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INFLUENCE OF ION VISCOSITY ON THE DISTRIBUTIONS OF PLASMA PARAMETERS IN STATIONARY GAS DISCHARGE

On the basis of hydrodynamic equations, the distributions of such plasma parameters as the electric potential, the ion and electron densities, and the ion flow velocity toward the wall in a plane layer of the stationary weakly ionized non-isothermal plasma confined between the dielectric walls have been obtained. The temperatures of ions and electrons, as well as the density of neutrals, are assumed to be constant. Instead of finding the eigenfunctions and eigenvalues of the described problem, the Cauchy problem is solved with the initial values corresponding to those that are in the plasma bulk. The wall position is determined from the balance condition for the ion and electron fluxes. A method to avoid the singularity in the system of hydrodynamic equations has been proposed. The influence of the ion viscosity in the equation of ion motion was estimated. The distributions of plasma parameters are obtained considering the ion viscosity in a quasineutral region.

K e y w o r d s: stationary gas discharge, viscosity, transient layer, hydrodynamic approximation, Debye radius.

1. Introduction

The problem of the steady state of a gas discharge has been studied for about a century. The interaction of plasma with surrounding surfaces occupies a special place in those researches. The interaction of plasma with a confining wall can be simply described as follows. Owing to a high mobility of electrons, the wall potential becomes negative with respect to the surrounding plasma. The repulsion of electrons leads to the formation of a region with a positive bulk charge that screens neutral plasma from the negatively charged wall. A typical width of this region is determined by a few Debye–Hückel screening radii of electrons, $r_{\mathrm{De}} = \sqrt{T_e/(4\pi e^2 n_{e0})}$, where T_e is the temperature of electrons, e the elementary charge, and n_{e0} the hydrodynamic concentration of electrons in the plasma bulk.

As a rule, the electron Debye radius r_{De} is small in comparison with other characteristic lengths such as the plasma size L or the free path of ions, which are determined by the ionization, recharging, and collision processes. In this case, plasma can be divided into two regions: a quasineutral region (the presheath) with the characteristic dimension L and the transient layer (the sheath) with the characteristic thickness r_{De}.

As was shown by Bohm [1], the formation of a stationary bulk charge layer is possible, only if the ions enter the transient layer region at a velocity not lower than the velocity of ion sound $v_{\mathrm{B}} = v_s = \sqrt{T_e/m_i}$, where T_i and m_i are the temperature and mass of ions, respectively. In the case of non-isothermal plasma ($T_e > T_i$), v_s is larger than the thermal velocity of ions. This condition was obtained in the case of cold ions ($T_i = 0$). At $T_i \neq 0$, the Bohm velocity equals $v_{\mathrm{B}} = v_s\sqrt{1+\tau}$, where $\tau = T_i/T_e \leqslant 1$. Hence, the ions are pre-accelerated by a self-consistent electric field in the quasineutral region.

If the ions move to the plasma-confining surfaces under the action of a self-consistent electric field, the potential in plasma must have a maximum. In the plane case, which is considered in this work, for the symmetry reasons, this maximum should be located in the middle plane of plasma. This plane is convenient to be chosen as the coordinate origin, $x = 0$. Then the dielectric walls confining plasma are located at $x = \pm L$. A generalization to the cylindrical or spherical case is not difficult.

Among the pioneering works on this issue, the most cited are works [2, 3]. The corresponding formulation of the problem is still often used. In work [2], it was assumed for the first time that the ion velocity is determined by a static self-consistent electric field, which is supported by the charge balance between electrons and ions. An integral equation describing the plasma-layer potential distribution was derived for various geometries, ion free path lengths, and ionization methods. The solution of this equation in the case of a short free path of ions in a cylinder with the ion generation proportional to the electron concentration brought about the same potential distribution as Schottky had found for a positive column in the framework of the ambipolar diffusion theory [3].

Among the later works, in which the hydrodynamic approximation was used, works [4, 5] should be mentioned. In work [4], the problem was solved analytically for the first time in the quasineutral region making allowance for the inertia of ions. In work [5], the dependence of the ionic mobility on the magnitude of the self-consistent field was taken into consideration.

The authors of many works (see, e.g., work [6]) assume that there are no collisions in plasma in the transient layer because of its small thickness as compared to the electron free path length. So, the terms describing collisions in the equations of ion motion and the continuity equations can be neglected. As a result, the corresponding solutions turn out simpler. However, there arise substantial difficulties associated with their matching with the solution in the quasineutral region. Numerous references can be found in works by Riemann [7, 8] who made a large contribution to the study of this problem.

In work [7], the problem of matching the solutions obtained for the quasineutral plasma region and the transient layer was considered. It was solved both analytically and numerically by explicitly constructing a consistent asymptotic expression and comparing it with exact solutions obtained for the hydrodynamic plane Tonks–Langmuir problem in the limiting case $r_{\mathrm{De}}/L \to 0$ ($T_i = 0$). However, the system of equations used in work [7] in the case of cold ions ($T_i = 0$) possesses a singularity in the middle of plasma layer, at $x = 0$. To overcome this difficulty, the dimensionless plasma potential Φ, the dimensionless concentration n, and the velocity of ions v were expanded in a vicinity of $x = 0$ in the following power series in the parameter $\bar{x} = (x\alpha_e)/v_s$, where α_e is the ionization frequency:

$$\Phi = (a_0 + a_{(2)}\bar{x}^2 + ...)\bar{x}^2,$$

$$n = c_0 + c_{(2)}\bar{x}^2 + ...,$$

$$v = (b_0 + b_{(2)}\bar{x}^2 + ...)\bar{x}.$$

The corresponding coefficients were determined in work [7] by substituting those series into the motion and continuity equations and grouping the terms with identical power exponents of $\bar{x}$.

In most works, the ion viscosity is considered to be a small parameter, so it is not taken into consideration in the equation of ionic motion. At the same time, there is no justification that such a viscosity neglecting is correct. In work [9], the following condition allowing the viscosity-associated effects in the transport equations to be neglected was presented: $v_i \ll \nu L_v$, where v_i is the hydrodynamic velocity of ions, ν the collision frequency, and L_v a characteristic scale of hydrodynamic velocity variations. This condition is definitely obeyed in the quasineutral region, but its validity for the transient layer remains uncertain.

The presented work is devoted to the elucidation of the role of the ion viscosity in the formation of distributions of plasma parameters in stationary gas discharges. Its structure is as follows. In Section 2, the problem is formulated, and the basic equations are derived. In Section 3, the basic system of equations is solved without taking viscosity into account. The quasineutral approximation is applied to the quasineutral region. In Section 4, the influence of the ion viscosity on the distributions of parameters in the nonisothermal weakly ionized stationary gas discharge plasma is analyzed using the solutions obtained in the previous section. Solutions with viscosity are also obtained for the quasineutral region. General conclusions are made in Section 5.

