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THE μ-DEFORMED EINSTEIN FIELD EQUATIONS WITH μ-DEPENDENT EFFECTIVE COSMOLOGICAL CONSTANT

In this paper, we derive the μ-deformed Einstein field equations from the generalized thermodynamic functions of the μ-deformed analog of Bose gas model, applying the (adapted) Verlinde's approach. The basic role of deformation parameter is shown: it provides the possibility to vary the value of the cosmological constant. Due to this, we suggest an interesting treatment of the cosmological constant (CC) problem within the framework of μ-deformation. Namely, viewing the derived μ-deformed CC as an effective one and varying the parameter μ appropriately, we gain the possibility to drastically reduce the CC, so as to get, for it, the realistic value. The relation to dark matter is of importance.

Keywords: μ-calculus, entropic force, gravity, μ-Bose gas model, effective cosmological constant, dark matter.

1. Introduction

Various deformed algebras have provided valuable insights into a wide range of problems across diverse branches of physics. Among different types of deformation, which are most conveniently and clearly characterized by the deformation structure function (DSF), see, e.g., [1], there are very popular and well-studied deformed algebras of the exponential type like the q-oscillators [2–4] or p, q-oscillators [5]. Unlike, the μ-deformation first introduced by Jannussis [6] belongs to a very different class of noncanonical Heisenberg algebras, namely, the class of rational deformations. That implies very different properties, and non-Fibonacci nature [7] is one of them. This extension has opened new perspectives for exploring quantum systems with modified algebraic structures, offering potential applications in such fields as high-energy physics, quantum gravity, and physics of dwarf galaxies.

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The μ-deformation was developed in several aspects and applied to various physical problems: the quasi-Fibonacci nature of the nonlinear μ-deformed oscillator has been established, as well as for the proposed extensions or hybrid cases $(\mu; p, q)$ [7]; the application to deformed versions of the non-relativistic Bose gas model has been constructed [8,10,11]; the intercepts of r-particle momentum correlation functions in the μ-Bose gas model have been derived [8,9]. Closely related deformations have been introduced and shown as able to account for compositeness and interaction of particles [12]. Relevant version of deformation was successfully applied to describe the entanglement entropy of composite (quasi)boson systems [13, 14]; it is worth noting the temperature dependence of virial coefficients and correlation function intercepts in the (μ, q)-Bose gas model [15, 16], the condensate of the μ-Bose gas as an effective model of galactic-halos dark matter [17, 18], the halo density profile of dwarf or low surface brightness galaxies and their rotation curves [19]. Also, one should mention the related research on deformed versions of Heisenberg algebra for the position and momentum operators. In particular, the three-parametric (p, q, μ)-deformed Heisenberg algebra [20] was shown to possess unusual properties, including pseudo-hermiticity of the involved operators. The special new

μ-deformed Heisenberg algebra has been derived [21] in the context of the approach to dark matter based on the μ-deformation [19].

Now, let us go over to one of well-known approaches to quantum gravity, namely, the induced gravity theory. As known, it branches into the Sakharov's quantum-field approach [22, 23], on one hand, and the emergent (or thermostatistical, or entropic) theory on the other. In general context of quantum gravity, Verlinde formulated a theory of induced gravity, wherein gravity is understood as an emergent phenomenon arising from entropic forces [24]. His approach is rooted in statistical mechanics and involves the holographic principle, as originally proposed by 't Hooft [25]; it is distinguished from Padmanabhan's thermostatistical perspective on gravity [26].

Verlinde's framework postulates that gravitational dynamics, as described by either Newtonian theory or General Relativity, are not fundamental interactions but emerge as effective macroscopic description from an underlying statistical system. This system is governed by classical statistics and holographic principle, where the thermodynamic quantities, such as entropy, play a pivotal role in the emergence of spacetime geometry and gravitational forces. The entropic force, in this view, can be interpreted as a manifestation of the underlying microscopic degrees of freedom encoded in the statistical properties of the system [24].

Furthermore, Verlinde's approach suggests that spacetime itself may be an emergent phenomenon, derived from the collective behavior of microscopic degrees of freedom, much like thermodynamic quantities emerge from the microscopic states of matter. This perspective has profound implications for the understanding of quantum gravity, providing a potential bridge between the thermodynamic description of space-time and the quantum mechanical description of fundamental interactions. The holographic principle further strengthens this connection, implying that the degrees of freedom (bits of information) describing the gravitational system are encoded on a lower-dimensional boundary, akin to the AdS/CFT correspondence in string theory [54, 55]. Thus, Verlinde's theory opens new avenues for exploring the relationship between gravity, thermodynamics, and quantum mechanics.

