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GENERALIZED UNCERTAINTY PRINCIPLE AND DELTA-FUNCTION POTENTIAL

In recent studies, the Heisenberg uncertainty principle has been modified into the generalized uncertainty principle (GUP) to explain gravity from a quantum mechanical perspective. Here, we study the GUP corrections to the bound-state energy eigenvalues for a delta-function potential well and a double delta-function potential well using nonrelativistic quantum mechanical tools. Transmission probabilities for scattering states have also been derived and compared with the unmodified cases for both systems.

Keywords: generalized uncertainty principle, delta-function potential well, minimal length.

1. Introduction

Explaining the gravitational interaction from the quantum mechanical perspective, as well as the unification of it with the other fundamental interactions, is a big challenge for modern theoretical physics. There are several approaches to quantum gravity such as the string theory [1], loop quantum gravity [2], doubly special relativity [3], *etc.* which lead to a minimum measurable length near the Planck scale. In a study, the Heisenberg uncertainty principle has been modified to the generalized uncertainty principle (GUP) [4] which predicts the existence of the minimal length scale. The consequent modification of the position-momentum commutation relation has been seen with a momentum-dependent result. The time-independent Schrödinger equation has also been modified accordingly. The GUP affects many quantum mechanical phenomena such as a harmonic oscillator [5], the hydrogen atom problem [6], angular momentum algebra [7], step and barrier potential wells [8], quantum tunneling [9], *etc.*

In 1995, Kempf, Mangano, and Mann first proosed a modified commutation relation between the one-dimensional position and the momentum as [10]

$$[x,p] = i\hbar\,[1+\beta p^2], \tag{1}$$

where β is a very small parameter with dimension of $momentum^{-2}$. It depends on Planck's constant and Planck's length. There was another group of scientists who tried to modify the standard Poisson bracket by deforming the classical Newtonian mechanics like a quantum commutator [11]

$$\{x,p\} = [1+\beta p^2]. \tag{2}$$

There are many approaches to the GUP. The 3D quantum gravity theories modify this quantum commutator as [8]

$$[x_i,p_j] = i\hbar\,[\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j]. \tag{3}$$

Following this modification, GUP becomes [8]

$$\triangle\, x_i \triangle\, p_i \geq \frac{\hbar}{2}\big[1+\beta\{(\triangle\, p)^2 + \langle p\rangle^2\} + +2\beta\{\triangle\, p_i^2 + \langle p_i\rangle^2\}\big]. \tag{4}$$

However, by defining

$$x_i = x_{0i}, \tag{5}$$

we have

$$p_i = p_{0i}\,(1+\beta p_0^2), \tag{6}$$

with $p_0^2 = \sum_{j=1}^{3} p_{0j}\,p_{0j}$, where x_{0i} and p_{0i} satisfy the well-known commutation relations

$$[x_{0i},p_{0i}] = i\hbar\delta_{ij} \quad \text{and} \quad p_{0i} = -i\hbar\frac{d}{dx_{0i}}. \tag{7}$$

If we consider a particle of mass m and energy E with a potential $V(x)$, the ordinary time-independent Schrödinger equation becomes

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\Psi = E\Psi, \tag{8}$$

where $\Psi(x)$ is the wave function associated to this particle. Using Eq. (7), the ordinary time-independent Schrödinger equation is modified by GUP (keeping only terms up to the order of β) [9] as

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{\beta}{m}\hbar^4\frac{d^4}{dx^4}+V(x)\right)\Psi=E\Psi. \qquad (9)$$

From Eq. (9), it is clear that, under the GUP, the time-independent Schrödinger equation acquires the extra term $\left(\frac{\beta}{m}\hbar^4\frac{d^4}{dx^4}\right)$ which can be treated as a perturbed Hamiltonian to find the solution of this GUP-modified time-independent Schrödinger equation.

This work is arranged as follows. In Sect. 2, we will discuss the GUP corrections to the bound-state energy eigenvalues for the delta-function potential. Some transmission probabilities for scattering states will be derived and compared with the unmodified cases. Following the same procedure, we study the case of a double delta-function potential well in Sect. 3. Finally, we present our conclusions in Sect. 4.

