doi:

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PACS 98.80.Jk

SOME SPHERICALLY SYMMETRIC R/W UNIVERSE INTERACTING WITH VACUUM B–D SCALAR FIELD

We study a spherically symmetric vacuum cosmological model of the Universe interacting with the Brans–Dicke (B–D) scalar field in the Robertson–Walker (R/W) metric. Exact time-dependent solutions of B–D vacuum field equations are obtained in two different cases. The physical and dynamical properties of the model are discussed in detail.

Keywords: Brans–Dicke theory, vacuum cosmological model, spherically symmetric scalar field.

1. Introduction

The Brans–Dicke (B–D) theory [1] describes most of the important features of the progress of the Universe during the late-time dynamical epoch. As a result, the B–D theory has attained a significant attention in recent years. The scalar-tensor theories are considered the simplest and best understood modification of gravity theory. The Brans–Dicke theory is, in fact, a modification of Einstein's General Relativity allowing the variable gravity with certain coupling parameter ω. It is somewhat classical in nature, for that reason it is expected to play a crucial role in the late-time evolution of the Universe. It is also realized that most of the inflationary models based on the B–D scalar theory overcharge many important elements about the evolution of the Universe [2, 3]. Hence, the B–D theory gives a connection between the accelerated expansion of the Universe and fundamental physics. Earlier, Brans and Dicke [1] obtained the vacuum solutions of B–D field equations followed by three more solutions for a spherically symmetric metric. Nariri [4] proposed a Hamiltonian approach to the dynamics of the expanding homogeneous Universe. Janis *et al.* [5] established a theorem to generate the B–D vacuum state solutions. Tabensky and Taub [6] obtained B–D vacuum static solutions with plane symmetric self-gravitating fluids. Rao *et al.* [7] discussed about cylindrically symmetric B–D fields. Various authors [8–13] discussed about vacuum solutions in the Brans–Dicke theory of gravitation for the metric tensors viz. plane symmetry, static cylindrical symmetry, zero-mass scalar field, conformal scalar field, for spatially homogeneous and anisotropic configuration, axisymmetric stationary and spherical symmetries, static fields, *etc.* Bhadra and Sarkar *et al.* [14] obtained that only two classes are independent among the four classes of static spherically symmetric solutions of the vacuum Brans–Dicke theory of gravity. Adhav *et al.* [15] obtained an exact solution of the vacuum Brans–Dicke field equations for the metric tensor of a spatially homogeneous and anisotropic model. Static, cylindrically symmetric vacuum solutions with and without a cosmological constant in the B–D theory were obtained by Baykal *et al.* [16]. Rai *et al.* [17] obtained an exact solution of the vacuum Brans–Dicke field equations for the metric tensor of a spatially homogeneous and anisotropic model. Here, we studied the problem of a B–D scalar field interacting with the spherically symmetric Robertson–Walker metric. The paper is organized as follows: in Section 2, we consider the metric and give solutions of the field equations in different cases; in Section 3, we give conclusion about the solutions.

2. Solutions of Field Equations

The vacuum Brans–Dicke field equations in the general form are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = \frac{\omega}{\phi^2}\left[\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}g^{sl}\phi_{,l}\phi_{,s}\right] - \\ - \frac{1}{\phi}[\phi_{,ij} - g_{ij}\phi^{;s}_{;s}], \quad (1)$$

$$(3+2\omega)\phi^{;s}_{;s} = 4\Lambda, \quad (2)$$

where ϕ is the scalar field, Λ is the cosmological constant, ω is the dimensionless Dicke coupling constant, R_{ij} is the Ricci tensor, R is the Riemann curvature scalar, g_{ij} is the metric tensor, $\Box\phi = \phi^{;s}_{;s}$, $\Box$ is the Laplace—Beltrami operator, and $\phi_{,i}$ is the partial differentiation with respect to the x^i coordinate.