In addition to classical statistics, deformed analogs of quantum statistics can be applied in the context of induced gravity [32, 33]. The corresponding physical systems (e.g., quantum black holes) will be described by deformed thermostatistical functions and, by applying the Verlinde's approach to them, one obtains deformed analogs of the Einstein field equations. Adopting Ubriaco's one-parameter quantum group $SU_q(2)$ interacting boson gas model results in the respective q-deformed Einstein field equations [27], the use of (p, q)-deformed Fermi gas model induces (p, q)-deformed Einstein field equations [28,29], and for Viswanathan–Parthasarathy–Jagannathan–Chaichian (VPJC) q-deformed fermion gas model there arises q-deformed Einstein field equations [30]; likewise, with q-deformed fermion gas model in two-dimensional space yet another type of q-deformed Einstein field equations does emerge [31].

Deformed Einstein equations for the q-deformed boson and q-fermion gas models at the high-temperature limit are studied in Ref. [34]. At last, in Ref. [37], two versions of modified Einstein equations were obtained basing on the GUP corrected Unruh temperature and the Verlinde's approach. Verlinde's approach has encountered an ambiguous reflections and some criticism, see, e.g., Ref. [38–45]. On the other hand, there is a wide range of modified entropic gravities, that includes noncommutativity, ungravity as conformal invariant fields, asymptotically safe gravity, Debye energy correction, surface entropic gravity, *etc.* [45–49, 53].

The paper is organized as follows. First, a brief review of the μ-Bose thermodynamics is given. Second, we will derive a μ-deformed analog of the Einstein equation following the logic of Verlinde's approach and basing on the μ-deformed Bose gas model. Note that, within the developed version of entropic gravity, the μ-deformed Friedmann equation can be inferred and its application to cosmology can be explored. Third, the implications for the obtained effective (depending on μ) extension of cosmological constant are analyzed in detail. The work is ended with concluding remarks. Throughout the paper, we set the units so that $c = \hbar = 1$.

2. The μ-Deformed Einstein Equations Using Verlinde Approach

2.1. The thermostatics of μ-Bose gas model

We start with recalling some facts, which are necessary for what follows [10, 11]. In the standard Bose gas model, the total number of particles is expressed as the Euler derivative of the logarithm of the grand partition function $\mathcal{Z}$, i.e

$$\mathcal{N} = z\frac{d}{dz}\ln\mathcal{Z}, \tag{1}$$

where the fugacity z as function of the chemical potential $\tilde{\mu}$ is given by $z = e^{\beta\tilde{\mu}}$, $\beta = \frac{1}{k_{\mathrm{B}}T}$ is the inverse temperature involving the Boltzmann constant k_{B}. The grand canonical partition function $\mathcal{Z}$ is treated through its logarithm by the following expression:

$$\ln\mathcal{Z} = -\sum_i \ln(1 - ze^{-\beta\epsilon_i}), \tag{2}$$

with ϵ_i being

$$\epsilon_i = \frac{|\mathbf{p}|^2}{2m} = \frac{p_i^2}{2m}. \tag{3}$$

From now on, we restrict ourselves to three-dimensional space. To formulate the thermodynamic framework for the μ-analog of the Bose gas model, the standard expression for the total number of particles $\mathcal{N}$ is supposed to incorporate the μ-dependence. Namely, the modified definition for the total number of particles is given as

$$\mathcal{N}^{(\mu)} = z\,\mathcal{D}_z^{(\mu)}\ln\mathcal{Z} = -z\,\mathcal{D}_z^{(\mu)}\sum_i \ln\left(1 - ze^{-\beta\epsilon_i}\right), \tag{4}$$

where we use the μ-derivative

$$\mathcal{D}_x^{(\mu)} f(x) = \int_0^1 f_x'(t^\mu x)\,dt, \quad f_x'(t^\mu x) = \frac{df(t^\mu x)}{dx}, \tag{5}$$

which acts on the monomials as

$$\mathcal{D}_x^{(\mu)} x^n = [n]_\mu x^{n-1}, \tag{6}$$

where

$$[n]_\mu \equiv \frac{n}{1+\mu n}, \quad 0 \le \mu \le 1. \tag{7}$$

Then the μ-deformed operator $\mathcal{D}^{(\mu)}$ is applied to the logarithm of the grand canonical partition function in Eq. (2), and the modified total number of particles reads

$$\mathcal{N}^{(\mu)} = z\sum_i\sum_{n=1}^{\infty} e^{-\beta\epsilon_i n}\frac{[n]_\mu}{n} z^{n-1} = \\ = \sum_i\sum_{n=1}^{\infty}\frac{[n]_\mu}{n}\left(e^{-\beta\epsilon_i}\right)^n z^n. \tag{8}$$

