the saddle point Q, when the relaxation time changes. In this connection, note that, owing to a specific form of the matrix A, the angular coefficients of the directions [20], along which the trajectories approach the stationary point Q, are equal to the eigenvalues $\lambda_{1,2}$. Therefore, the direction of

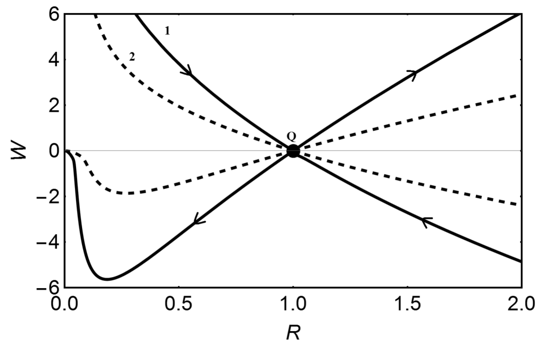

Fig. 1. Phase portrait of the dynamic system (12). Solid curves *1* demonstrate separatrices of the stationary point Q at $\tau_1 = 0.01$, and dashed curves *2* at $\tau_2 = 0.05$

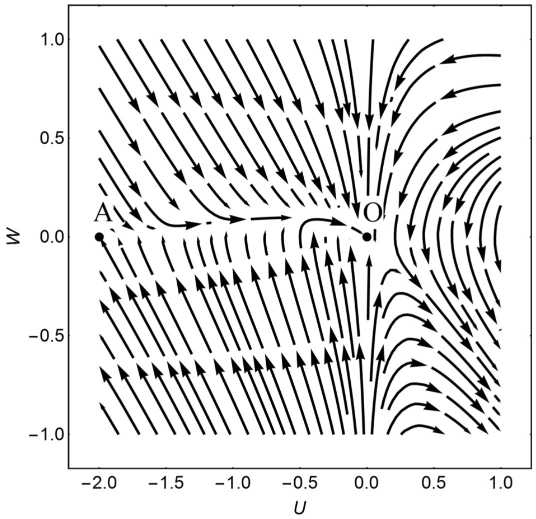

Fig. 2. Phase portrait of the dynamic system (14) at $m = 0.35$, $\tau = 0.5$, and $\theta = 2$

entering a separatrix is determined by the quantity $\lambda(\tau) = \beta - \sqrt{D}$. The sign of its derivative

$$\frac{d\lambda}{d\tau} = -\frac{\theta}{\tau^2 s\sqrt{D}}\left[\sqrt{D} - \left(\beta - \frac{2ms}{G_0\theta}\right)\right]$$

depends on the sign of the expression in the brackets. Subjecting this expression to a number of equivalence conversions, we obtain that

$$\operatorname{sign}\left(\frac{d\lambda}{d\tau}\right) = \operatorname{sign}(1-\theta).$$

Therefore, if $\theta < 1$, the function $\lambda(\tau)$ is increasing, and we have $0 > \lambda(\tau_2) > \lambda(\tau_1)$ for $\tau_2 > \tau_1$, which corresponds to the separatrix rotation counterclockwise (Fig. 1). If $\theta < 1$, the function $\lambda(\tau)$ is decreasing, and the separatrix rotates in the opposite direction.

4.2. Polynomial solutions

With the help of the variable separation method, one can ascertain that system (10) has solutions of the type

$$\rho = R\,(t)\,x^2, \quad u = U\,(t)\,x. \tag{13}$$

Substituting them into Eq. (10) and equating the coefficients in terms with the same powers of x, we obtain a nonlinear system of dynamic equations for the quantities R and U in the form

$$m\frac{dR}{dt} + 3RU = 0,$$
$$\tau\left(\frac{dU}{dt} + U^2 + 2k^\infty c^2\left(\frac{dR}{dt} + UR\right)\right) = -U - 2k^0c^2R.$$

In order to reduce the number of parameters, let us perform the rescaling transformation $R = W/2c^2k^\infty$, which brings about the system

$$\begin{aligned} &m\frac{dW}{dt} + 3WU = 0,\\ &\tau\left(\frac{dU}{dt} + U^2 + \frac{dW}{dt} + UW\right) = -U - \theta W, \end{aligned} \tag{14}$$

where $\theta = k^0/k^\infty$. This system has two stationary points: $A\,(-1/\tau, 0)$ and $O\,(0, 0)$ (see Fig. 2). At point A, the eigenvalues of the matrix of a linearized system equal $\lambda_A = 3/m\tau$ and $1/\tau$, which testifies to the saddle character of this point. At the same time, at point O, the eigenvalues are$\lambda_O = -1/\tau$ and 0, and the point is complex.

