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SPIN IN UNIFORM GRAVITY, HIDDEN MOMENTUM, AND THE ANOMALOUS HALL EFFECT[1]

We review the recent discussions concerning the absence of the spin Hall effect in a uniform gravitational field, pointing out differences from the anomalous spin Hall effect in ferromagnetic systems despite a similar form of the Hamiltonian.

Keywords: spin Hall effect, gravitational field, anomalous spin Hall effect.

1. Introduction

Spin–orbit phenomena probe how internal angular momentum couples to background fields through the particle's motion, i.e. its orbital degrees of freedom. In condensed matter, this coupling produces, for example, the anomalous Hall effect (AHE): a transport current transverse to an applied electric field in ferromagnets, described by an antisymmetric conductivity σ_{ij}^{H}, analysed by Karplus and Luttinger (KL) [1] and subsequently interpreted in a Berry curvature formulation (see [2] and references therein; see also [3, 4]; excellent reviews are [5–7]). In contrast, a recent claim of a "gravitational spin Hall effect" (SHE) in a uniform field for Dirac wave packets asserts a polarization-dependent transverse deflection without any transport geometry (no driven current) [8].

Our recent analysis [9] resolves this tension. First, an object with spin $\boldsymbol{S}$ in a uniform gravitational field $\boldsymbol{g}$ carries a hidden momentum $\mathbf{p}_{\text{hidden}} \sim \boldsymbol{S}\times\boldsymbol{g}/c^2$ that modifies the relation between canonical momentum and velocity. Second, for Dirac wave packets evolved with the Foldy–Wouthuysen (FW) Hamiltonian in a linear potential, the canonical transverse momentum is conserved, while the velocity operator contains a spin term. The "at rest" initial state with $\langle\mathbf{p}\rangle = 0$ in [8] is therefore not at mechanical rest; preparing true rest requires a spin-dependent phase in the initial packet. With that preparation, any $O(g)$ transverse drift vanishes. Third, using a FW transformation (linear in g, all orders in $1/c$), we showed that spin-dependent transverse motion can appear no earlier than $O(g^2)$ for a broad class of states. These points together exclude a gravitational SHE at linear order in a uniform field.

A second theme of this note is to contrast this outcome with the KL AHE. The KL current is a linear-response, preparation-independent transport effect in crystals: $J_i = \sum_j \sigma_{ij}E_j$, with an intrinsic contribution whose microscopic origin relies on Bloch periodicity: the interband matrix elements of the periodic part of $-e\mathbf{E}\cdot\mathbf{r}$ (KL's $\mathcal{H}_1''$) are essential. In a uniform gravitational field there is no lattice periodicity (the natural basis is plane waves), so the KL objects $J_b^{nn'}(\mathbf{k})$ and $\mathcal{H}_1''$ have no counterpart and the intrinsic Hall pathway is absent.

Below we summarise the hidden-momentum mechanism and its role in uniform gravity, restate and streamline the $O(g)$ no-go result, and we contrast elements of the KL mechanism with the uniform gravity situation, highlighting the missing ingredients for Hall transport in the latter.

2. What Constitutes a Hall Effect?

A Hall effect is a transverse transport response to a longitudinal drive, involving, among other characteristics,

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[1] This work is based on the results presented at the 2025 "New Trends in High-Energy Physics" Conference.

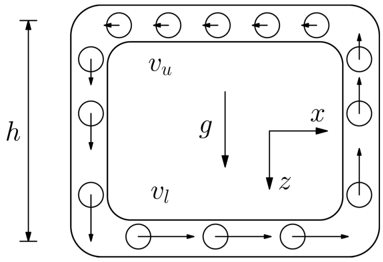

Fig. 1. Masses circulating in a frictionless pipe bent into a rectangle. The rectangle is placed in a constant gravitational field with field strength g near the Earth

1. Driver and linear response. A uniform field $\mathbf{E}$ drives a current $J_i = \sum_j \sigma_{ij} E_j$; the antisymmetric part $\sigma^{\mathrm{H}}_{ij} = \frac{1}{2}(\sigma_{ij} - \sigma_{ji})$ is responsible for the Hall effect.