For the series to converge, the following condition for the product $e^{-\beta\epsilon_i}z$ must hold:

$$\lim_{n\to\infty}\sum_i\sum_{n=1}^{\infty}\frac{[n]_\mu}{n}\left(e^{-\beta\epsilon_i}\right)^n z^n < \infty \Rightarrow 0 \le |ze^{-\beta\epsilon_i}| < 1. \tag{9}$$

Separate the contribution of the $p_i = 0, i = 0$ term from the remaining sum. This yields:

$$\mathcal{N}^{(\mu)} = {\sum_i}'\sum_{n=1}^{\infty}\frac{[n]_\mu}{n}\left(e^{-\beta\epsilon_i}\right)^n z^n + \sum_{n=1}^{\infty}\frac{[n]_\mu}{n}z^n. \tag{10}$$

As is well known, for a large volume V and a large number of particles $\mathcal{N}$, the spectrum of single-particle states becomes nearly continuous. Due to this, we replace the summation in Eq. (8) by a 3-integral over the 3-momentum space

$$\sum_i \to \frac{V}{(2\pi\hbar)^3}\int d^3k. \tag{11}$$

Then we obtain

$$\mathcal{N}^{(\mu)} = \frac{V}{\lambda_T^3}\sum_{n=1}^{\infty}\frac{[n]_\mu}{n^{5/2}}z^n + \mathcal{N}_0^{(\mu)}, \quad \mathcal{N}_0^{(\mu)} \equiv \sum_{n=1}^{\infty}\frac{[n]_\mu}{n}z^n. \tag{12}$$

Here the thermal wavelength λ_T is defined in terms of the particle mass m and the temperature T (viewed as the thermodynamic temperature):

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}. \tag{13}$$

The standard Bose–Einstein function is given by the series

$$g_\ell(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^\ell}. \tag{14}$$

As a generalization of the latter, the μ-polylogarithm is introduced, namely,

$$g_\ell^{(\mu)}(z) = \sum_{n=1}^{\infty}\frac{[n]_\mu}{n^{\ell+1}}z^n. \tag{15}$$

Then the Eq. (10) becomes

$$\mathcal{N}^{(\mu)} = \frac{V}{\lambda_T^3}g_{3/2}^{(\mu)}(z) + g_0^{(\mu)}(z), \tag{16}$$

where

$$g^{(\mu)}_{3/2}(z) = \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{5/2}} z^n, \quad g^{(\mu)}_0(z) = \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n. \quad (17)$$

The μ-deformed $\ln \mathcal{Z}^{(\mu)}$ is obtained by applying the inverse of the Euler derivative, namely,

$$\ln \mathcal{Z}^{(\mu)} = \left(z\frac{d}{dz} \right)^{-1} \mathcal{N}^{(\mu)}. \quad (18)$$

After inserting $\mathcal{N}^{(\mu)}$

$$\left(z\frac{d}{dz} \right)^{-1} \left(\frac{V}{\lambda_T^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{5/2}} z^n + \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} z^n \right) = = \frac{V}{\lambda_T^3} \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{5/2}} \left(z\frac{d}{dz} \right)^{-1} z^n + \sum_{n=1}^{\infty} \frac{[n]_\mu}{n} \left(z\frac{d}{dz} \right)^{-1} z^n, \quad (19)$$

we get $\ln \mathcal{Z}^{(\mu)}$ in the form

$$\ln \mathcal{Z}^{(\mu)} = \frac{V}{\lambda_T^3} g^{(\mu)}_{5/2} + g^{(\mu)}_1. \quad (20)$$

The internal energy $\mathcal{U}^{(\mu)}$ is determined through the relation

$$\mathcal{U}^{(\mu)} = -\left(\frac{\partial}{\partial \beta} \ln \mathcal{Z}^{(\mu)} \right)_{z,V}. \quad (21)$$