To analyze the behavior of the system in a vicinity of the coordinate origin, let us construct a system constraint on the central manifold [21]. For this purpose, let us transform system (14) to the canonical form with the help of the linear transformation

$$\begin{pmatrix} W\\ U\end{pmatrix} = T\begin{pmatrix} x\\ y\end{pmatrix}, \quad T = \begin{pmatrix} 1/\tau & 0\\ -\theta/\tau & 1\end{pmatrix}.$$

As a result, we obtain

$$\frac{d}{dt}\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} 0\\ -y/\tau\end{pmatrix} + T^{-1}F\left(T\begin{pmatrix} x\\ y\end{pmatrix}\right), \tag{15}$$

where F is a column matrix of the right-hand sides of system (14). According to the properties of a central manifold, it can have the form $h(x) = dx^n + \dots$. Substituting $h(x)$ into the second equation in system (15) and analyzing the monomials of the highest degree, we obtain

$$n = 2, \quad d = \frac{(3-m)(\theta-1)\theta}{2m\theta}.$$

The reduction of system (15) on the central manifold leads to the equation $dx/dt = \frac{3\theta}{m\tau}x^2$.

Thus, since $0 < m < 1$, the sign of α depends only on the value of $\theta - 1$. If $\theta > 1$, the central manifold is a parabola with upward branches; if $\theta < 1$, this is a parabola with branches directed downward (Fig. 3). The qualitative behavior of trajectories in a vicinity of point O is governed by the solutions of the reduced equation and characterizes this point as a stable node in the left semiplane and as a saddle point in the right one.

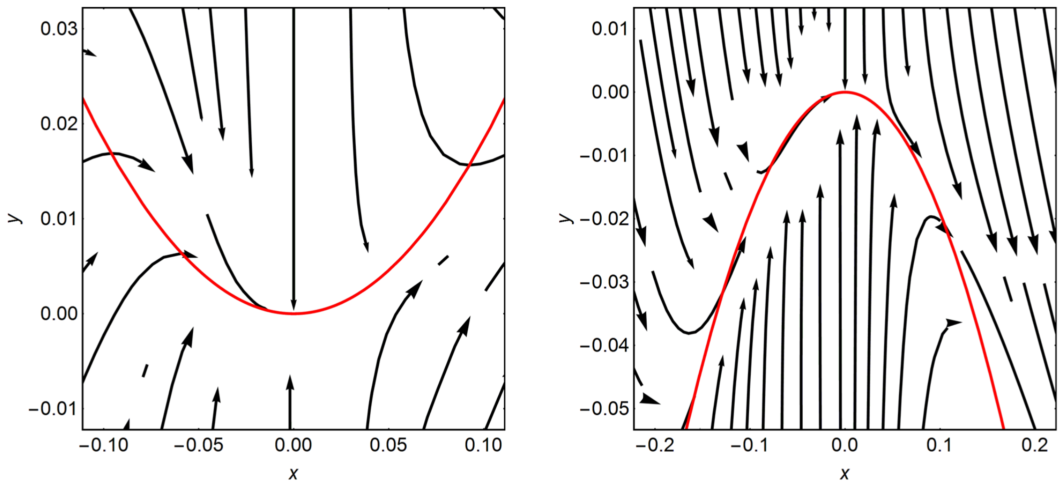

Fig. 3. Central manifold and the structure of the phase plane in a vicinity of point O of the dynamic system ((15) at $\theta > 1$ (left panel) and $\theta < 1$ (right panel)

