https://doi.org/10.15407/ujpe64.11.1064

V.M. SIMULIK

Institute of Electron Physics, Nat. Acad. of Sci. of Ukraine
(21, Universytets'ka Str., Uzhgorod 88017, Ukraine; e-mail: vsimulik@gmail.com)

RELATIVISTIC EQUATIONS FOR ARBITRARY SPIN, ESPECIALLY FOR THE SPIN $s = 2$ [1]

The further approbation of the equation for the particles of arbitrary spin introduced recently in our papers is under consideration. The comparison with the known equations suggested by Bhabha, Pauli–Fierz, Bargmann–Wigner, Rarita–Schwinger (for spin s = 3/2) and other authors is discussed. The advantages of the new equations are considered briefly. The advantage of the new equation is the absence of redundant components. The important partial case of spin s = 2 is considered in details. The 10-component Dirac-like wave equation for the spin s = (2,2) particle-antiparticle doublet is suggested. The Poincaré invariance is proved. The three-level consideration (relativistic canonical quantum mechanics, canonical Foldy–Wouthuysen-type field theory, and locally covariant field theory) is presented. The procedure of our synthesis of arbitrary spin covariant particle equations is demonstrated on the example of spin s = (2,2) doublet.

Keywords: Dirac equation, relativistic quantum mechanics, arbitrary spin, graviton, spin (2,2) particle-antiparticle doublet.

1. Introduction

Recently, the equation for a particle-antiparticle doublet of arbitrary spin without redundant components was suggested [1] (see also [2]). The reason for our interest follows from difficulties of the known approaches of other authors.

The start of an arbitrary spin consideration was given in [3]. Today, the Pauli–Fierz [4], Bhabha [5], and Bargmann–Wigner [6] equations and their contemporary modifications (see, e. g. [7]) are most often considered. A more detailed review can be found in [8]. Here, in [1] and [2], only the approach started in [9] and [10] is the basis for the further application.

Note only some general deficiencies of the known equations in the case of an arbitrary spin. The consideration of the partial cases, when the substitution of a fixed value of spin is fulfilled, is not successful in all cases. For example, for the spin $s > 1$, the available equations have the redundant components and should be complemented by some additional conditions. Indeed, the known Pauli–Fierz [4] and Rarita–Schwinger [11] equations for the spin $s = 3/2$ (and their confirmation in [12]) should be essentially complemented by the additional conditions. The main difficulty in the models of an arbitrary spin is the interaction between the fields of higher-spin [13]. Even the quantization of higher-spin fields generated the questions [14]. These and other deficiencies of the known equations for higher-spin are under consideration till recent years (see, e.g., [15]) (a brief review of deficiencies can be found in [16] or in [17]). It gives the place for our investigation.

Our program of synthesis of arbitrary spin equations is based on a step-by-step transition as follows: *relativistic canonical quantum mechanics → canonical Foldy–Wouthuysen-type field theory → locally covariant field theory*. This gives a possibility to start from the quantum mechanical equations without redundant components and to finish without such components, which determines our perspectives.

The equation for the particle-antiparticle doublet of arbitrary spin is given by

$$\left[i\partial_0 - \Gamma^0_{2\mathrm{N}}(\mathbf{\Gamma}_{2\mathrm{N}} \cdot \widehat{\mathbf{p}} + m)\right] \psi(x) = 0, \tag{1}$$

where $x \in \mathrm{M}(1,3)$, $\partial_\mu \equiv \partial/\partial x^\mu$, $\mu = \overline{0,3}$, $j = 1,2,3$, and $\mathrm{M}(1,3) = \{x \equiv (x^\mu) = (x^0 = t, \mathbf{x} \equiv (x^j))\}$

[1] This work is based on the results presented at the XI Bolyai–Gauss–Lobachevskii (BGL-2019) Conference: Non–Euclidean, Noncommutative Geometry and Quantum Physics.

is the Minkowski space-time, and the 2N-component function $\psi(x)$ belongs to the rigged Hilbert space

$$\mathrm{S}^{3,2\mathrm{N}} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^{2\mathrm{N}} \subset \mathrm{H}^{3,2\mathrm{N}} \subset \mathrm{S}^{3,2\mathrm{N}*}. \qquad (2)$$