2. Transport coefficient. The transverse deflection is caused by σ^{H}_{ij} (or ρ_{ij}), not by an initial-state-dependent initial velocity.

These conditions are satisfied by KL's AHE but fail in a uniform gravitational field for neutral Dirac packets, where neither a transverse driver nor σ_{xy} are present.

3. Spinning Particle in a Gravitational Field

In this section, we use a simple model to show that a spinning object gains a "hidden momentum" that modifies the relation between velocity and momentum in a gravitational field. However, the trajectory of the object in a uniform gravitational field is not modified by the spin.

3.1. The hidden momentum

The idea of the hidden momentum first appeared in a 1967 paper by Shockley and James [10] in the context of electromagnetism. A concise discussion and a list of key references up to 2012 can be found in [11]. Following the electromagnetism discussion in [12], we consider a circulating mass model in a gravitational field, as shown in Fig. 1.

In the rectangular pipe, the upper section contains n_u masses moving with speed v_u and momenta p_u, the lower section contains n_l masses moving with speed v_l and momenta p_l, and the masses accelerate/decelerate in the left/right vertical sections that connect the upper and lower pipes. Current conservation imposes a condition on the velocity $n_u v_u = n_l v_l$, whereas the total momentum reduces to a difference between the momenta in the lower and upper section $p_{\mathrm{tot}} = n_l p_l - n_u p_u$. Without relativistic effects, the relation $p = mv$ guarantees that the momentum within the pipe adds up exactly to zero; however, when relativistic effects are accounted for, the momenta gain an extra factor γ due to gravitational and special relativity time dilation:

$$p_u = m\gamma_u v_u, \quad p_l = m\gamma_l v_l, \tag{1}$$

$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{gz}{c^2}\right)^2 - v^2/c^2}} \simeq 1 + \frac{gz}{c^2} + \frac{v^2}{2c^2}, \tag{2}$$

where the origin is chosen at the center of the rectangular loop, and the z-direction is along the same direction as $\boldsymbol{g}$. With the difference in γ between the upper and lower sections, the total momentum gains a "hidden" x-component, in the sense that does not vanish even when there is no overall linear motion of the system:

$$p_{\mathrm{hidden}} = n_l m v_l \left(\frac{gh}{c^2} + \frac{v_l^2 - v_u^2}{2c^2}\right) = $$
$$= 2n_l m v_l \frac{gh}{c^2}. \tag{3}$$

To make manifest the connection between angular motion and hidden momentum, the result can be rewritten in terms of the total angular momentum of the system:

$$\boldsymbol{p}_{\mathrm{hidden}} = \frac{\boldsymbol{L} \times \boldsymbol{g}}{c^2}. \tag{4}$$

3.2. Trajectory in a uniform gravitational field

It is claimed in [8] that particles carrying spin will be deflected in a spin-dependent way in a uniform gravitational field. Here we take another look at the discussion from the viewpoint of hidden momentum.

In [8], the Dirac Hamiltonian coupled to the uniform gravitational field is Foldy–Wouthuysen-transformed into a non-relativistic single-particle Hamiltonian at leading order in g. For positive energy states, this Hamiltonian is written as

$$\mathcal{H}_{\mathrm{FW}} = Vmc^2 + \frac{\boldsymbol{p} \cdot (V\boldsymbol{p})}{2m} +$$
$$+ \frac{g\hbar}{4mc^2} (\sigma_x p_y - \sigma_y p_x), \tag{5}$$

$$V = 1 - \frac{gz}{c^2}. \tag{6}$$

By evaluating the time evolution of the expectation value of the position operators for a Gaussian wave packet with $\langle \boldsymbol{x}(0)\rangle = 0$, $\langle \boldsymbol{p}(0)\rangle = 0$, and $\langle \sigma_y\rangle = \pm 1$, [8] concluded that particles with spin along the $\pm y$ direction are deflected in the x direction with the velocity $v_x = \mp g\hbar/(4mc^2)$. However, as shown in [9], a look at the classical Hamilton's equations

$$m\frac{dx}{dt} = m\frac{\partial \mathcal{H}_{\rm FW}}{\partial p_x} = Vp_x - \frac{g\hbar}{4c^2}\sigma_y. \tag{7}$$

shows that the particle carries a hidden momentum, which indicates that the deflection is due to a nonzero initial velocity rather than a dynamical evolution of the wave packet.