2. Basic Equations

In order to solve the problem of the stationary distribution of plasma parameters in gas discharges, the hydrodynamic approximation will be used. This approach can be applied when the macroscopic plasma parameters, such as the hydrodynamic velocity v and the particle concentration n, vary rather slowly in space and time; namely, the characteristic distances at which the valuesof macroscopic quantitieschange should be much longer than the average free path length. The hydrodynamic approximation is also valid in the case of collisionless plasma, if the thermal motion of the particles can be neglected, i.e. plasma must be sufficiently cold [10]. However, even if those conditions are not satisfied, the hydrodynamic approach can be used for the qualitative analysis of plasma parameters.

The general continuity and motion equations for a unit volume with gas particles of the l-th kind look like [9]

$$\frac{\partial n_l}{\partial t} + \nabla\left(n_l \mathbf{v}_l\right) = \frac{\delta n_l}{\delta t}, \tag{1}$$

$$m_l n_l \left[\frac{\partial v_{l\alpha}}{\partial t} + \left(\mathbf{v}_l \nabla\right) v_{l\alpha}\right] = Zen_l E_\alpha -$$
$$- \frac{\partial p_l}{\partial x_\alpha} - \frac{\partial \pi_{l\alpha\beta}}{\partial x_\beta} + m_l n_l \frac{\delta v_{l\alpha}}{\delta t}, \tag{2}$$

where $p_l = n_l T_l$; $E_\alpha = -\partial\varphi/\partial x_\alpha$; v_l and n_l are the hydrodynamic velocity and the concentration, respectively, of the particles of the l-th kind, E is the self-consistent electric field, and φ is its the potential; p_l, T_l, m_l, and $\pi_{l\alpha\beta}$ are the pressure, temperature, mass, and tensor of viscous stresses, respectively, for the l-th particles; Z is the ion charge, ∇ the differentiation operator, and the subscripts α and β denote the coordinate components x_α and x_β. The second term on the left-hand side of Eq. (1) describes a change in the concentration of particles owing to their hydrodynamic directional motion, whereas the right-hand side describes the appearance or disappearance of particles as a result of collision processes. The second term on the left-hand side of Eq. (2) describes the variation of the particle velocity as a result of the plasma displacement, whereas the right-hand side represents the force acting on the particles of the l-th kind in a unit volume. Its first component is the electric force, the second one is the force arising owing to the pressure gradient of the l-th plasma particles, the third one is the force arising under the action of viscous stresses, and the fourth one is the friction force associated with collision processes.

We suppose that plasma consists of electrons and ions of atomic hydrogen. Let us consider the stationary variant of the problem ($\partial/\partial t = 0$). Let the magnetic field be absent ($\mathbf{H} = 0$), and let the temperatures of the plasma particles and the concentration of hydrogen atoms do not depend on coordinates [1]. The particle concentration changes due to the electron-impact ionization,

$$\frac{\delta n_{i,e}}{\delta t} = \alpha_e n_e,$$

and the hydrodynamic velocity of ions due to the recharging and ionization processes,

$$\frac{\delta v_{i\alpha}}{\delta t} = \left(\nu_{\mathrm{ex}} + \frac{\alpha_e n_e}{n_i}\right) v_{i\alpha},$$

where n_i and n_e are the hydrodynamic concentrations of ions and electrons, respectively; and ν_{ex} is the frequency of the ion recharging at hydrogen atoms. The electron concentration is assumed to be determined by the Boltzmann formula $n_e = n_{e0}\exp(e\varphi/T_e)$. This formula can be easily obtained from Eq. (2), if we divide the latter by the characteristic values of parameters and neglect the terms containing the electron-to-ion mass ratio m_e/m_i. We also suppose that, when reaching the plasma–insulator boundary (the wall), the electrons and ions completely recombine.

In the absence of a magnetic field, the ion viscous stress tensor $\pi_{i\alpha\beta}$ is expressed via the shear rate tensor $W_{\alpha\beta}$ as follows:

$$\pi_{i\alpha\beta} = -\eta_i W_{\alpha\beta},$$
$$W_{\alpha\beta} = \frac{\partial v_{i\alpha}}{\partial x_\beta} + \frac{\partial v_{i\beta}}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\frac{\partial v_{i\gamma}}{\partial x_\gamma}, \tag{3}$$

where η_i is the dynamic ion viscosity coefficient, $\delta_{\alpha\beta}$ the Kronecker delta, and the subscript γ means the relation to the coordinate component x_γ. Let us divide the equation of motion (2) by the product $m_i n_i$. Then, in the one-dimensional case, the motion, continuity, and Poisson equations can be written in

[1] If the particle temperature is determined self-consistently, it is necessary to consider the equation of thermal conductivity for particles in the presence of cooling and heating sources, which makes the problem substantially more complicated. Moreover, to a great extent, the problem solution will be determined by the method of plasma heating.

the form

$$v_i \frac{dv_i}{dx} = -\frac{e}{m_i}\frac{d\varphi}{dx} - \left(\nu_{\rm ex} + \alpha_e \frac{n_{e0}\exp(e\varphi/T_e)}{n_i}\right) v_i -$$

$$- \frac{T_i}{m_i n_i}\frac{dn_i}{dx} + \frac{4}{3}\bar{\eta}_i \frac{d^2 v_i}{dx^2}, \qquad (4)$$

$$\frac{d(n_i v_i)}{dx} = \alpha_e n_{e0}\exp(e\varphi/T_e), \qquad (5)$$

$$\frac{d^2\varphi}{dx^2} = 4\pi e\,(n_{e0}\exp(e\varphi/T_e) - n_i), \qquad (6)$$

respectively, where $\bar{\eta}_i = \eta_i/(m_i n_i)$ is the coefficient of kinematic ion viscosity [11].

The recharging and ionization frequencies were calculated making use of the relations $\alpha_e = \sigma_i v_{Te} n_n$ and $\nu_{\rm ex} = \sigma_{\rm ex} v_{Ti} n_n$, where σ_i and $\sigma_{\rm ex}$ are the cross-sections of electron-impact ionization and ion recharging at hydrogen atoms, respectively; $v_{Te} = \sqrt{T_e/m_e}$ and $v_{Ti} = \sqrt{T_i/m_i}$ are the thermal velocities of electrons and ions, respectively; and n_n is the concentration of neutral particles. The dependence of the average product of the ionization cross-section σ_i (in cm^{-3}/s units) and the thermal velocity of electrons on their temperature T_e (in eV units) is determined by the formula

$$\overline{\sigma_i v_e} = 10^{-5}\frac{\Theta^{1/2}}{I^{3/2}\,(6+\Theta)}\exp\left(-\Theta^{-1}\right),$$

where $I = 13.6$ eV and $\Theta = T_e/I$ [12]. The quantity $\sigma_{\rm ex}$ is determined by the formula [13]

$$\sigma_{\rm ex} = 4.9\times 10^{-15}\left[1 + 0.15\times\ln\left(\frac{1}{T_i}\right)\right]^2\ {\rm cm}^2,$$

where T_i is reckoned in electronvolt units.