Note that, due to a special role of the time variable $x^0 = t \in (x^\mu)$ (in the obvious analogy with non-relativistic theory), one can use the quantum-mechanical rigged Hilbert space (2) in the general consideration. Here, the Schwartz test function space $\mathrm{S}^{3,2\mathrm{N}}$ is the core (i.e., it is dense both in $\mathrm{H}^{3,2\mathrm{N}}$ and in the space $\mathrm{S}^{3,2\mathrm{N}*}$ of the 2N-component Schwartz generalized functions), and $\mathrm{H}^{3,2\mathrm{N}}$ is the quantum-mechanical Hilbert space of 2N-component functions over $\mathrm{R}^3 \subset \mathrm{M}(1,3)$. The space $\mathrm{S}^{3,2\mathrm{N}*}$ is conjugate to the Schwartz test functions space $\mathrm{S}^{3,2\mathrm{N}}$ by the corresponding topology (see, e.g., [18]).

In order to finish with notations, assumptions, and definitions, let us note that the system of units $\hbar = c = 1$ is chosen here, the metric tensor in the Minkowski space-time $\mathrm{M}(1,3)$ is given by

$$g^{\mu\nu} = g_{\mu\nu} = g^\mu_\nu,\ (g^\mu_\nu) = \mathrm{diag}\,(1,-1,-1,-1)\,; \qquad (3)$$

$x_\mu = g_{\mu\nu}x^\mu$, and the summation over the twice repeated indices is implied.

The Γ matrices in (1) are taken in the form

$$\Gamma^0_{2\mathrm{N}} \equiv \sigma^3_{2\mathrm{N}} = \begin{vmatrix} \mathrm{I}_\mathrm{N} & 0 \\ 0 & -\mathrm{I}_\mathrm{N} \end{vmatrix}, \quad \Gamma^j_{2\mathrm{N}} = \begin{vmatrix} 0 & \Sigma^j_\mathrm{N} \\ -\Sigma^j_\mathrm{N} & 0 \end{vmatrix}. \qquad (4)$$

Further, there is a degree of freedom in the choice of Σ^j_N matrices in (4). This freedom started from the case of 4×4 Σ^j_4 matrices, which can be chosen in both forms

$$\Sigma^j_4 = \begin{vmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{vmatrix}, \qquad (5)$$

where $\{\sigma^j\}$ are the standard Pauli matrices, and

$$\Sigma^1_4 = \begin{vmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{vmatrix}, \Sigma^2_4 = \begin{vmatrix} 0 & -i\mathrm{I}_2 \\ i\mathrm{I}_2 & 0 \end{vmatrix}, \Sigma^3_4 = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{vmatrix}. \qquad (6)$$

Below, the consideration of the partial spin $s = (2,2)$ case is presented.

2. Spin $s = (2,2)$ Particle-Antiparticle Bosonic Doublet in the Relativistic Canonical Quantum Mechanics

The corresponding Schrödinger–Foldy equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \mathrm{column}\,\left|f^1, f^2, ..., f^{10}\right|, \quad (7)$$

where the pseudodifferential operator $\widehat{\omega}$ is given by

$$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}. \qquad (8)$$

In (7), the 10-component wave function is the direct sum of the particle and antiparticle wave functions. According to the quantum-mechanical tradition, the antiparticle wave function is put into the bottom part of the 10-column.

Therefore, the general solution of the Schrödinger–Foldy equation (7) has the form

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{-ikx} g^\mathrm{A}(\mathbf{k})\mathrm{d}_\mathrm{A}, \qquad (9)$$

where $\mathrm{A} = \overline{1,10}$ and d_A are the orts of the 10-component Cartesian basis. Hence, the space of the states of a spin $s = (2,2)$ particle-antiparticle doublet is the rigged Hilbert space $\mathrm{S}^{3,10} \subset \mathrm{H}^{3,10} \subset \mathrm{S}^{3,10*}$, i.e., it is the direct sum of two spaces $\mathrm{S}^{3,5} \subset \mathrm{H}^{3,5} \subset \mathrm{S}^{3,5*}$.