Further, we show that this spin-dependent deflection of the trajectory is unobservable in a free-fall experiment, to the order in g to which the Hamiltonian (6) is valid. To illustrate our point, we consider the trajectory of three particles: two Dirac particles with the spin along the $\pm y$ direction, and one scalar that does not carry spin. With the three trajectories initially parallel to each other and carrying the same initial velocity along the z direction, a spin-dependent deflection would cause the trajectories to deflect from each other, as shown in Fig. 2, *a*; however, the condition that the three trajectories are initially parallel to each other implies

$$p_x(0) = \frac{g\hbar}{4c^2}\sigma_y \tag{8}$$

for the Dirac particles, and

$$p_x(0) = 0 \tag{9}$$

for the scalar particle. Therefore, due to the cancellation between the initial (total) momentum and the hidden momentum, the subsequent motions of the three particles are completely identical, as shown in Fig. 2, *b*. While we discuss the classical trajectory for simplicity, a complete analog of the discussion can be obtained in quantum mechanics as in [9].

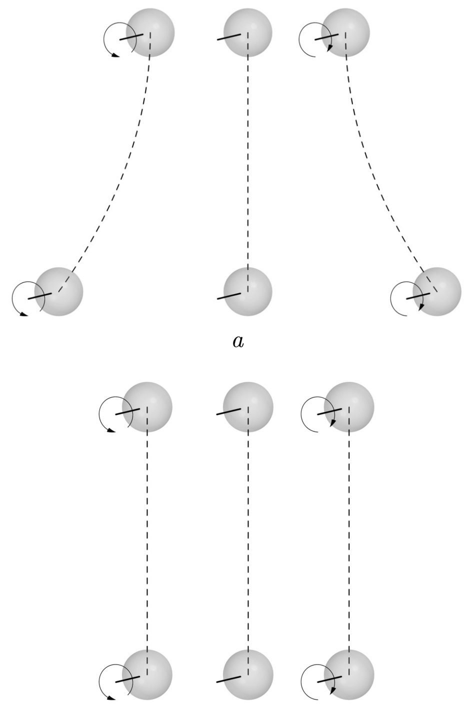

Fig. 2. Trajectories for particles with spin $\sigma_y = +1$ (left), $\boldsymbol{\sigma} = 0$ (middle), and $\sigma_y = -1$ (right) when (*a*) the trajectories experience a spin-dependent deflection; (*b*) the trajectories are independent of the spin. The initial velocity is along the z-direction (vertical) and is chosen to be the same for all particles for the ease of comparison

4. Comparison with Hall Effects in Ferromagnetics

While the spin-orbit coupling in the uniform gravitational field does not change the orbit, there has been a belief since the 1950s that a similar type of coupling is responsible for the anomalous Hall effect in ferromagnetics [1]. In this section we briefly review this mechanism and discuss the difference in the consequences.

As considered in [1], the Hamiltonian for an electron in ferromagnetics with spin polarized along the direction of the magnetization $\boldsymbol{M}$ in an external electric field $\boldsymbol{E}$ can be written as

$$\begin{aligned} &\mathcal{H}_T = \mathcal{H}_0 + \mathcal{H}' + \mathcal{H}'', \\ &\mathcal{H}_0 = \frac{p^2}{2m} + U(\boldsymbol{r}), \\ &\mathcal{H}' = \frac{1}{4m^2c^2}\frac{(\boldsymbol{M}\times\boldsymbol{\nabla}U)\cdot\boldsymbol{p}}{M_s}, \\ &\mathcal{H}'' = -e\boldsymbol{E}\cdot\boldsymbol{r}, \end{aligned} \tag{10}$$