The coefficient of kinematic ion viscosity is equal to the product of the characteristic ion velocity v_{Ti} and the free path length l (see, e.g., work [9]),

$$\bar{\eta}_i = v_{Ti} l = \frac{v_{Ti}^2}{\alpha_e + \nu_{\rm ex} + \nu_{\rm Ci}},$$

where $\nu_{\rm Ci} = \sigma_{\rm Ci} v_{Ti} n_{i0}$ is the frequency of Coulomb ion collisions, n_{i0} the hydrodynamic ion concentration at $x = 0$, $\sigma_{\rm Ci} = \pi(e^2/T_i)^2 \Lambda_i$ is the cross-section of Coulomb ion collisions, $\Lambda_i = \ln(r_{Di}/r_{\min\ i})$ is the Coulomb logarithm, $r_{\rm Di} = \sqrt{T_i/(4\pi e^2 n_{i0})}$ is the ion

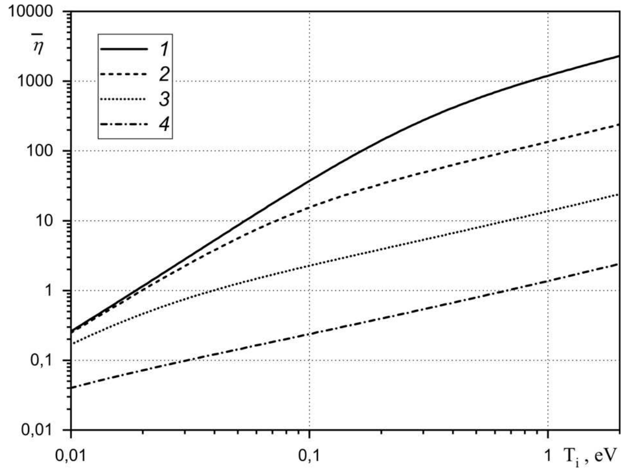

Fig. 1. Dependences of the dimensionless kinematic ion viscosity η on the ion temperature T_i (in electronvolts) calculated for $n_{e0} = 10^{10}$ cm^{-3}, $T_e = 2$ eV, and various concentrations of neutral particles $n_n = 10^{13}$ (*1*), 10^{14} (*2*), 10^{15} (*3*), and 10^{16} cm^{-3} (*4*)

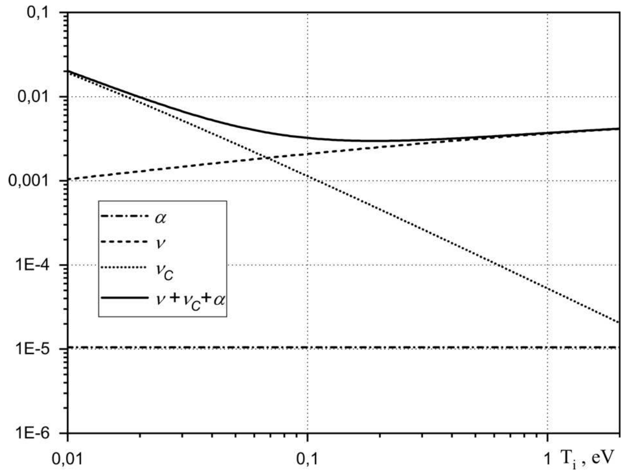

Fig. 2. Dependences of the dimensionless frequencies of Coulomb collisions, ν_C, recharging, ν, and ionization, α, on the ion temperature T_i calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, and $T_e = 2$ eV

Debye radius, and $r_{\min\ i} = e^2/T_i$ is the impact parameter for the short-distance interaction.

In Fig. 1, the dependences of the dimensionless kinematic ion viscosity $\bar{\eta} = \bar{\eta}_i/(v_s r_{\rm De}) = \tau/(\nu + \nu_C + \alpha)$ on the ion temperature T_i calculated for $T_e = 2$ eV and various concentrations of neutral particles n_n are depicted. Here, $\alpha = \alpha_e/\omega_{pi}$, $\nu = \nu_{\rm ex}/\omega_{pi}$, and $\nu_C = \nu_{\rm Ci}/\omega_{pi}$ are the dimensionless frequencies, and $\omega_{pi} = v_s/r_{\rm De}$ is the ion plasma frequency. Note

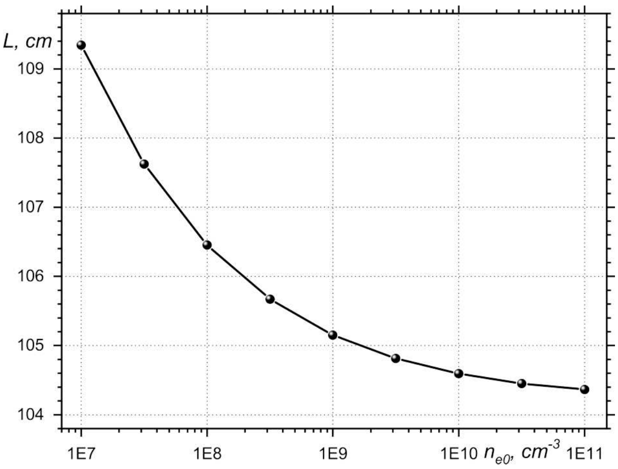

Fig. 3. Dependence of the plasma size L on the electron concentration at the plasma center calculated for $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV

that the kinematic ion viscosity increases considerably with the growth of T_i and the reduction of n_n.

In Fig. 2, the dependences of the dimensionless frequencies of Coulomb collisions, ν_C, recharging, ν, and ionization, α, on the ion temperature T_i calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, and $T_e = 2$ eV are shown. This figure makes it possible to estimate which of the processes gives the main contribution to the dimensionless parameter of the kinematic ion viscosity. As the concentration n_n of neutral particles decreases, the point where the plots for the recharging, ν, and Coulomb collision, ν_C, frequencies intersect becomes shifted toward higher ion temperatures. The figure makes it evident that the main contribution to η is given by the recharging frequency ν, if the ion temperature T_i is high and by the Coulomb collision frequency ν_C at very low T_i's. At the same time, the electron-impact ionization frequency α weakly affects the dimensionless kinematic ion viscosity at any T_i-value.