2. GUP and Delta-Function Potential

The delta-function is an infinitesimally narrow spike at the origin ($x=0$) with the infinite height. Its definite integral is unity by definition:

$$\int\limits_{-\infty}^{\infty}\delta(x)\,dx=1. \qquad (10)$$

So, the spike of a unit area at $x=a$ is denoted as $\delta(x-a)$. For any arbitrary function $f(x)$,

$$\int\limits_{-\infty}^{\infty}f(x)\,\delta(x-a)\,dx=f(a). \qquad (11)$$

Let us consider the potential of the form

$$V(x)=-\alpha\,\delta(x), \qquad (12)$$

where α is some positive constant with the dimension of [energy × length]. Now, the time-independent Schrödinger equation under GUP for a delta-function well reads

$$\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}-\frac{\beta}{m}\hbar^4\frac{d^4\Psi}{dx^4}+(E+\alpha\,\delta(x))\Psi=0, \qquad (13)$$

$$\frac{d^2\Psi}{dx^2}-l_{pl}^2\frac{d^4\Psi}{dx^4}+\frac{2m}{\hbar^2}(E+\alpha\,\delta(x))\Psi=0, \qquad (14)$$

where the Planck length $(l_{\rm pl})=\sqrt{2\beta\hbar^2}\backsim 10^{-35}$ m. From this time-independent Schrödinger equation under GUP, the solutions for both bound states ($E<0$) and scattering states ($E>0$) can be derived as follows.

Let us start with bound states ($E<0$). For both regions $x<0$ and $x>0$, the potential $V(x)=0$, and, at $x=0$, the potential $V(x)$ has an infinite spike. As $\left(\frac{\beta}{m}\hbar^4\frac{d^4\Psi}{dx^4}\right)$ is the perturbed Hamiltonian $H^{'}$, the solutions for the unperturbed bound-state energy (E_0) and the wave function (Ψ_0) are [12]

$$E_0=-\frac{m\alpha^2}{2\hbar^2},\quad \Psi_0=\frac{\sqrt{m\alpha}}{\hbar}\,e^{-\frac{m\alpha}{\hbar^2}|x|}. \qquad (15)$$

Now, the first-order correction to the energy,

$$E'=\langle\Psi_0\mid H'\mid\Psi_0\rangle=4\beta m\left(\frac{m\alpha^2}{2\hbar^2}\right)^2. \qquad (16)$$

Therefore, only one bound-state energy is allowed:

$$E=E_0+E^{'}=-\frac{m\alpha^2}{2\hbar^2}+4\beta m\left(\frac{m\alpha^2}{2\hbar^2}\right)^2. \qquad (17)$$

Now, at the scattering states ($E>0$), for both the regions $x<0$ and $x>0$, the potential $V(x)=0$. So, the GUP-modified time-independent Schrödinger equation (TISE) reads

$$\frac{d^2\Psi_1}{dx^2}-l_{pl}^2\frac{d^4\Psi_1}{dx^4}+k^2\Psi_1=0:\quad x<0, \qquad (18)$$

$$\frac{d^2\Psi_2}{dx^2}-l_{pl}^2\frac{d^4\Psi_2}{dx^4}+k^2\Psi_2=0:\quad x>0, \qquad (19)$$

where $k^2=\frac{2m}{\hbar^2}E$. Following the method of [8] in solving this fourth-order differential equation in a general way by neglecting the exponentially decaying term ($\sim e^{-|x|/l_{\rm pl}}$) from the wave function due to the quick drop for a very small value of $l_{\rm pl}$ or Planck length, we find the following physical solution set for both of the regions:

$$\Psi_1=Ae^{ik'x}+Be^{-ik'x}:\quad x<0, \qquad (20)$$

$$\Psi_2=Fe^{ik'x}+Ge^{-ik'x}:\quad x>0, \qquad (21)$$

where $k'=k(1-\beta\hbar^2k^2)$. Applying the boundary conditions.