Let us consider the R/W space time metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \quad (3)$$

where $R(t)$ is the scale factor, and k is the curvature index, which can take up the values $(-1, 0, +1)$ for open, flat, and closed models of the Universe, respectively. Corresponding to metric (3), the Brans–Dicke field equation (1) becomes

$$\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} - \Lambda = \frac{\omega}{2\phi^2}\left[\frac{(kr^2-1)}{R^2}\phi'^2 - \dot{\phi}^2\right] + \\ + \frac{1}{\phi}\left[\frac{2(1-kr^2)}{R^2 r}\phi' - \frac{2\dot{R}\dot{\phi}}{R} - \ddot{\phi}\right], \quad (4)$$

$$\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} - \Lambda = \frac{\omega}{2\phi^2}\left[\frac{(1-kr^2)}{R^2}\phi'^2 - \dot{\phi}^2\right] + \\ + \frac{1}{\phi}\left[\frac{(1-kr^2)}{R^2}\phi'' - \frac{(2kr^2-1)}{R^2 r}\phi' - \frac{2\dot{R}\dot{\phi}}{R} - \ddot{\phi}\right], \quad (5)$$

$$3\left(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2}\right) - \Lambda = \frac{\omega}{2\phi^2}\left[\dot{\phi}^2 + \frac{(1-kr^2)}{R^2}\phi'^2\right] + \\ + \frac{1}{\phi}\left[\frac{(1-kr^2)}{R^2}\phi'' - \frac{(3kr^2-2)}{R^2 r}\phi' - \frac{3\dot{R}\dot{\phi}}{R}\right], \quad (6)$$

and

$$\frac{\omega}{\phi^2}\phi'\dot{\phi} + \frac{\dot{\phi}'}{\phi} - \frac{\dot{R}\phi'}{R\phi} = 0. \quad (7)$$

From Eq. (2), we get

$$\left[-\frac{(1-kr^2)}{R^2}\phi'' + \frac{(3kr^2-2)}{R^2 r}\phi' + \frac{3\dot{R}\dot{\phi}}{R} + \ddot{\phi}\right] = \\ = \frac{4\Lambda}{(3+2\omega)}, \quad (8)$$

where a dot (.) and dash (′) denote the differentiation with respect to the time t and r, respectively. From Eqs. (4) and (5), we obtain the relation

$$\frac{\phi''}{\phi'} + \omega\frac{\phi'}{\phi} = \frac{1}{r} + \frac{kr}{1-kr^2} \quad (9)$$

under the conditions $\phi' \neq 0, 1-kr^2 \neq 0$. Integrating Eq. (9), we get

$$\phi^{\omega+1} = B\sqrt{1-kr^2} + D \quad (10)$$

provided $k \neq 0$, where B and D are arbitrary functions of time t.

Using (10) in (4) and (5), we obtain

$$\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} - \Lambda = -\frac{\dot{L}}{(1+\omega)\phi^{1+\omega}} - \\ - \frac{\omega B^2 k^2 r^2}{2R^2(1+\omega)^2(\phi^{1+\omega})^2} + \frac{\omega L^2}{2(1+\omega)^2(\phi^{1+\omega})^2} - \\ - \frac{2Bk\sqrt{1-kr^2}}{R^2(1+\omega)\phi^{1+\omega}} - \frac{2\dot{R}L}{R(1+\omega)\phi^{1+\omega}}, \quad (11)$$

where $L = \dot{B}\sqrt{1-kr^2} + \dot{D}$, $\dot{L} = \ddot{B}\sqrt{1-kr^2} + \ddot{D}$.

Using (10) in (6), we obtain

$$3\left(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2}\right) - \Lambda = \frac{\omega L^2}{2(1+\omega)(\phi^{1+\omega})^2} - \\ - \frac{\omega B^2 k^2 r^2}{2R^2(1+\omega)^2(\phi^{1+\omega})^2} - \frac{3Bk\sqrt{1-kr^2}}{R^2(1+\omega)\phi^{1+\omega}} - \\ - \frac{3\dot{R}L}{R(1+\omega)\phi^{1+\omega}}. \quad (12)$$

Using (10) in (8), we obtain

$$\left[\frac{3Bk\sqrt{1-kr^2}}{R^2(1+\omega)\phi^{1+\omega}} + \frac{\omega B^2 k^2 r^2}{R^2(1+\omega)^2(\phi^{1+\omega})^2} + \right. \\ + \frac{3\dot{R}L}{R(1+\omega)\phi^{1+\omega}} - \frac{\omega L^2}{(1+\omega)^2(\phi^{1+\omega})^2} + \\ \left. + \frac{\dot{L}}{(1+\omega)\phi^{1+\omega}}\right] = \frac{4\Lambda}{(3+2\omega)}. \quad (13)$$

Using (10) in (7), we obtain

$$\frac{\dot{B}}{B} = \frac{\dot{R}}{R}. \tag{14}$$

Now, we shall determine the values of five unknowns B, ω, R, Λ, and D, by using four equations (11), (12), (13), and (14). Since the number of unknowns is more than the number of equations, this is a case of underdeterminacy, so it is reasonable to assume a physical relation to solve the field equations. Now, we try to solve the field equations under different physical situations.