Let us change to the dimensionless variables $v = v_i/v_s$, $n = n_i/n_{e0}$, and $\Phi = e\varphi/T_e$ in the set of equations (4)–(6). As a result, we obtain

$$vv' = -\Phi' - \left(\nu + \alpha\frac{\exp(\Phi)}{n}\right)v - \tau\frac{n'}{n} + \frac{4}{3}\bar{\eta}v'', \quad (7)$$

$$n'v + nv' = \alpha\exp(\Phi), \quad (8)$$

$$\Phi'' = \exp(\Phi) - n, \quad (9)$$

where the primed quantities mean their derivative with respect to the dimensionless coordinate x/r_{De}.

The system of differential equations (7)–(9), in which the ion viscosity is neglected, is a nonlinear fourth-order system for the sought functions. It must be supplied with boundary conditions. For reasons of symmetry at the plasma center ($x = 0$), we have

$$v(0) = n'(0) = \Phi'(0) = 0. \quad (10)$$

We should also remember that the potential is defined with accuracy to an additive constant. Let us take $\Phi(0) = 0$. Another boundary condition concerns the plasma-wall boundary: the hydrodynamic flow of ions is equal to the flow of electrons in the direction of growing x. Besides that, it is assumed that the electrons are distributed according to the Maxwell–Boltzmann law, whereas the effects of electron reflection from the wall and electron emission by the wall are absent [14]. As a result, we obtain

$$\Gamma(L) = n(L)v(L) = \sqrt{\frac{m_i}{2\pi m_e}}\exp\left(\Phi(L)\right). \quad (11)$$

Hence, we have a system of fourth-order equations with four boundary conditions, i.e. an eigenvalue problem. For instance, if the parameters L, T_e, T_i, and n_n are known, the stationary gas discharge is possible at a certain value of n_{e0}, which has to be determined by solving the indicated system.

3. Solution of the System of Basic Equations in the Quasineutral Approximation

Finding the eigenfunctions and eigenvalues for the nonlinear system of equations (7)–(9) is a rather complicated task. Therefore, it was analyzed with the help of an alternative approach. Boundary conditions (10) were considered as initial ones. They were appended with an arbitrary initial n_{e0}-value, and system (7)–(9) was integrated in the direction of positive x's, i.e. the Cauchy problem was solved (see, e.g., works [7, 15], where the case of cold ions was considered). Integration was terminated at a point $x = \hat{L}$, where condition (11) is satisfied. As a rule, $\hat{L} \neq L$ for the first attempt. Then, the n_{e0}-value was changed, and a new integration was carried out. The procedure was repeated until $\hat{L} = L$ was obtained (see Fig. 3). This method (the so-called shooting method) was also used below when analyzing the influence of the ion viscosity on the distribution of parameters in stationary discharge plasma.

Let us apply Eq. (7) with the excluded last term and Eq. (8) to determine the dimensionless derivatives of the ion velocity and concentration. We obtain

$$v' = -\frac{\Phi' v + \left(\nu + \alpha\frac{\exp(\Phi)}{n}\right) v^2 + \tau\alpha\frac{\exp(\Phi)}{n}}{v^2 - \tau}, \tag{12}$$

$$n' = \frac{\left[\Phi' + \left(\nu + \alpha\frac{\exp(\Phi)}{n}\right) v\right] n}{v^2 - \tau}. \tag{13}$$

One can see that this system of equations has a singularity at the point, where the hydrodynamic velocity of ions and their thermal velocity are identical, $v = \sqrt{\tau}$. In the case of strongly non-isothermal plasma, the singularity point is located in the quasineutral region, where the plasma parameters vary at distances of the order of plasma size, $\Delta x \sim L$, i.e. rather far from the transient layer. The indicated singularity prohibits the direct integration of system (7)–(9).

To avoid difficulties associated with the singularity, let us take advantage of the fact that the second derivative of the potential in Eq. (9) is small: $\Phi'' \ll n \exp \Phi$. Such an approach will be called the "quasineutral approximation". After the first iteration ($\Phi'' = 0$), from Eq. (9), we obtain

$$n_{(1)} = \exp \Phi_{(1)}, \quad \Phi'_{(1)} = \frac{n'_{(1)}}{n_{(1)}}. \tag{14}$$

Equation (14) makes it possible to exclude the potential from the system of equations (7) and (8) so that

$$v_{(1)}v'_{(1)} = -(1+\tau)\frac{n'_{(1)}}{n_{(1)}} - (\nu+\alpha)\, v_{(1)} + \frac{4}{3}\bar{\eta} v''_{(1)}, \tag{15}$$

$$n'_{(1)}v_{(1)} + n_{(1)}v'_{(1)} = \alpha n_{(1)}. \tag{16}$$

Equations (15) and (16) give us the dimensionless derivatives of the ion velocity and concentration without making allowance for the ion viscosity:

$$v'_{(1)} = \frac{(1+\tau)\alpha + (\nu+\alpha)v^2_{(1)}}{1+\tau - v^2_{(1)}}, \tag{17}$$

$$n'_{(1)} = -\frac{(\nu+2\alpha)n_{(1)}v_{(1)}}{1+\tau - v^2_{(1)}}. \tag{18}$$

From whence, one can see that the system of equations (17) and (18) has a singularity at a point, where the hydrodynamic ion velocity is equal to the Bohm velocity, $v_{(1)} = \sqrt{1+\tau}$. Therefore, we solve this system in the region $0 < v_{(1)} < \sqrt{1+\tau}$ using the Cauchy method with the initial conditions at the point $x = 0$. After passing the point $v_{(1)} = \sqrt{\tau}$, the determined values of $v_{(1)}$ and $n_{(1)}$, as well as the values of $\Phi_{(1)}$ and $\Phi'_{(1)}$ calculated using $v_{(1)}$ and $n_{(1)}$, are applied as the initial conditions for integrating Eqs. (7)–(9) in the interval $\sqrt{\tau} < v_{(1)} < \sqrt{1+\tau}$.

This method can be used to obtain a smooth matching for the main plasma parameters. However, substantial oscillations of the second derivative of the velocity take place in this case, whereas the values of this parameter are required to evaluate the ion viscosity. Therefore, in order to make the matching of the solutions of the systems of equations (9), (12), (13) and (17), (18) more accurate, the next iteration for the second derivative of the potential is used. In the second iteration, $\Phi'' = \Phi''_{(1)}$, and Eqs. (14)–(16) read

$$\exp \Phi_{(2)} = n_{(2)} + \Phi''_{(1)}, \qquad \Phi'_{(2)} = \frac{n'_{(2)} + \Phi'''_{(1)}}{n_{(2)} + \Phi''_{(1)}}, \tag{19}$$

$$v_{(2)}v'_{(2)} = -\frac{n'_{(2)} + \Phi'''_{(1)}}{n_{(2)} + \Phi''_{(1)}} - \left(\nu + \alpha\frac{n_{(2)} + \Phi''_{(1)}}{n_{(2)}}\right) v_{(2)} - \tau\frac{n'_{(2)}}{n_{(2)}} + \frac{4}{3}\bar{\eta} v''_{(2)}, \tag{20}$$