Thus, in the model under consideration, information about the positive and equal masses of the particle and antiparticle is inserted. Further, information about that the observer sees the antiparticle as the mirror reflection of the particle is also inserted, see formula (10) below. Therefore, the charge of the antiparticle should be opposite in sign to that of the particle (in the case of the existence of the charge), and the spin projection of the antiparticle should be opposite in sign to the spin projection of the particle.

Therefore, according to these principles, the corresponding SU(2)-spin generators are taken in the form

$$\mathbf{s}_{10} = \begin{vmatrix} \mathbf{s}_5 & 0 \\ 0 & -C\mathbf{s}_5C \end{vmatrix}, \qquad (10)$$

where the 5×5-matrices $\mathbf{s}_5$ are given by

$$s^1 = \frac{1}{2}\begin{vmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{vmatrix},$$

$$s^2 = \frac{i}{2}\begin{vmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{vmatrix},$$

$$s^3 = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} \qquad (11)$$

and $C\mathrm{I}_5$ is the 5×5 diagonal matrix operator of complex conjugation. It is easy to verify that, for operators (10), the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n}s^n$ of the SU(2)-algebra are valid. The Casimir operator for this reducible representation of the SU(2)-algebra is given by $\mathbf{s}^2 = 6\mathrm{I}_{10} = 2\,(2+1)\,\mathrm{I}_{10}$, where I_{10} is the 10×10 unit matrix.

Solution (9) is associated with the stationary complete set $\widehat{\mathbf{p}}$, $s^3 = s_z$ of the momentum and spin projection operators of the spin $s = (2, 2)$ bosonic particle-antiparticle doublet. The equations for the momentum operator eigenvalues have the form

$$\widehat{\mathbf{p}} e^{-ikx}\mathrm{d}_\mathrm{A} = \mathbf{k} e^{-ikx}\mathrm{d}_\mathrm{A}, \quad \mathrm{A} = \overline{1, 10}, \qquad (12)$$

and the equations for the spin projection operator s^3_{10} (10) eigenvalues are given by

$$\begin{gathered} s^3_{10}\mathrm{d}_1 = 2\mathrm{d}_1,\ s^3_{10}\mathrm{d}_2 = \mathrm{d}_2,\ s^3_{10}\mathrm{d}_3 = 0,\ s^3_{10}\mathrm{d}_4 = -\mathrm{d}_4, \\ s^3_{10}\mathrm{d}_5 = -2\mathrm{d}_5, \quad s^3_{10}\mathrm{d}_6 = -2\mathrm{d}_6, \quad s^3_{10}\mathrm{d}_7 = -\mathrm{d}_7, \qquad (13) \\ s^3_{10}\mathrm{d}_8 = 0, \quad s^3_{10}\mathrm{d}_9 = \mathrm{d}_9, \quad s^3_{10}\mathrm{d}_{10} = 2\mathrm{d}_{10}. \end{gathered}$$

Therefore, the functions $g^1(\mathbf{k})$, $g^2(\mathbf{k})$, $g^3(\mathbf{k})$, $g^4(\mathbf{k})$, $g^5(\mathbf{k})$ in solution (9) are the momentum-spin amplitudes of the particle (boson) with the momentum $\widehat{\mathbf{p}}$ and spin projection eigenvalues (+2, +1, 0, –1, –2), respectively, and the functions $g^6(\mathbf{k})$, $g^7(\mathbf{k})$, $g^8(\mathbf{k})$, $g^9(\mathbf{k})$, $g^{10}(\mathbf{k})$ are the momentum-spin amplitudes of the antiparticle with the momentum $\widehat{\mathbf{p}}$ and spin projection eigenvalues (–2, –1, 0, +1, +2), respectively.

The Schrödinger–Foldy equation (7) and the set $\{\mathrm{f}\}$ of its solutions (9) are invariant with respect to the reducible unitary bosonic representation $(a, \omega) \rightarrow$

$$U(a, \omega) = \exp(-ia^0\widehat{p}_0 - i\mathbf{a}\cdot\widehat{\mathbf{p}} - \frac{i}{2}\omega^{\mu\nu}\widehat{j}_{\mu\nu}) \qquad (14)$$

of the Poincaré group $\mathcal{P}$. The corresponding 10×10 matrix-differential generators are given by

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \qquad (15)$$

$$\begin{gathered} \widehat{j}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n}, \\ \widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \breve{s}_\ell\right), \qquad (16) \end{gathered}$$

whereas the spin $s = (2, 2)$ SU(2) generators $\mathbf{s} = (s^{\ell n}) \equiv \mathbf{s}_{10}$ are given in (10).