Case I: Taking the arbitrary constant $D = 0$ and using Eq. (14) in (11), (12), and (13), we obtain the relations

$$\left(\frac{\dot{R}}{R}\right)^2 \left[\frac{2(1+\omega)^2 + 3\omega + 4}{2(1+\omega)^2}\right] + \frac{3+2\omega}{1+\omega}\frac{\ddot{R}}{R} = = \Lambda - \frac{k}{R^2}\left[\frac{\omega k r^2}{2(1+\omega)^2(1-kr^2)} + \frac{3+\omega}{1+\omega}\right], \tag{15}$$

$$\left(\frac{\dot{R}}{R}\right)^2 \left[3 - \frac{\omega}{2(1+\omega)^2} + \frac{3}{1+\omega}\right] - \Lambda = - = -\frac{k}{R}\left[3 + \frac{\omega k r^2}{2(1+\omega)^2(1-kr^2)} + \frac{3}{1+\omega}\right], \tag{16}$$

$$\left[\frac{3k}{R^2(1+\omega)} + \frac{\omega k^2 r^2}{R^2(1+\omega)^2(1-kr^2)} + + \frac{3+2\omega}{(1+\omega)^2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{R}}{R(1+\omega)}\right] = \frac{4\Lambda}{(3+2\omega)}. \tag{17}$$

To obtain the exact solutions from Eqs. (15), (16), and (17), we consider a case where the coupling constant $\omega = 0$. Then Eqs. (15), (16), and (17) are reduced to the following forms:

$$3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{\ddot{R}}{R} - \Lambda = -\frac{3k}{R^2}, \tag{18}$$

$$3\left(\frac{\dot{R}}{R}\right)^2 - \frac{\Lambda}{2} = -\frac{3k}{R^2}, \tag{19}$$

$$3\left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{R}}{R} + \frac{3k}{R^2} = 4\Lambda. \tag{20}$$

Corresponding to $k = -1$, Eqs. (18), (19), and (20) imply that $\Lambda = 0$ and $R = t$. In this case, the value of ϕ from Eqs. (10) is given by

$$\phi = t\sqrt{1+r^2}. \tag{21}$$

From Eqs. (14) and (21), we observe that the expansion parameter is purely a function of the time t, while the B–D scalar ϕ is a function of both r and t. Here, $r \to \infty$, $\phi \to \infty$, while R remains finite. However, as $t \to \infty$, both ϕ and R tends to ∞. We can further conclude that, corresponding to $k = -1$ and $\omega = 0$, the B–D scalar ϕ is an increasing function of both r and t, since the B–D scalar ϕ and the gravitational variable G [18] are related by the relation

$$G = \frac{1}{\phi}\left(\frac{4+2\omega}{3+2\omega}\right). \tag{22}$$

So, the gravitational variable

$$G \propto \frac{1}{\phi}, \tag{23}$$

i.e., G decreases, as t (or r) increases. From Eq. (14), we further observe that, at the initial stage (i.e., when $t = 0$), the radius of the Universe is zero, thereby showing that the Universe was concentrated to a mass point and expands gradually till it becomes infinitely large, which supports the present finding for the accelerated expansion of the Universe. This is in conformity with the steady state theory of the cosmological Universe. The corresponding deceleration parameter is zero.

Case II: Taking $\phi' = 0$ and $3 + 2\omega \neq 0$ in the field equations, we obtain

$$3\frac{\dot{R}}{R}\dot{\phi} + \ddot{\phi} = \frac{4\Lambda}{3+2\omega}, \tag{24}$$

$$\frac{k}{R^2} + \left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} - \Lambda = -\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - 2\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi}, \tag{25}$$