I. Ψ will be always continuous at $x=0$;

$$\Psi_1(0)=\Psi_2(0)\Rightarrow A+B=F+G. \qquad (22)$$

II. $\frac{d\Psi}{dx}$ is also continuous except at the point $x = 0$, where the potential is infinite. Now, integrating the GUP-modified TISE from $-\epsilon$ to $+\epsilon$ and later setting the limit of ϵ tends to zero, we get:

$$\lim_{\epsilon\to 0}\Bigg[\int_{-\epsilon}^{\epsilon}\frac{d^2\Psi}{dx^2}\,dx - l_{pl}^2\int_{-\epsilon}^{\epsilon}\frac{d^4\Psi}{dx^4}\,dx + \\ + \frac{2m}{\hbar^2}\int_{-\epsilon}^{\epsilon}(E+\alpha\delta(x))\Psi(x)\,dx\Bigg] = 0.$$

So,

$$\lim_{\epsilon\to 0}\triangle\left(\frac{d\Psi}{dx}\right) - l_{pl}^2\lim_{\epsilon\to 0}\triangle\left(\frac{d^3\Psi}{dx^3}\right) = \\ = -\frac{2m\alpha}{\hbar^2}\lim_{\epsilon\to 0}\Bigg[\int_{-\epsilon}^{\epsilon}\delta(x)\Psi(x)\,dx\Bigg].$$

Equation (11) gives

$$\lim_{\epsilon\to 0}\triangle\left(\frac{d\Psi}{dx}\right) - l_{pl}^2\lim_{\epsilon\to 0}\triangle\left(\frac{d^3\Psi}{dx^3}\right) = -\frac{2m\alpha}{\hbar^2}\Psi(0), \quad (23)$$

where

$$\lim_{\epsilon\to 0}\triangle\left(\frac{d\Psi}{dx}\right) = i\,k'(F-G-A+B);$$

$$\lim_{\epsilon\to 0}\triangle\left(\frac{d^3\Psi}{dx^3}\right) = i\,k'^3(F-G-A+B).$$

Meanwhile, $\Psi_0 = A + B$. So, we get

$$(F-G-A+B) = i\,\frac{2m\alpha}{\hbar^2 k'}\,\frac{1}{(1+l_{pl}^2k'^2)}(A+B). \quad (24)$$

Let us consider $\xi = \frac{m\alpha}{\hbar^2 k'}\frac{1}{(1+l_{pl}^2k'^2)}$,

$$F - G = A(1+i2\xi) - B(1-i\,2\xi). \quad (25)$$

Here, A is the amplitude of the incident wave, B is the amplitude of the reflected wave, and F is the amplitude of the transmitted wave. As there is no reflection of the wave from the right, G will be zero: $G = 0$.

Solving Eqs. (22) and (25), we obtain

$$B = \frac{i\xi}{1-i\xi}A; \quad F = \frac{1}{1-i\xi}A. \quad (26)$$

Now, the reflection coefficient ($R_{\rm GUP}$) and transmission coefficient ($T_{\rm GUP}$) are

$$R_{\rm GUP} = \left(\frac{B}{A}\right)^*\left(\frac{B}{A}\right) = \frac{\xi^2}{1+\xi^2} = \frac{1}{1+\frac{\hbar^4(k'+l_{pl}^2k'^3)^2}{m^2\alpha^2}}, \quad (27)$$

$$T_{\rm GUP} = \left(\frac{F}{A}\right)^*\left(\frac{F}{A}\right) = \frac{1}{1+\xi^2} = \frac{1}{1+\frac{m^2\alpha^2}{\hbar^4(k'+l_{pl}^2k'^3)^2}}. \quad (28)$$

The sum of these two probabilities

$$R_{\rm GUP} + T_{\rm GUP} = 1.$$

Now, if we take some approximation due to the very small value of $l_{\rm pl}$, we find the expressions of the above probabilities:

$$R_{\rm GUP} = \frac{1}{1+\frac{\hbar^4(k^2+l_{pl}^2k^4-4l_{pl}^4k^6)}{m^2\alpha^2}}, \quad (29)$$

$$T_{\rm GUP} = \frac{1}{1+\frac{m^2\alpha^2}{\hbar^4(k^2+l_{pl}^2k^4-4l_{pl}^4k^6)}}. \quad (30)$$

From the above expressions, we can get more than one value of the energy for which $R_{\rm GUP}$ becomes 1, and $T_{\rm GUP}$ becomes 0, as $k' = k(1-\beta\hbar^2k^2)$ and $k^2 = \frac{2m}{\hbar^2}E$. To find those values, we are interested to use only the accurate expression of Eqs. (27) and (28). From them, we found that there is a drastic change in the probabilities for a specific value of the energy, $E_c = \frac{1}{2m\beta}$ for the GUP cases. Furthermore, for $\beta = 0$, these two probabilities are reduced to the non-GUP case [12] with

$$R = \frac{1}{1+\frac{2\hbar^2E}{m\alpha^2}}, \quad (31)$$

$$T = \frac{1}{1+\frac{m\alpha^2}{2\hbar^2E}}. \quad (32)$$

To compare the relative behaviour of these probabilities, we represent Eqs. (27) and (28) in Figs. 1 and 2 using Planck units with $\hbar = 1$, $m = 1$, $\alpha = 1$ with the energy on the horizontal axis.