With the usual β-derivative, we obtain the expression for the μ-deformed internal energy

$$-\frac{\partial}{\partial \beta} \left(\frac{V}{\lambda_T^3} g^{(\mu)}_{5/2} + g^{(\mu)}_1 \right)_{z,V} \Rightarrow \mathcal{U}^{(\mu)} = -\tilde{\mu} \mathcal{N}^{(\mu)}. \quad (22)$$

The equation of state then reads

$$\frac{(PV)_\mu}{k_{\mathrm{B}} T} = \ln \mathcal{Z}^{(\mu)} = \frac{V}{\lambda_T^3} g^{(\mu)}_{5/2}(z) + g^{(\mu)}_1(z). \quad (23)$$

The Helmholtz free energy of the model results as

$$\mathcal{H}^{(\mu)} = \tilde{\mu} \mathcal{N}^{(\mu)} - (PV)_\mu. \quad (24)$$

The μ-deformed entropy can be obtained from the relation $\mathcal{S}^{(\mu)} = \frac{1}{T_{\mathrm{U}}} \left(\mathcal{U}^{(\mu)} - \mathcal{H}^{(\mu)} \right)$ (with T_{U} viewed as the *Unruh temperature* [55]) to yield

$$\mathcal{S}^{(\mu)} = \frac{1}{T_{\mathrm{U}}} \left(-2\tilde{\mu} \frac{V}{\lambda_T^3} g^{(\mu)}_{3/2}(z) - 2\tilde{\mu} g^{(\mu)}_0(z) + E \frac{V}{\lambda_T^3} \times \times g^{(\mu)}_{5/2}(z) + E g^{(\mu)}_1(z) \right). \quad (25)$$

In the μ-deformed extension of Verlinde's emergent gravity, we use the identification $E = k_{\mathrm{B}} T$ rather than $\frac{1}{2} k_{\mathrm{B}} T$, assuming each μ-boson encodes two fundamental bits of information. Physically, μ-bosons may be viewed as composite Bose-like particles built of correlated or entangled pairs of elementary constituents (of either Fermi or Bose type, see [12–16] for the former one). Consequently, each μ-boson carries twice the energy of a single bit in standard equipartition, leading to the total energy $E = N k_{\mathrm{B}} T$ and preserving thermodynamic consistency when μ-deformation modifies the entropy-energy relation along with the emergent gravitational coupling. Explicitly rewriting the expression for the thermal wavelength and chemical potential in terms of $E = k_{\mathrm{B}} T$,

$$\lambda_T^3 = \left(\frac{2\pi\hbar^2}{mE} \right)^{3/2}, \quad \frac{\tilde{\mu}}{E} = \ln z, \quad (26)$$

we get the general expression for μ-deformed entropy as function of E:

$$\mathcal{S}^{(\mu)} = \frac{1}{T_U} \left(-2V \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} g^{(\mu)}_{3/2}(z) E^{5/2} \ln z - 2E \times \times g^{(\mu)}_0(z) \ln z + V \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} g^{(\mu)}_{5/2}(z) E^{5/2} + E g^{(\mu)}_1(z) \right). \quad (27)$$

This is of principal importance for what follows.

2.2. Derivation of the μ-deformed Einstein equations using Verlinde approach

In Verlinde's approach to entropic gravity, the system approaches to statistical equilibrium, when the entropic force stemming from the changes in entropy, becomes balanced with the forces that contribute to an increase in the entropy. Accordingly, the formulation suggests that gravity itself can be understood as an emergent force arising from the entropic dynamics of underlying microscopic degrees of freedom, rather than a fundamental interaction. At equilibrium, the total entropy $\mathcal{S}$ of the system remains constant, and it reaches an extreme value. This is a direct consequence of the second law of thermodynamics, which states that entropy tends to increase until a maximum is reached under the given constraints. In what follows, the entropy $\mathcal{S}^{(\mu)}(E, x^\nu)$ satisfies:

$$\frac{d}{dx^\nu} \mathcal{S}^{(\mu)}(E, x^\nu) = 0. \quad (28)$$

From Verlinde's perspective, the system's equilibrium is not only a condition of the force balance, but also a reflection of the deeper thermodynamic nature of gravity

$$\frac{\partial \mathcal{S}^{(\mu)}}{\partial E}\frac{\partial E}{\partial x^{\nu}} + \frac{\partial \mathcal{S}^{(\mu)}}{\partial x^{\nu}} = 0, \tag{29}$$

where the components represent the energy gradient and an entropic force

$$\frac{\partial E}{\partial x^{\nu}} = -\mathcal{F}_{\nu}, \quad \frac{\partial \mathcal{S}}{\partial x^{\nu}} = \nabla_{\nu}\mathcal{S}. \tag{30}$$