The validity of this assertion is verified by three following steps. (i) The calculation of that the $\mathcal{P}$-generators (15) and (16) commute with the operator $i\partial_0 - \widehat{\omega}$ of the Schrödinger–Foldy equation (7). (ii) The verification of that the $\mathcal{P}$-generators (15) and (16) satisfy the commutation relations of the Lie algebra of the Poincaré group $\mathcal{P}$. (iii) The proof of that generators (15) and (16) realize the spin $s(s+1)$ irreducible representation of this group [or the spin $2s(s+1)$ reducible representation in the case of doublet]. Therefore, the Bargmann–Wigner classification on the basis of calculations of the corresponding Casimir operators should be given. These three steps can be made by direct non-cumbersome calculations.

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_{10}, \qquad (17)$$

$$W = w^\mu w_\mu = m^2 \mathbf{s}^2_{10} = 2\,(2+1)\,m^2 \mathrm{I}_{10}, \qquad (18)$$

where I_{10} is the 10×10 unit matrix.

Hence, a brief consideration of the relativistic canonical quantum mechanics foundations of the particle-antiparticle doublet with the mass $m > 0$ and the spin $s = (2, 2)$ has been given above. In the limit $m = 0$, this model describes the partial case of a graviton-antigraviton doublet. The hypothesis about the massive graviton and other tiny problems of the gravity are not the subject of this consideration.

3. Spin $s = (2, 2)$ Particle-Antiparticle Bosonic Doublet in the Foldy–Wouthuysen Canonical Field Representation

Thus, the Schrödinger–Foldy equation (7) and its solution (9) are linked with the canonical field theory equation

$$(i\partial_0 - \Gamma^0_{10}\widehat{\omega})\phi(x) = 0, \quad \Gamma^0_{10} = \begin{vmatrix} \mathrm{I}_5 & 0 \\ 0 & -\mathrm{I}_5 \end{vmatrix}, \qquad (19)$$

and its solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^\mathrm{A}(\mathbf{k})\mathrm{d}_\mathrm{A} + e^{ikx} g^{*\mathrm{B}}(\mathbf{k})\mathrm{d}_\mathrm{B}\right], \qquad (20)$$

($\mathrm{A} = \overline{1, 5}$; $\mathrm{B} = \overline{6, 10}$) by the operator $v^{-1}_{10} = v^\dagger_{10} = v_{10}$,

$$v_{10} = \begin{vmatrix} \mathrm{I}_5 & 0 \\ 0 & C\mathrm{I}_5 \end{vmatrix}, \quad v_{10}v_{10} = \mathrm{I}_{10}, \qquad (21)$$

 ISSN 2071-0194. Ukr. J. Phys. 2019. Vol. 64, No. 11

(the orts are Cartesian), C is the operator of the complex conjugation, and I_5 is the 5×5 unit matrix.

The operator v_{10} (21) transforms an arbitrary operator $\hat{q}_{\rm qm}$ of the 10-component relativistic canonical quantum mechanics (considered in the previous section) into the corresponding operator $\hat{q}_{\rm cf}$ of the canonical field theory and *vice versa*. Operator (21) gives a link between solutions (9) and (20) as well. Note that this transition is valid only for the operators taken in anti-Hermitian (prime) form. The mathematical correctness of such choice of operators and the physical interpretation are explained in [19] and [20]. The return to the Hermitian operators is very easy.

The spin operators of the canonical field theory found from the corresponding quantum mechanical SU(2) spin (10) on the basis of transformation (21) satisfy the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n}s^n$ and have the form

$$\mathbf{s}_{10} = \begin{vmatrix} \mathbf{s}_5 & 0 \\ 0 & \mathbf{s}_5 \end{vmatrix}, \tag{22}$$

where the 5×5 spin $s=2$ SU(2) generators are given in (11). The corresponding Casimir operator is given by $\mathbf{s}_{10}^2 = 6\mathrm{I}_{10} = 2(2+1)\mathrm{I}_{10}$, where I_{10} is the 10×10 unit matrix.