$$n'_{(2)}v_{(2)} + n_{(2)}v'_{(2)} = \alpha(n_{(2)} + \Phi''_{(1)}). \tag{21}$$

If the last term on the right hand side of Eq. (20) is omitted, we obtain

$$v'_{(2)} = \left\{\left(1 + \tau\frac{n_{(2)} + \Phi''_{(1)}}{n_{(2)}}\right)\alpha + \left(\nu + \alpha\frac{n_{(2)} + \Phi''_{(1)}}{n_{(2)}}\right) v^2_{(2)} + \frac{\Phi'''_{(1)}v_{(2)}}{n_{(2)} + \Phi''_{(1)}}\right\} \times \left\{n_{(2)}n_{(2)} + \Phi''_{(1)} + \tau - v^2_{(2)}\right\}^{-1}, \tag{22}$$

$$n'_{(2)} = \frac{\alpha(n_{(2)} + \Phi''_{(1)}) - n_{(2)}v'_{(2)}}{v_{(2)}}, \tag{23}$$

where $\Phi''_{(1)}$ and $\Phi'''_{(1)}$ are known functions. Note that, in the course of the second iteration, the singularity became slightly shifted toward the plasma-wall boundary. When the ion temperature approaches the

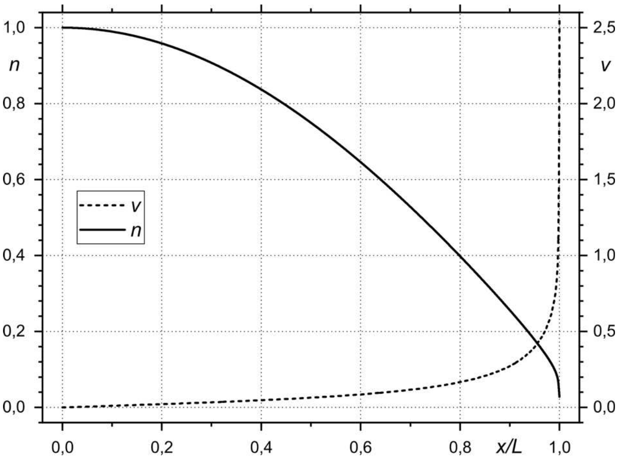

Fig. 4. Dependences of the dimensionless ion velocity v and concentration n on the x/L-coordinate calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV

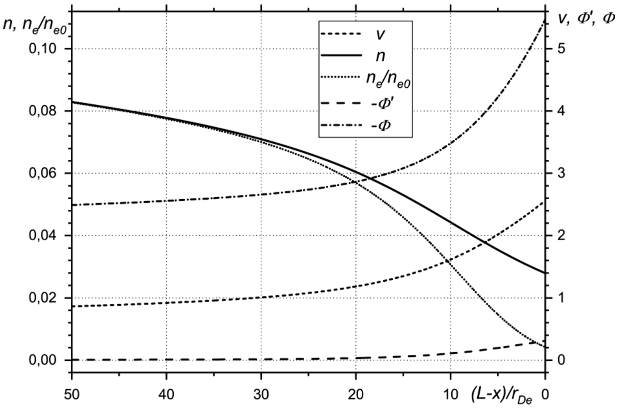

Fig. 5. Dependence of the dimensionless ion velocity v and concentration n, the electron concentration n_e/n_{e0}, the electric field $-\Phi'$, and the potential Φ in the transient layer on the coordinate $(L-x)/r_{De}$ calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV

electron temperature, the singularity points in the systems of equations (9), (12), (13) and (17), (18) come closer to each other, so further iterations in the framework of the quasineutral approximation are required.

Figure 4 illustrates the solution matching for the systems of equations (9), (12), (13) and (22), (23) in the whole plasma volume. The solutions were matched at the point $x/L = 0.85$. The solutions of the system of equations (22) and (23) are shown in the interval $0 < x/L < 0.85$; and the solutions of the system of equations (9), (12), and (13) in the interval $0.85 < x/L < 1$. The solutions of the system of equations (9), (12), and (13) in the transient layer are shown in Fig. 5. The following values of dimensional and dimensionless quantities were used in calculations: $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, $T_i = 0.1$ eV, $v_{Te} = 5.93 \times 10^7$ cm/s, $v_{Ti} = 3.09 \times 10^5$ cm/s, $v_s = 1.38 \times 10^6$ cm/s, $r_{\rm De} = 1.05 \times 10^{-2}$ cm, $\omega_{pi} = 1.32 \times 10^8$ s^{-1}, $\alpha_e = 1.39 \times 10^3$ s^{-1}, $\nu_{\rm ex} = 2.74 \times 10^5$ s^{-1}, $\nu_{{\rm C}i} = 1.49 \times 10^5$ s^{-1}, $\bar{\eta}_i = 2.25 \times 10^5$ cm^2/s, $L = 104.6$ cm, $\alpha = 1.05 \times 10^{-5}$, $\nu = 2.08 \times 10^{-3}$, $\nu_C = 1.13 \times 10^{-3}$, and $\bar{\eta} = 15.5$. The parameter values typical of a glow discharge were chosen. Note that, for a better perception, the distance is reckoned in the units of the plasma size L in Fig. 4 and in Debye radii $r_{\rm De}$ in Fig. 5. The reference point corresponds to the plasma-confining surface.

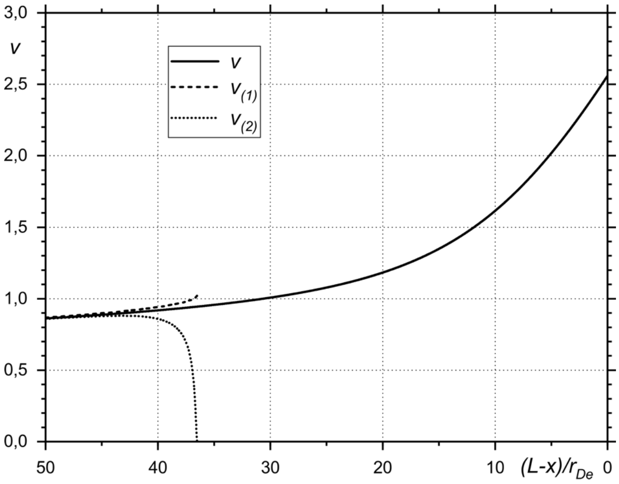

Fig. 6. Dependences of the dimensionless ion velocities v, $v_{(1)}$, and $v_{(2)}$ in the transient layer on the coordinate $(L-x)/r_{\rm De}$ calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV

From Figs. 4 and 5, one can see that the solution obtained for the systems of equations is continuous and has no peculiarities. Expectedly, the solutions change slowly in the quasineutral region. At the same time, in the transient layer, the size of which is approximately equal to 3 cm $[(25{\div}40) \times r_{De}]$, there appears a bulk charge and the plasma parameters vary drastically.