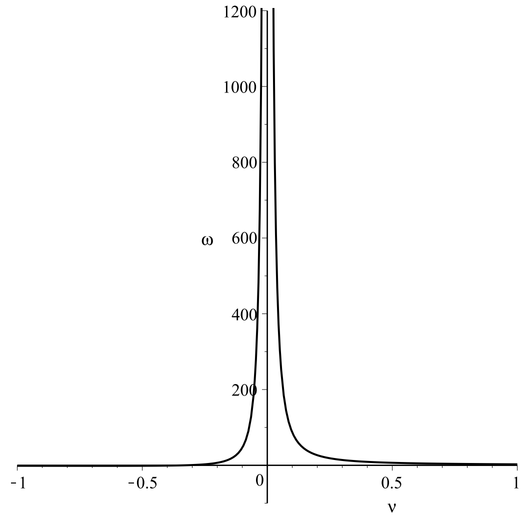
and

$$\frac{3k}{R^2} + 3\left(\frac{\dot{R}}{R}\right)^2 - \Lambda = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 - 3\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi}. \tag{26}$$

Under the conditions $\Lambda = 0$ and $k = 0$, relations (24)–(26) become

$$\frac{\dot{R}}{R} = -\frac{1}{3}\frac{\ddot{\phi}}{\dot{\phi}}, \tag{27}$$

$$\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} = -\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - 2\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi}, \tag{28}$$



Variation of ω for various values of ν according to (37)

$$3\left(\frac{\dot{R}}{R}\right)^2 = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 - 3\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi}. \tag{29}$$

Adding (28) and (29), we get

$$4\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} = -5\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi}. \tag{30}$$

Integrating (27) and (30), we get

$$R^3\dot{\phi} = a = \text{const}, \tag{31}$$

$$\phi\frac{d}{dt}(R^3) = b = \text{const}. \tag{32}$$

The sum of Eqs. (31) and (32) becomes

$$\frac{d}{dt}(\phi R^3) = a + b = c = \text{const}. \tag{33}$$

Integrating, we get

$$\phi = \frac{ct + l}{R^3}, \tag{34}$$

where c and l are constants. Moreover, from (31) and (32), we get

$$\frac{\dot{\phi}}{\phi} = 3\nu\frac{\dot{R}}{R}, \tag{35}$$

where $\nu = \frac{a}{b} = \text{const}$.

Using (35) in (29), we have

$$\left(1 + 3\nu - \frac{3}{2}\omega\nu^2\right)\left(\frac{\dot{R}}{R}\right)^2 = 0. \tag{36}$$

Since $\frac{\dot{R}}{R} \neq 0$, Eq. (36) becomes

$$\omega = \frac{2}{3}\left(\frac{1 + 3\nu}{\nu^2}\right). \tag{37}$$

The variation of ω according to (37) for variuos values of ν has been shown in Figure.

For $\nu = -\frac{1}{3}$ and $\omega = 0$, we get that there is neither expansion nor contraction of the Universe, where the B–D scalar ϕ decreases with time, till it vanishes as $t \to \infty$.

In addition, when $\nu = -1$ and $\nu = -\frac{1}{2}$, we get $\omega = -\frac{4}{3}$, which implies that the B–D scalar ϕ and the gravitational variable G will remain finite for all finite values of the time t. Here, corresponding to $\omega = 0$ and $\omega = -\frac{4}{3}$ from Eq. (22), we find that $G \propto \frac{1}{\phi}$, as in Eq. (23), which implies that ϕ and G will remain finite for all finite values of time t, and the gravitational variable G will be an increasing function of the time.

3. Conclusion

Here, we have seen that the role played by the scalar ϕ relating to the contraction and the expansion of the Universe consists in that the B–D scalar ϕ, which is an increasing function of the time, can be treated as something reflecting the contraction of the Universe, while the B–D scalar ϕ which is a decreasing function of the time may be treated as something reflecting the expansion of the Universe.

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Received 11.04.15

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СФЕРИЧНО-СИМЕТРИЧНИЙ
Р/У ВСЕСВІТ, ВЗАЄМОДІЮЧИЙ
З ВАКУУМНИМ Б–Д СКАЛЯРНИМ ПОЛЕМ

Р е з ю м е

Розглянуто сферично-симетричну вакуумну космологічну модель Всесвіту, взаємодіючу зі скалярним Бранса–Діке (Б–Д) полем в метриці Робертсона–Уолкера (Р/У). Отримано точні залежні від часу рішення Б–Д вакуумних польових рівнянь для двох різних випадків. Докладно обговорюються фізичні і динамічні властивості моделі.