The stationary complete set of operators is given by the operators $\widehat{\mathbf{p}}$, $s_{10}^3 = s_z$ of the momentum and spin projection. The equations for the eigenvectors and eigenvalues of the operators $\widehat{\mathbf{p}}$ and $s_{10}^3 = s_z$ from (22) have the form

$$\begin{gathered} \widehat{\mathbf{p}}e^{-ikx}\mathrm{d}_{\rm A} = \mathbf{k}e^{-ikx}\mathrm{d}_{\rm A}, \quad \widehat{\mathbf{p}}e^{ikx}\mathrm{d}_{\rm B} = -\mathbf{k}e^{ikx}\mathrm{d}_{\rm B}, \\ s_{10}^3\mathrm{d}_1 = 2\mathrm{d}_1,\ s_{10}^3\mathrm{d}_2 = \mathrm{d}_2,\ s_{10}^3\mathrm{d}_3 = 0,\ s_{10}^3\mathrm{d}_4 = -\mathrm{d}_4, \\ s_{10}^3\mathrm{d}_5 = -2\mathrm{d}_5,\ s_{10}^3\mathrm{d}_6 = 2\mathrm{d}_6,\ s_{10}^3\mathrm{d}_7 = \mathrm{d}_7, \\ s_{10}^3\mathrm{d}_8 = 0,\ s_{10}^3\mathrm{d}_9 = -\mathrm{d}_9,\ s_{10}^3\mathrm{d}_{10} = -2\mathrm{d}_{10}, \end{gathered} \tag{23}$$

and determine the interpretation of the amplitudes in the general solution (20). Note that the direct quantum-mechanical interpretation of the amplitudes $g^{\rm A}(\mathbf{k})$, $g^{\rm B}(\mathbf{k})$ in solution (20) should be taken from the quantum-mechanical equations (12), (13) and is given in the section above.

The relativistic invariance of the canonical field equation (19) follows from the corresponding invariance of the Schrödinger–Foldy equation (7) and transformation (21) (for the anti-Hermitian operators). The explicit form of the corresponding generators follows from of the explicit form generators (15) and (16) with the spin matrices (10) after transformation (21).

Thus, the canonical field equation (19) and the set $\{\phi\}$ of its solutions (20) are invariant with respect to the reducible unitary bosonic representation (14) of the Poincaré group $\mathcal{P}$, whose Hermitian 10×10 matrix-differential generators are given by

$$\begin{gathered} \widehat{p}^0 = \Gamma_{10}^0\widehat{\omega} \equiv \Gamma_{10}^0\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \\ \widehat{j}^{\ell n} = x^\ell\widehat{p}^n - x^n\widehat{p}^\ell + s_{10}^{\ell n} \equiv \widehat{m}^{\ell n} + s_{10}^{\ell n}, \\ \widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\Gamma_{10}^0\left\{x^\ell, \widehat{\omega}\right\} + \Gamma_{10}^0\frac{(\mathbf{s}_{10}\times\widehat{\mathbf{p}})^\ell}{\widehat{\omega}+m}, \end{gathered} \tag{24}$$

where the spin $s=(2,2)$ SU(2) generators $\mathbf{s}_{10} = (s_{10}^{\ell n})$ have the form (22) and Γ_{10}^0 matrix is given in (19).

The proof is similar to that given in the previous subsection. The corresponding Casimir operators have the eigenvalues similar to (17) and (18).

Thus, due to the eigenvalues in Eqs. (23), positive- and negative-frequency forms of solution (20), and the Bargmann–Wigner analysis of the Casimir operators, one can come to a conclusion that Eq. (19) describes the canonical field (the bosonic particle-antiparticle doublet) with the spins $s=(2,2)$ and $m>0$. The transition to the $m=0$ limit leads to the canonical field equation for the graviton-antigraviton field [if the graviton is massless (?)].