In addition to Figs. 4 and 5, Fig. 6 demonstrates the first and second iterations for the ion velocity. One can see that the iterations $v_{(1)}$ and $v_{(2)}$, which were calculated under the quasineutrality condition, begin to deviate from the "true" solution v at a distance of about $40r_{\rm De}$ from the plasma-confining surface. This is slightly larger than the transient-layer thickness, if the latter is defined as the distance from the surface to the point, where $v = (1+\tau)^{1/2}$. From the first iteration of the Poisson equation, we determine the velocity at which the quasineutrality approximation is violated. It evidently occurs, when the space-charge term $\Phi''_{(1)}$ comprises a considerable fraction of the electron- or ion-concentration terms, $\Phi''_{(1)} \approx 0.01 n_{(1)}$.

In order to find $\Phi''_{(1)}$, let us firstly take the derivative of the last expression in Eqs. (14). Then, we have to determine $(n'_{(1)}/n_{(1)})^2$ and $n''_{(1)}/n_{(1)}$. For this purpose, we use Eq. (18). Ultimately, we obtain

$$\left|\frac{\Phi''_{(1)}}{n_{(1)}}\right| \approx \frac{(\nu+2\alpha)\,(\nu+5\alpha)\,(1+\tau)^2}{n_{(1)}\left(1+\tau-n^2_{(1)}\right)^3}. \tag{24}$$

By substituting the parameter values at which the numerical calculations were performed into Eq. (24), we arrive at the conclusion that the quasineutrality approximation is violated at $v_{(1)} \approx 0.93$, which is in good agreement with the result shown in Fig. 6.

In Fig. 7, the distributions of the potential, self-consistent electric field, and space charge in the plasma volume are exhibited. Note that the ion concentration exceeds that of electrons within the entire plasma volume by the quantity Φ'', which can be calculated at the coordinate origin by expanding it in a series in x, so we obtain $\Phi''_{x=0} \approx -[(\nu + 2\alpha)\alpha]/2$. In this case, the hydrodynamic velocity of electrons, which can be obtained from the continuity equation for ions and the stationary condition, $v_e/v_s = nv/(n+\Phi'')$, turns out some higher than the ion velocity. In spite of the presence of the space charge, the electric field becomes appreciable only in the transient layer, unlike the potential that changes rather strongly throughout the whole plasma volume.

4. Estimates of the Influence of Ion Viscosity in the Ion Motion Equation

Let us estimate the last term in the right-hand side of Eq. (15). This term describes the viscosity of ions in the quasineutral approximation. Let us divide it by the first term (with $n'_{(1)}/n_{(1)}$), which includes the electric field. The expression for $v''_{(1)}$ can be obtained from Eq. (17). As a result, we have

$$\left|\frac{4\bar{\eta} n_{(1)} v''_{(1)}}{3n'_{(1)}}\right| = \frac{8\bar{\eta}(1+\tau)}{3\left(1+\tau-v^2_{(1)}\right)^2} \times$$
$$\times \left[(\nu+2\alpha)(1+\tau) - (\nu+\alpha)\left(1+\tau-v^2_{(1)}\right)\right]. \tag{25}$$

From whence, it follows that, at the plasma center, where $v_{(1)} \ll 1$,

$$\left|\frac{4\bar{\eta} n_{(1)} v''_{(1)}}{3n'_{(1)}}\right| \approx \frac{8}{3}\bar{\eta}\alpha. \tag{26}$$

As one can see from Figs. 1 and 2, in the case of plasma parameters that were used in numerical calculations, the contribution of the ion viscosity (26) to Eq. (15) is small at the plasma center, being of an order of 4×10^{-4}. At the boundary between the quasineutral plasma and the space-charge layer, $v_{(1)} \lesssim \sqrt{1+\tau}$, and Eq. (25) implies that

$$\left|\frac{4\bar{\eta} n_{(1)} v''_{(1)}}{3n'_{(1)}}\right| \approx \frac{8\bar{\eta}(\nu+2\alpha)(1+\tau)^2}{3\left(1+\tau-v^2_{(1)}\right)^2}. \tag{27}$$

This result demonstrates that the contribution of the ion viscosity increases rapidly, as the hydrodynamic ion velocity approaches the ion sound velocity. But, in this case, the system of equations (9),

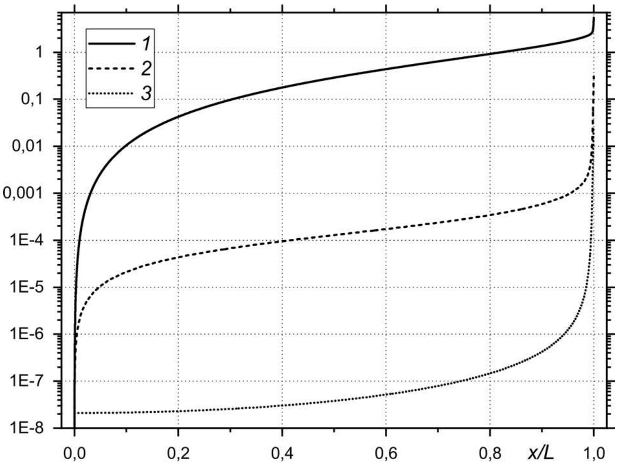

Fig. 7. Dependences of the dimensionless potential Φ (*1*), electric field $-\Phi'$ (*2*), and space charge $-\Phi'' = n_i - n_e$ (*3*) on the coordinate x/L calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV

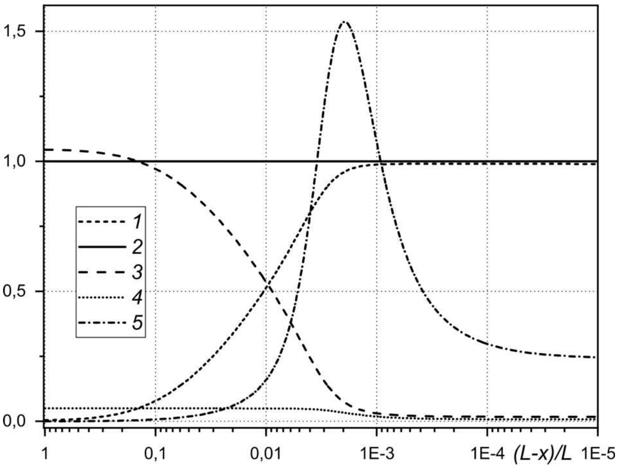

Fig. 8. Dependences of the ratios of the terms in the equations of motion (20) and (7) to the term containing the electric field on the coordinate $(L-x)/L$ calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV: $|vv'/\varPhi'|$ (*1*), $|\varPhi'/\varPhi'| = 1$ (*2*), $|(\nu + \alpha\exp(\varPhi)/n)v/\varPhi'|$ (*3*), $|\tau(n'/n)/\varPhi'|$ (*4*), and $|(4/3)\bar{\eta}v''/\varPhi'|$ (*5*)