3. GUP and Double Delta-Function Potential

Let us consider the equation with the attractive double delta-function potential well as

$$V(x) = -\alpha\,[\delta(x+a) + \delta\,(x-a)], \quad (33)$$

where α is the same positive constant that has the dimension of [energy $\times$ length]. Now, we have to solve the GUP-modified time-independent Schrödinger equation (9) following the method given in [8]. First, we consider the bound state ($E < 0$). The solution takes the form

$$\Psi(x) = \begin{cases} Ae^{-k'x}: & x > a, \\ Be^{k'x} + Ce^{-k'x}: & -a \le x \le a, \\ De^{k'x}: & x < -a, \end{cases} \tag{34}$$

where $k' = k(1 + \beta\hbar^2 k^2)$, and $k \equiv \frac{\sqrt{-2mE}}{\hbar}$. Now, in view of the boundary condition, the wave function Ψ will be continuous at $x = \pm a$, while there must be a discontinuity at $x = \pm a$ for the derivatives of the wave function: $\left(\frac{d\Psi}{dx}\right)$. Applying the boundary conditions, we have

$$\begin{cases} Ae^{-k'a} = Be^{k'a} + Ce^{-k'a}, \\ De^{-k'a} = Be^{-k'a} + Ce^{k'a}, \\ Be^{k'a} - Ce^{-k'a} = (2\gamma - 1)Ae^{-k'a}, \\ -De^{-k'a} + (Be^{-k'a} - Ce^{k'a}) = (1 - 2\gamma)De^{-k'a}, \end{cases}$$

where $\gamma = \frac{m\alpha}{\hbar^2(k' - 2\beta\hbar^2 k'^3)}$. Solving this set of equations, we get

$$C^2 = B^2 \Rightarrow C = \pm B. \tag{35}$$

Here, the parity comes due to the symmetry of the potential. So, we have both even and odd solutions. For the even solution ($C = B$), we get $D = A$, and the corresponding solution is given by

$$\Psi(x) = \begin{cases} Ae^{-k'x}: & x > a, \\ B(e^{k'x} + e^{-k'x}): & -a \le x \le a, \\ Ae^{k'x}: & x < -a. \end{cases} \tag{36}$$

Applying the boundary condition, this set of even solutions gives a transcendental equation

$$e^{-2k'a} = \frac{1}{\gamma} - 1 = \frac{\hbar^2 k'}{m\alpha} - \frac{2\beta\hbar^4 k'^3}{m\alpha} - 1. \tag{37}$$

For $2k'a = z$, Eq. (37) becomes

$$e^{-z} = cz - pz^3 - 1, \tag{38}$$

with $c = \frac{\hbar^2}{2am\alpha}$, $p = \frac{\beta\hbar^2}{2a^2}c$. These coefficients become unity at the value of $\alpha = \frac{\hbar^2}{2ma}$ and $\beta = \frac{2a^2}{\hbar^2}$. Now, Eq. (38) becomes

$$e^{-z} = z - z^3 - 1. \tag{39}$$

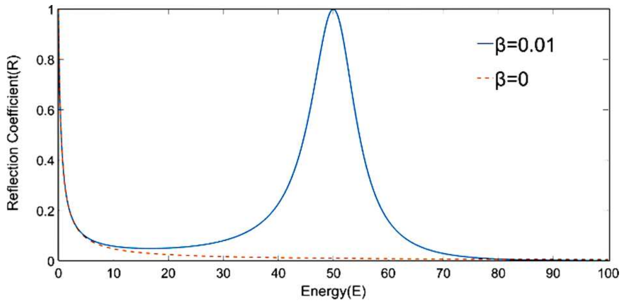

Fig. 1. Reflection coefficient for $\beta = 0.01$ (blue line) and $\beta = 0$ (red line) [Eq. (27)]