4. Equation for a Spin $s = (2,2)$ Particle-Antiparticle Bosonic Doublet in the Locally Covariant Field Representation

The final transition to the locally covariant field model is performed by the inverse Foldy–Wouthuysen-type transformation. The corresponding field equation has the form

$$\begin{aligned} & i\partial_0\psi^1 - p^3\psi^6 - m\psi^1 = 0, \\ & i\partial_0\psi^2 - p^3\psi^7 + ip^2\psi^{10} - p^1\psi^{10} - m\psi^2 = 0, \\ & i\partial_0\psi^3 - p^3\psi^8 + ip^2\psi^9 - p^1\psi^9 - m\psi^3 = 0, \\ & i\partial_0\psi^4 + p^3\psi^9 - ip^2\psi^8 - p^1\psi^8 - m\psi^4 = 0, \\ & i\partial_0\psi^5 + p^3\psi^{10} - ip^2\psi^7 - p^1\psi^7 - m\psi^5 = 0, \\ & i\partial_0\psi^6 - p^3\psi^1 + m\psi^6 = 0, \\ & i\partial_0\psi^7 - p^3\psi^2 + ip^2\psi^5 - p^1\psi^5 + m\psi^7 = 0, \\ & i\partial_0\psi^8 - p^3\psi^3 + ip^2\psi^4 - p^1\psi^4 + m\psi^8 = 0, \\ & i\partial_0\psi^9 + p^3\psi^4 - ip^2\psi^3 - p^1\psi^3 + m\psi^9 = 0, \\ & i\partial_0\psi^{10} + p^3\psi^5 - ip^2\psi^2 - p^1\psi^2 + m\psi^{10} = 0, \\ & -ip^2\psi^1 - p^1\psi^1 = 0, \quad -ip^2\psi^6 - p^1\psi^6 = 0, \end{aligned} \tag{25}$$

or the Dirac-like form

$$\left[i\partial_0 - \Gamma_8^0(\boldsymbol{\Gamma}_8 \cdot \widehat{\mathbf{p}} + m)\right]\psi(x) = 0, \tag{26}$$

$$(i\partial_0 - \sigma^1 p^3 - \sigma^3 m)\chi = 0, \quad (p^1 + ip^2)\chi = 0,$$

where $\psi = \text{column}(\psi^1, \psi^2, ..., \psi^8)$,

$$\chi = \begin{vmatrix} \psi^9 \\ \psi^{10} \end{vmatrix}, \Gamma_8^0 = \begin{vmatrix} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{vmatrix}, \Gamma_8^j = \begin{vmatrix} 0 & \Sigma_4^j \\ -\Sigma_4^j & 0 \end{vmatrix},$$

and Σ_4^j are the 4×4 Pauli matrices

$$\Sigma_4^1 = \begin{vmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{vmatrix}, \Sigma_4^2 = \begin{vmatrix} 0 & -i\mathrm{I}_2 \\ i\mathrm{I}_2 & 0 \end{vmatrix}, \Sigma_4^3 = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{vmatrix}.$$

The forms (25) and (26) are linked by the ordinary linear transformation.

5. Brief Discussions

It is easy to see that our equations do not coincide with known approaches. Indeed, the Rarita–Schwinger equation [11] for spin $s = 3/2$ contains 16 components, whereas our equation (1) for spin $s = (3/2, 3/2)$ particle-antiparticle doublet contains 8 components. The Bargmann–Wigner equation [6] has 12 components in the partial case $s = 3/2$. Bhabha himself [21] analyzed the partial case $s = 3/2$ for his equation [5]. He found [21] that, in this case, his equation [5] coincides with the Rarita–Schwinger equation, i.e., has 16 components, *etc.*

The important partial example of the spin $s = 2$ case is considered in details. The 10-component Dirac-like wave equation for the spin $s = (2, 2)$ particle-antiparticle doublet is suggested. The Poincaré invariance is proved. The three-level consideration (relativistic canonical quantum mechanics, canonical Foldy–Wouthuysen-type field theory, and locally covariant field theory) is presented.