From the derivative of $\mathcal{S}^{(\mu)}$, we have

$$\frac{\partial \mathcal{S}^{(\mu)}}{\partial E} = \frac{1}{T_U}\left(\frac{5V}{2} g^{(\mu)}_{5/2}(z)\left(\frac{mE}{2\pi\hbar^2}\right)^{3/2} - 5V g^{(\mu)}_{3/2}(z) \times \right.$$
$$\left. \times \left(\frac{mE}{2\pi\hbar^2}\right)^{3/2} \ln z + g^{(\mu)}_1(z) - 2g^{(\mu)}_0(z)\ln z\right). \tag{31}$$

For simplicity, let us use the notation

$$G_1(\mu; z) = \frac{5}{2}\frac{V}{\lambda_T^3}\left(g^{(\mu)}_{5/2}(z) - 2\ln z g^{(\mu)}_{3/2}(z)\right), \tag{32}$$

$$G_2(\mu; z) = g^{(\mu)}_1(z) - 2g^{(\mu)}_0(z)\ln z. \tag{33}$$

The formula for the derivative then takes the form

$$\frac{\partial \mathcal{S}^{(\mu)}}{\partial E} = \frac{1}{T_{\rm U}}\Big(G_1(\mu; z) + G_2(\mu; z)\Big). \tag{34}$$

By inserting formulae (34) and (30) in Eq. (29), we obtain the relation

$$\left(G_1(\mu; z) + G_2(\mu; z)\right)\mathcal{F}_{\nu} = T_{\rm U}\nabla_{\nu}\mathcal{S}. \tag{35}$$

Here the entropic force $\mathcal{F}_{\nu} = -me^{\varphi}\nabla_{\nu}\varphi$ and φ is a GR generalization of Newton's potential:

$$\varphi = \frac{1}{2}\log(-\xi^{\nu}\xi_{\nu}). \tag{36}$$

The change of entropy through the holographic screen resulting from a displacement of particle (bit) by one thermal wavelength λ_T along some normal direction $\mathbf{n}$ to the screen determined by $n_{\nu}(\nu = 0, 1, 2, 3)$ reads

$$\nabla_{\nu}\mathcal{S} = -2\pi\frac{m}{\hbar}n_{\nu}. \tag{37}$$

In Verlinde's emergent gravity framework, the derivation of Einstein's field equations is performed in thermodynamic terms by employing the holographic principle, the Unruh effect, and the Bekenstein entropy bound, all of which can be naturally interpreted within the AdS/CFT correspondence. Hence, the accelerating observer may be regarded as intrinsically coupled to the framework of induced gravity, since the Unruh temperature perceived by such an observer constitutes a macroscopic manifestation of the same underlying microscopic degrees of freedom whose collective dynamics gives rise to the emergent gravitational field. Then (on equating the thermodynamic temperature to the Unruh temperature), we obtain the relation that involves the μ-deformed temperature of the form:

$$T_{\rm U} = (G_1(\mu; z) + G_2(\mu; z))e^{\varphi}\nabla_{\nu}\varphi\frac{\hbar}{2\pi}n_{\nu}. \tag{38}$$

With a screen located on a closed surface $\tilde{\mathcal{S}}$ at a "constant redshift" the Komar mass can be written as

$$\mathcal{M} = \frac{1}{2}\int_{\tilde{\mathcal{S}}} T_{\rm U} d\mathcal{N}. \tag{39}$$