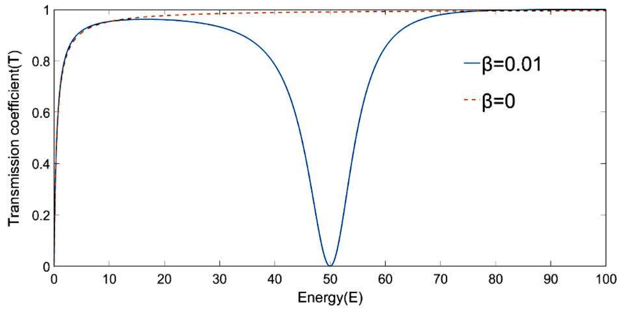

Fig. 2. Transmission coefficient for $\beta = 0.01$ (solid line) and $\beta = 0$ (red line) [Eq. (28)]

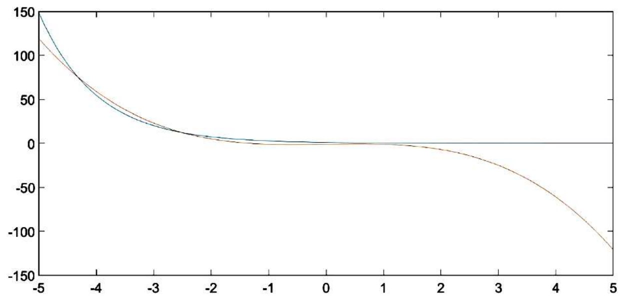

Fig. 3. Graphs of Eq. (39) taking e^{-z}(blue) and $z - z^3 - 1$ (red)

(See Fig. 3). Solving the above equation, we get two approximate solutions:

$$z_1 \approx -4.325: \quad z_2 \approx -2.51.$$

Further, in view of Eq. (38), we note that four possibilities depend on the values of positive parameters c and p. Now, we can find the proper solutions of that equation as (i) $c < 1$ and $p < 1 (c < p$ or $c > p)$, (ii) $c < 1$ and $p > 1$, (iii) $c > 1$ and $p < 1$ and (iv) $c > 1$ and $p > 1 (c < p$ or $c > p)$. Inspecting each of the above cases, we found that there is no possible

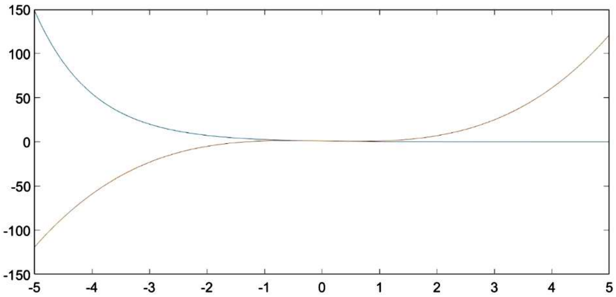

Fig. 4. Graphs of Eq. (43) taking e^{-z} (blue) and $1-z+$ $+z^3$ (red)

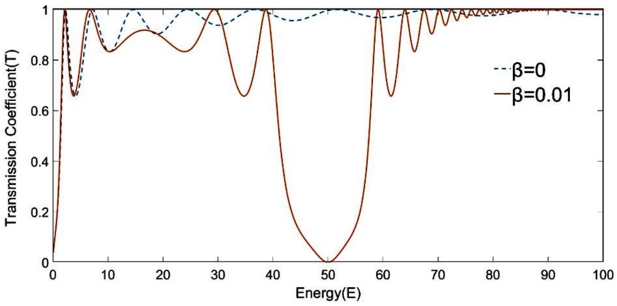

Fig. 5. Transmission coefficient for $\beta = 0.01$ (blue line) GUP case and $\beta = 0$ (red line) non-GUP case [Eq. (45)]

solution of Eq. (38) for case (i), but we have two real possible solutions for the other three remaining cases (ii), (iii), and (iv).