4.3. Self-similar solutions of the filtration model with the nonlinear dynamic Darcy's equation

As a rule, the account for relaxation processes makes the set of exact solutions of the model narrower. However, in our case – the filtration model with the nonlinear dynamic Darcy's equation in form (9) – self-similar solutions can be determined by applying the methods of symmetry analysis [22], in contrast to the case of models with Eq. (7). By analyzing rescaling transformations allowable by the system

$$m\rho_t + (\rho u)_x = 0,$$
$$\tau \left(u^n + k^\infty c_2^2 \rho_x\right)_t = -u^\ell - k^0 c_1^2 (\rho^s)_x, \qquad (16)$$

one can make sure that the latter is invariant with respect to the operator

$$\hat{X} = \alpha t \frac{\partial}{\partial t} + \beta x \frac{\partial}{\partial x} + r\rho \frac{\partial}{\partial \rho} + su \frac{\partial}{\partial u},$$

where $\alpha = (n - \ell)s$, $\beta = (1 - \ell + n)s$, and $r = n + 1$ if, additionally,

$$n(1 - 2s) + s(\ell - 1) + 1 = 0.$$

The invariants of the operator $\hat{X}$ determine the form of self-similar solutions,

$$\rho = R(\omega) t^{r/\alpha}, \quad u = U(\omega) t^{s/\alpha}, \quad \omega = x^\alpha t^{-\beta}.$$

As a result, the system of partial differential equations (16) is reduced to a system of ordinary differential equations

$$m\left(-\beta\omega R' + R r \alpha\right) + \alpha \omega^{1-1/\alpha} (UR)' = 0,$$
$$\tau \left(\alpha n U^{n-1} U' \omega^{1/\alpha} + \alpha k^\infty c_2^2 [\alpha \omega R'' + (\alpha - 1) R']\right) \times$$
$$\times \omega^{1-2/\alpha} = -U^\ell - s\alpha k^0 c_1^2 R^{s-1} R' \omega^{1-1/\alpha},$$

where the prime denotes the derivative with respect to ω. Since this system is nonautonomous and nonlinear, its analysis is a complicated and still unresolved problem. It should also be noted that if $n = \ell$, then the parameter $\alpha = 0$, and system (16) has no self-similar solutions of the indicated type.

5. Conclusions

Our study, which was carried out in the framework of the relaxation formalism, allowed the filtration laws to be generalized with regard for nonlinear and spatial-temporal nonlocal effects that occur in nonequilibrium porous media. As a result, dynamic filtration laws were derived in the case of equilibrium or frozen processes, with the classical Darcy's law or its nonlinear modifications being their asymptotics. Together with the mass conservation law and the equations of state for a moving substance, those equations seem to be promising for the development of the model of elastic filtration mode applicable at high filtration velocities and a high inhomogeneity of the porous space.

In the linear approximation, Darcy's relaxation equations were substantiated in statistical physics and proved themselves in practice, while describing the filtration of non-Newtonian fluids in porous media, where microscopic processes on a pore scale have a significant effect. Those relationships were obtained by linearizing nonlinear Darcy's equations. They make it possible to describe the propagation of small wave-like perturbations and to analyze the influence of relaxation processes on the phase propagation velocity of waves.

Although the account for the nonlinear nonequilibrium character of the system made the analytical consideration of filtration models more complicated, we managed to analyze a number of simple solutions. In particular, the filtration model is demonstrated to possess solutions in the form of nonlinear waves that propagate at a constant velocity, as well as polynomial and self-similar solutions. On the basis of the methods of qualitative analysis of nonlinear dynamical systems, the structure of the phase spaces of reduced systems, their dependence on the relaxation time, and the ratio between the equilibrium and frozen permeability parameters are determined.

The proposed models and their solutions can be useful, when studying the parameters of fluid flow fields in rocks, which is important for the development of hydrocarbon extraction technologies [2, 14].

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РОЗВ'ЯЗКИ МОДЕЛІ ПРУЖНОГО РЕЖИМУ ФІЛЬТРАЦІЇ РІДИН ТА ГАЗІВ З ДИНАМІЧНИМ ЗАКОНОМ ФІЛЬТРАЦІЇ

Резюме

У статті розробляється модель фільтрації з узагальненим законом Дарсі, який містить опис нелокальних та нелінійних ефектів. Вигляд такого закону отримано засобами релаксаційного формалізму нерівноважної термодинаміки. В рамках побудованої моделі проаналізовано вплив релаксаційних ефектів на фазову швидкість поширення малих хвильових збурень, встановлено характер нелінійних біжучих хвиль, а також досліджено властивості поліноміальних та автомодельних розв'язків.