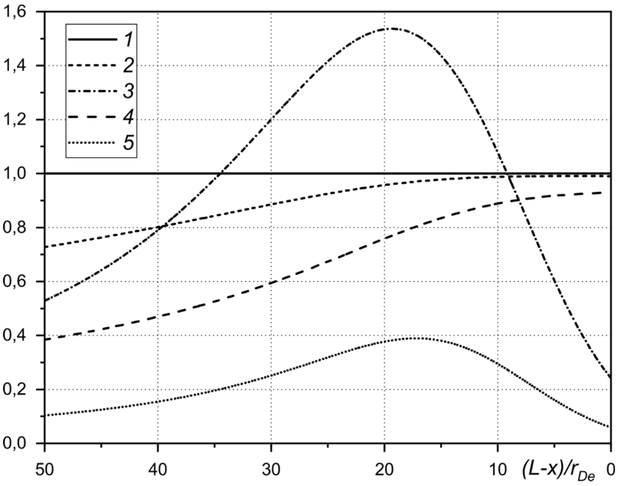

Fig. 9. Dependences of the ratios of the terms in the equations of motion (20) and (7) to the term containing the electric field on the coordinate $(L-x)/r_{De}$ in the transient layer calculated for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, $T_i = 0.1$ eV, and $n_n = 10^{14}$ (*2, 3*) and 5×10^{14} cm^{-3} (*3, 4*): $|vv'/\varPhi'|$ (*2, 4*), $|\varPhi'/\varPhi'| = 1$ (*1*), $|(4/3)\bar{\eta}v''/\varPhi'|$ (*3, 5*)

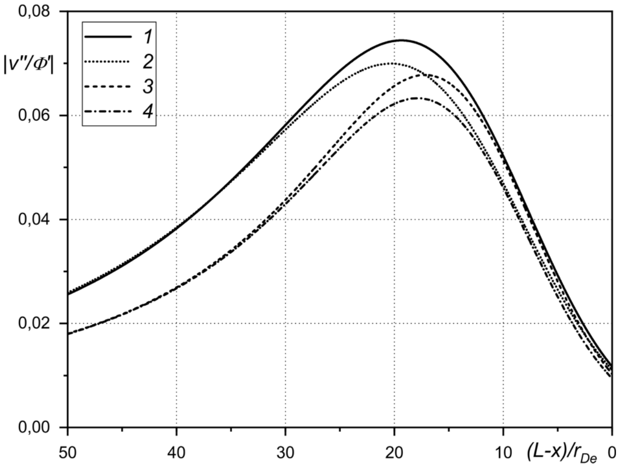

Fig. 10. Dependence of the ratio $|v''/\varPhi'|$ on the coordinate $(L-x)/r_{De}$ in the transient layer calculated for $n_{e0} = 10^{10}$ cm^{-3}, $T_e = 2$ eV, $T_i = 0.1$ (*1, 3*) and 0.3 eV (*2, 4*), and $n_n = 10^{14}$ (*1, 2*) and 5×10^{14} cm^{-3} (*3, 4*)

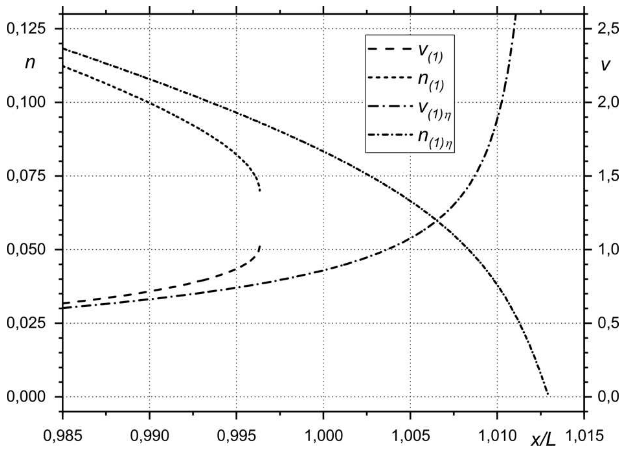

Fig. 11. Dependences of the ion concentration and velocity on the coordinate x/L calculated in the quasineutral approximation taking ($v_{(1)\eta}$, $n_{(1)\eta}$) and not taking ($v_{(1)}$, $n_{(1)}$) the ion viscosity into account for $n_{e0} = 10^{10}$ cm^{-3}, $n_n = 10^{14}$ cm^{-3}, $T_e = 2$ eV, and $T_i = 0.1$ eV

(12), and (13) has to be applied instead of Eqs. (17) and (18).

Let us estimate the contribution of the ion viscosity to the equation of ion motion (7), if $v > \sqrt{\tau}$. For this purpose, we should determine the second derivative of the ion velocity taking the Poisson equation (9) into account. The first derivatives of the ion velocity and concentration are determined by Eqs. (12) and (13), respectively. Then we obtain

$$v'' = -\left\{\varPhi''v + \left[\varPhi' + 2\left(\nu + \alpha\frac{\exp\varPhi}{n} - v'\right)v\right] \times \right.$$
$$\left. \times\, v' + \alpha\frac{\exp\varPhi}{n}\left(\varPhi' - \frac{n'}{n}\right)(v^2+\tau)\right\} \times \{v^2-\tau\}^{-1}. \quad (28)$$

Finally, the contribution of the ion viscosity to the equation of ion motion (7) in the region, where

$v > \sqrt{\tau}$, amounts to

$$\left|\frac{4\bar{\eta}v''}{3\Phi'}\right| = -\left\{\bar{\eta}\left|\Phi''v + \left[\Phi' + 2\left(\nu + \alpha\frac{\exp\Phi}{n} - v'\right)v\right]\times\right.\right.$$
$$\left.\left.\times\, v' + \alpha\frac{\exp\Phi}{n}\left(\Phi' - \frac{n'}{n}\right)(v^2+\tau)\right|\right\}\times$$
$$\times\left\{3\,|\Phi'|\,(v^2-\tau)\right\}^{-1}. \qquad (29)$$

In the case of strongly non-isothermal plasma, expression (28) for the transient layer can be simplified. Then we obtain

$$\left|\frac{4\bar{\eta}v''}{3\Phi'}\right| \approx \frac{4\bar{\eta}|\Phi''v + [\Phi' - 2vv']\,v'|}{3\,|\Phi'|\,(v^2-\tau)}. \qquad (30)$$

In Fig. 8, the ratios of all terms in the equation of motion (7) to the term containing the electric field are depicted. From this figure, one can see that the ion viscosity has a little effect in the quasineutral region, but this effect increases in the transient layer and must be taken into consideration.