The bound-state energy for the even wave function can be calculated from the equation $2k'a = z$,

$$\left(\frac{8\beta m^2}{\hbar^2}\right) E^2 - \left(\frac{2m}{\hbar^2}\right) E - \frac{z^2}{4a^2} \approx 0. \tag{40}$$

Solving the above quadratic equation, we have only one possible bound-state energy, which is

$$E \approx -\frac{z^2\hbar^2}{8ma^2}. \tag{41}$$

Now, let us consider Eq. (38). The possible energy eigenvalues of the even wave function for the double delta-function potential well are:

$$E_1^{\text{even}} = -(4.325)^2 \frac{\hbar^2}{8ma^2}, \quad E_2^{\text{even}} = -(2.51)^2 \frac{\hbar^2}{8ma^2}.$$

Similarly, for the odd solution ($C = -B$), we get $D = -A$, and the corresponding solution is given by

$$\Psi(x) = \begin{cases} Ae^{-k'x}: & x > a, \\ B(e^{k'x} + e^{-k'x}): & -a \le x \le a, \\ -Ae^{k'x}: & x < -a. \end{cases} \tag{42}$$

With regard for the boundary conditions, this set of odd solutions gives a transcendental equation

$$e^{-z} = 1 - cz + pz^3; \quad z = 2k'a. \tag{43}$$

Putting the same conditions as previously, we can again inspect each of the cases. Then we found that there is only one possible solution of Eq. (43) for $c < 1$, and we have three possible solutions for $c > 1$. For $c = 1$ and $p = 1$, Eq. (43) has the solution: $z = 0$. So, for odd one, the possible energy eigenvalue is: $E^{\text{odd}} = 0$.

Similarly to the above case, if we go to the scattering state and solve Eq. (9), we can get the solution as:

$$\Psi(x) = \begin{cases} Fe^{ik'x} + Ge^{-ik'x}: & x < -a, \\ He^{ik'x} + Ie^{-k'x}: & -a \le x \le a, \\ Je^{ik'x}: & x > a, \end{cases} \tag{44}$$

where $k' = k(1-\beta\hbar^2k^2)$. Applying the boundary conditions, we get the transmission coefficient

$$T_{\text{GUP}} = \frac{8g'^4}{(8g'^4+4g'^2+1) + (4g'^2-1)\cos\phi - 4g'\sin\phi}, \tag{45}$$

where

$$g' \equiv \frac{\hbar^2(k' + 2\beta\hbar^2k'^3)}{2m\alpha} \approx \frac{\hbar^2(k + \beta\hbar^2k^3 - 6\beta^2\hbar^4k^5)}{2m\alpha}$$

and

$$\phi \equiv 4a(k' + 2\beta\hbar^2k'^3) \approx 4a(k + 2\beta\hbar^2k^3 - 6\beta^2\hbar^4k^5).$$

Here, we also find a similar drop in the transmission coefficient, as we got it under the variation of the single delta-function. The energy is $E_c = \frac{1}{2m\beta}$ for GUP.

To compare the relative behavior of T_{GUP}, we plot the Eq. (45) in Fig. 5 using Planck units with $\hbar = 1$, $m = 1$, $\alpha = 1$ (with the energy on the horizontal axis).

4. Conclusions

In this paper, we have tried to explore quantum gravity effects through GUP with consideration of the momentum equation (6) which is based on non-local quantum mechanics. Using the GUP-modified time-independent Schrödinger equation, the delta-function potential has been used to find the bound-state energy and to derive the scattering-state transmission coefficient. Later, as a special case, the GUP-shifted bound state energy and GUP-shifted scattering-state transmission coefficient have been calculated for the double delta-function potential well. We have found an interesting result that, at a particular value of the energy, $E_c = \frac{1}{2m\beta}$, the probabilities change drastically for both the cases. We hope that this modification will help the scientific community in their future work on GUP and will give some more interesting results.

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УЗАГАЛЬНЕНИЙ ПРИНЦИП
НЕВИЗНАЧЕНОСТІ ТА ПОТЕНЦІАЛ
У ВИГЛЯДІ ДЕЛЬТА-ФУНКЦІЇ

Нещодавно принцип невизначеності Гайзенберга було узагальнено (УПН) для того, щоб пояснити гравітацію в межах квантової механіки. Використовуючи УПН та методи нерелятивістичної квантової механіки, ми знайшли власні значення енергії зв'язаних станів для потенціальної ями, що має вигляд дельта-функції або подвійної дельта-функції. Отримано ймовірності проходження для станів розсіювання та проведено їх порівняння з результатами, отриманими у випадках без модифікації принципу невизначеності для обох систем

Ключові слова: узагальнений принцип невизначеності, потенціальна яма у вигляді дельта-функції, мінімальна довжина.