where $U(\boldsymbol{r})$ is the periodic crystal potential, M_s is the maximum magnetization, and e is the charge of the electron. Under the eigenbasis $\phi_{(n,\boldsymbol{k})}$ of $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$, $\mathcal{H}''$ is found to have both a singular part $\mathcal{H}_i''$ responsible for the usual conductivity effects, whose matrix element is

$$\langle n\boldsymbol{k}|\mathcal{H}_1''|n'\boldsymbol{k}'\rangle = -ie\delta_{nn'}\boldsymbol{E}\cdot\boldsymbol{\nabla}_k\delta_{\boldsymbol{k}\boldsymbol{k}'}, \tag{11}$$

and a regular periodic part $\mathcal{H}_1''$ responsible for the anomalous Hall effect which gives the matrix element

$$\langle n\boldsymbol{k}|\mathcal{H}_1''|n'\boldsymbol{k}'\rangle = -ie\delta_{\boldsymbol{k}\boldsymbol{k}'}\boldsymbol{E}\cdot\boldsymbol{J}^{nn'}, \tag{12}$$

where

$$\boldsymbol{J}^{nn'}(\boldsymbol{k}) = \int w_{n\boldsymbol{k}}^*(r)\boldsymbol{\nabla}_k w_{n'\boldsymbol{k}}(r)d^3x, \tag{13}$$

and $\phi_{(n,k)} = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}w_{n\boldsymbol{k}}(\boldsymbol{r})$. Averaging the velocity over the Fermi distribution ρ then gives

$$\bar{v}_a = -ieE_b\sum_l \rho_0'(\epsilon_l)v_b(l)J_a(l), \tag{14}$$

where a and b are vector indices, ρ_0 is the distribution with respect to $\mathcal{H}$, $\rho_0' = \partial\rho_0/\partial\epsilon$, and $J_a(l) = J_b^{nn}(k)$. The average velocity here then gives rise to a transverse current $\boldsymbol{J} = N_d e\bar{\boldsymbol{v}} = r\boldsymbol{M}\times\boldsymbol{E}$, where the coefficient r depends on the material properties.

Comparing with the discussion for the spin-orbit coupling in uniform gravitational field in Sec. 3, we note that, different from ferromagnetics where the lattice potential $U(\boldsymbol{r})$ gives rise to a non-trivial eigenbasis $\phi_{(n,\boldsymbol{k})}$, the eigenbasis for the Hamiltonian in Eq. (6) in the absence of the gravitational field are plane waves. Therefore, the regular periodic perturbation $\mathcal{H}_1''$ does not have a counterpart in the uniform gravitational field. Indeed, as already noted in [1], when the band becomes almost free, its contribution to the average velocity becomes almost zero. Therefore, while these two effects look very similar, the transverse velocity in ferromagnetics does not occur in the uniform gravitational field due to the absence of the essential lattice potential.

5. Summary

In this note, we reviewed the recent discussion for the hidden momentum and the trajectory of Dirac particles in the uniform gravitational field, and compared the results obtained therein with the Hall effects in ferromagnetics.

Using a circulating mass model, we have shown that a spinning object obtains a hidden momentum in a gravitational field and this hidden momentum modifies the relation between velocity and canonical momentum. However, with a given trajectory, it is not possible to tell the spin of the particle, and therefore, up to the linear in g approximation adopted here, the spin Hall effect does not exist in the uniform gravitational field in the usual sense. While a similar Hamiltonian leads to the anomalous spin Hall effect in ferromagnetics, the lattice potential is crucial for this phenomenon and the same conclusion does not apply to free particles in the uniform gravitational field.

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СПІН В ОДНОРІДНОМУ ГРАВІТАЦІЙНОМУ ПОЛІ, ПРИХОВАНИЙ ІМПУЛЬС ТА АНОМАЛЬНИЙ ЕФЕКТ ХОЛЛА

Ми розглядаємо нещодавні дискусії щодо відсутності спінового ефекту Холла в однорідному гравітаційному полі, вказуючи на відмінності від аномального спінового ефекту Холла у феромагнетиках, незважаючи на подібність форми гамільтоніана.

Ключові слова: спіновий ефект Холла, гравітаційне поле, аномальний спіновий ефект Холла.