1. V.M. Simulik. Derivation of the Dirac and Dirac-like equations of arbitrary spin from the corresponding relativistic canonical quantum mechanics. *Ukr. J. Phys.* **60**, 985 (2015).
2. V.M. Simulik. Link between the relativistic canonical quantum mechanics of arbitrary spin and the corresponding field theory. *J. Phys: Conf. Ser.* **670**, 012047 (2016).
3. P.A.M. Dirac. Relativistic wave equations. *Proc. Roy. Soc. Lond. A.* **155**, 447 (1936).
4. M. Fierz, W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc. R. Soc. Lond. A.* **173**, 211 (1939).
5. H.J. Bhabha. Relativistic wave equations for the elementary particles. *Rev. Mod. Phys.* **17**, 200 (1945).
6. V. Bargmann, E.P. Wigner. Group theoretical discussion of relativistic wave equations. *Proc. Nat Acad. Sci. USA.* **34**, 211 (1948).
7. A. Zecca. Massive field equations of arbitrary spin in Schwarzschild geometry: separation induced by spin-3/2 case. *Int. J. Theor, Phys.* **45**, 2241 (2006).
8. V.M. Simulik. On the relativistic canonical quantum mechanics and field theory of arbitrary spin. *Univ. J. Phys. Appl.* **11**, 202 (2017).
9. L.L. Foldy, S.A. Wouthuysen. On the Dirac theory of spin 1/2 particles and its non-relativistic limit. *Phys. Rev.* **78**, 29 (1950).
10. L.L. Foldy. Synthesis of covariant particle equations. *Phys. Rev.* **102**, 568 (1956).
11. W. Rarita, J. Schwinger. On a theory of particles with half-integral spin. *Phys. Rev.* **60**, 61 (1941).
12. A.S. Davydov. Wave equations of a particle having spin 3/2 in the absence of a field. *J. Exper. Theor. Phys.* **13**, 313 (1943) (in Russian).
13. G. Velo, D. Zwanziger. Propagation and quantization of Rarita–Schwinger waves in an external electromagnetic potential. *Phys. Rev.* **186**, 1337 (1969).
14. J.D. Kimel, L.M. Nath. Quantization of the spin-$\frac{3}{2}$ field in the presence of interactions. *Phys. Rev. D.* **6**, 2132 (1972).
15. A.E. Kaloshin, V.P. Lomov. Rarita–Schwinger field and multi-component wave equation. *Phys. Part. Nucl. Lett.* **8**, 517 (2011).
16. A.Z. Capri, R.L. Kobes. Further problems in spin-3/2 field theories. *Phys. Rev. D.* **22**, 1967 (1980).
17. V.M. Red'kov. Particle with spin $S = 3/2$ in Riemannian space-time. arXiv: 1109.3871v1.
18. V.S. Vladimirov. *Methods of the Theory of Generalized Functions* (Taylor and Francis, 2002) [ISBN: 9780429153013].
19. B. Wybourne. *Classical Groups for Physicists* (Wiley, 1974) [ISBN-13: 978-0471965053].
20. J. Elliott, P. Dawber. *Symmetry in Physics, Vol. 1* (Macmillan Press, 1979) [ISBN-13: 978-0333382707].
21. H.J. Bhabha. On a class of relativistic wave equations of spin 3/2. *Proc. Indian Acad. Sci. A* **34**, 335 (1951).

Received 31.08.19

В.М. Симулик

РЕЛЯТИВІСТСЬКІ РІВНЯННЯ
ДЛЯ ДОВІЛЬНОГО СПІНУ, ЗОКРЕМА
ДЛЯ СПІНУ $s = 2$

Р е з ю м е

Продовжено апробацію запропонованого нами рівняння для частинок довільного спіну. Проведено порівняння з відомими рівняннями Баба, Паулі–Фірца, Баргмана–Вігнера, Раріти–Швінгера (для спіну $s = 3/2$) та інших авторів. Показано, що перевагою нового рівняння є відсутність у ньому зайвих компонент. Детально розглянуто важливий частинний випадок спіну $s = 2$. Запропоновано 10-компонентне діракоподібне хвильове рівняння для дублета частинка-античастинка спінів $s = (2, 2)$. Доведено його Пуанкаре інваріантність. Аналіз виконано на трьох рівнях: релятивістська канонічна квантова механіка, канонічна теорія поля типу Фолді–Ваутхайсена, локально коваріантна теорія поля. На прикладі дублета $s = (2, 2)$ продемонстровано схему синтезу релятивістських польових рівнянь довільного спіну.