For the sake of comparison, the same ratios but for $n_n = 5\times10^{14}$ cm^{-3} in the transient layer are shown in Fig. 9. As the concentration of neutrals n_n decreases, the ion viscosity coefficient η increases so that the ion viscosity contribution exceeds that of the electric-field term in the transient layer. At the same time, the contribution of the ion-viscosity term remains small throughout the other plasma volume.

In Fig. 10, the spatial profiles of the ratio $|v''/\Phi'|$, which were calculated for $T_e = 2$ eV and various values of T_i and n_n, are demonstrated. As one can see, if T_e is fixed, the magnitude and the behavior of the quantity $|v''/\Phi'|$ only slightly differs for various T_i- and n_n-values. From whence, it follows that the magnitude of the contribution $|(4/3)\bar{\eta}v''/\Phi'|$ is determined by the value of $(4/3)\bar{\eta}$ only.

Figure 11 illustrates the distributions of the ion concentration and velocity calculated in the quasineutral approximation, taking and not taking into account the term describing the ion viscosity in Eqs. (20) and (15). As one can see, since the magnitude of this term is small in the volume of quasineutral plasma, its consideration has a little effect on the plasma size L.

The exact solution of the system of equations (7)–(9) taking the viscous term in the transient layer into account is the subject of further researches.

5. Conclusions

It is well known that, when studying the distribution of plasma parameters in a stationary gas discharge, the plasma volume can be divided into two regions. One of them includes the main volume of plasma, where the quasineutrality condition is satisfied almost exactly, i.e. the concentration of ions is almost equal to the concentration of electrons distributed in the plasma potential according to the Boltzmann distribution. The potential distribution in plasma is determined by the exit of electrons to the plasma-confining surface and the subsequent screening of this surface by ions. The other region is the region, where ions screen the surface, and the quasineutrality condition is violated. An approximate “boundary” between those regions is a point, where the velocity of ions accelerated toward the surface is equal to the Bohm velocity. The distance of this point from the surface is about 30 electron Debye radii. The task of determining the parameters for a stationary gas discharge is an eigenfunction and eigenvalue problem. It is usually solved in the form of a Cauchy problem with the initial conditions given at the plasma center, whereas the coordinate of the boundary surface is determined from the balance condition for the ion and electron fluxes. A “true” position of the surface can be obtained by varying, e.g., the electron concentration at the plasma center provided that the other parameters are fixed.

When determining the distribution of plasma parameters in a stationary gas discharge, the account for the ion viscosity was usually neglected. In this work, a method aimed at determining the parameter distributions in strongly non-isothermal ($T_e \gg T_i$) plasma making allowance for the finite ion temperature has been proposed and applied. In particular, the method was used to estimate the account for the ion viscosity in the equation of ion motion. It was shown that the corresponding effect is small in the quasineutrality region, i.e. in almost the whole plasma volume. It does not result in an appreciable change in the plasma concentration in bulk. On the other hand, the calculations showed that the ion viscosity substantially affects the distribution of plasma parameters in the transient layer. The exact solution of the system of equations taking the ion viscosity in the transition layer into account is the subject of further researches.

1. D.J. Bohm. Minimum ionic kinetic energy for a stable sheath. In: *The Characteristics of Electrical Discharges in Magnetic Fields.* Edited by A. Guthrie, R.K. Wakerling (McGraw Hill, 1949), p. 77.
2. I. Langmuir, L. Tonks. A general theory of the plasma of an arc. *Phys. Rev.* **34**, 876 (1929).
3. W. Schottky. Wondstrem und theorie der positiven saule. *Phys. Zeits.* **25**, 342 (1924).
4. K.B. Persson. Inertia controlled ambipolar diffusion. *Phys. Fluids* **5**, 1625 (1962).
5. V.S. Golubev. Positive column of glow discharge and low-pressure arcs. In *Electric Current in Gas. Steady-State Current.* Edited by V.L. Granovskii (Nauka, 1971), p. 235 (in Russian).
6. I.D. Kaganovich. How to patch active plasma and collisionless sheath: practical guide. *Phys. Plasmas* **9**, 4788 (2002).
7. K.-U. Riemann. The Bohm criterion and sheath formation. *J. Phys. D* **24**, 493 (1991).
8. K.-U. Riemann, J. Seebacher, D.D. Tskhakaya, S. Kuhn. The plasma-sheath matching problem. *Plasma Phys. Control. Fusion* **47**, 1949 (2005).
9. V.E. Golant, A.P. Zhilinskii, I.E. Sakharov. *Fundamentals of Plasma Physics* (Wiley, 1980).
10. A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Sitenko, K.N. Stepanov. *Plasma Electrodynamics. Vols. 1 and 2* (Pergamon Press, 1975).
11. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, 1993).
12. R.W.P. McWhirter. Spectral intensities. In *Plasma Diagnostic Techniques.* Edited by R.H. Huddlestone, S.L. Leonard. (Academic Press, 1965).
13. B.M. Smirnov. *Physics of Weakly Ionized Gases (Problems and Solutions)* (Mir, 1981).
14. Yu.M. Kagan, V.I. Perel. Probe methods of plasma research. *Usp. Fiz. Nauk* **81**, 409 (1963) (in Russian).
15. R.N. Franklin, J.R. Ockerdon. Asymptotic matching of plasma and sheath inactive law pressure discharge. *J. Plasma Phys.* **4**, 371 (1970).

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ВПЛИВ В'ЯЗКОСТІ ІОНІВ
НА РОЗПОДІЛ ПАРАМЕТРІВ ПЛАЗМИ
В СТАЦІОНАРНИХ ГАЗОВИХ РОЗРЯДАХ

Для плоского шару стаціонарної слабоіонізованої сильнонеізотермічної плазми, обмеженого діелектричними стінками, на основі рівнянь гідродинаміки отримано розподіли параметрів плазми – потенціалу, густини іонів і електронів та швидкості потоку іонів у напрямку стінок. Припускалось, що температури іонів і електронів та густина нейтралів є постійними. При цьому замість знаходження власних функцій і власних значень цієї задачі, розв'язувалась задача Коші для початкових значень, які є заданими в центрі плазми. Положення стінки визначалось з умови рівності потоків іонів і електронів. Запропоновано метод розв'язання проблеми сингулярності, що присутня в системі рівнянь гідродинаміки. Проведено оцінки ефекту в'язкості іонів у рівнянні руху іонів. Отримано розподіли параметрів плазми з урахуванням в'язкості іонів в області квазінейтральності.

Ключові слова: стаціонарний газовий розряд, в'язкість, перехідний шар, гідродинамічне наближення, радіус Дебая.