The infinitesimal change of the number of bits can be written, according to Bekenstein idea [32], as

$$d\mathcal{N} = \frac{d\mathcal{A}}{G\hbar}. \tag{40}$$

By using Eq. (40) and Eq. (38) in Eq. (39), we obtain the total mass

$$\mathcal{M} = \frac{1}{4\pi G}\int_{\tilde{\mathcal{S}}} \mathcal{G}_V(\mu; z)e^{\varphi}\nabla_{\nu}\varphi n_{\nu}\, d\mathcal{A}, \tag{41}$$

where

$$\mathcal{G}_V(\mu; z) \equiv G_1(\mu; z) + G_2(\mu; z). \tag{42}$$

Next, from the Stokes' theorem, the Killing equation, and the relation $\nabla_{\nu}\nabla^{\nu}\xi^{\mu} = -\mathcal{R}^{\mu}_{\nu}\xi^{\nu}$, we obtain the desired μ-dependent total mass in the form

$$\mathcal{M} = \frac{1}{4\pi G}\int_{V} \mathcal{G}_V(\mu; z)n^{\nu}\xi^{\mu}\,\mathcal{R}_{\mu\nu}\, dV. \tag{43}$$

On the other hand, there is an alternative expression for the Komar mass in terms of energy-momentum tensor:

$$\mathcal{M} = 2\int_{V}\left(\mathcal{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{T} + \frac{\Lambda}{8\pi G}g_{\mu\nu}\right)n^{\nu}\xi^{\mu}\, dV. \tag{44}$$

In this framework, the μ-deformed mass refers to the thermodynamic energy of a μ-Bose gas distributed across the holographic volume. The Komar mass, by contrast, is an μ-deformed emergent gravitational quantity that inherits deformation effects from the underlying thermodynamic system via the AdS/CFT correspondence. Then from Eqs. (44), (43), we infer

$$2\int_V \left(\mathcal{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{T} + \frac{\Lambda}{8\pi G}g_{\mu\nu}\right) n^\nu \xi^\mu \, dV = \\ = \frac{\mathcal{G}_V(\mu;z)}{4\pi G}\int_V \mathcal{R}_{\mu\nu} n^\nu \xi^\mu \, dV. \quad (45)$$

From this we obtain the μ-deformed Einstein equations describing μ-induced gravity acting on deformed matter fields

$$2\left(\mathcal{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{T} + \frac{\Lambda}{8\pi G}g_{\mu\nu}\right) = \frac{\mathcal{G}_V(\mu;z)}{4\pi G}\mathcal{R}_{\mu\nu}. \quad (46)$$

At last, taking the trace of Eq. (46), yields the desired μ-deformed Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \frac{\Lambda}{\mathcal{G}_V(\mu;z)}g_{\mu\nu} = \frac{8\pi G}{\mathcal{G}_V(\mu;z)}\mathcal{T}_{\mu\nu}. \quad (47)$$

As seen in this equation, the modified (or *effective* $\Lambda_{\text{eff}} \equiv \frac{\Lambda}{\mathcal{G}_V(\mu;z)}$) *cosmological constant* (ECC) has appeared, along with *modified gravitational constant.*

In the both deformed ($\mu > 0$) and undeformed ($\mu = 0$) situations, the ECC and also the gravitational constant (GC) in the resulting μ-deformed Einstein Eqs. depend on the *variable* fugacity z of underlying inducing system. To get rid of z, natural way is to fix z as $z = 1$ (or some value very close to 1). That implies dealing with Bose-like condensate. Now, the crucial point is that the $\mu = 0$ (i.e. pure Bose gas) case, for fixed $z = 1$, results in vanishing CC and GC, what is clearly unphysical. Thus, pure Bose gas cannot serve as underlying (or inducing) system while the (μ-)deformed extension will be of principal importance, as will be seen from the treatment below.

3. Analysis of the Effective Cosmological Constant

We study the behavior of effective cosmological constant, with more spectacular plotting of the factor $\mathcal{G}_V(\mu;z) \equiv G_1(\mu;z) + G_2(\mu;z)$. The functions $G_1(\mu;z)$ and $G_2(\mu;z)$ defined by (32)–(33) involve slow convergent μ-polylogarithmic series $g_s^{(\mu)}(z) = \sum_{n=1}^{\infty}(1+\mu n)^{-1}\frac{z^n}{n^s}$, for $s = 0, 1, \frac{3}{2}, \frac{5}{2}$, see e.g. [11] for a definition and some uses. In view of calculation nontriviality of the $g_s^{(\mu)}(z)$ series at z close to 1, and small μ, we applied *Riemann sum approximation* to appropriate terms of $G_2(\mu;z)$ described below, as well as *Euler–Maclaurin formula* [60] when treating $G_1(\mu;z)$. Besides, we use the expansion of polylogarithm $\mathrm{Li}_s(z)$ in $(\ln z)^n$

$$\mathrm{Li}_s(z) = \Gamma(1-s)\left(\ln\frac{1}{z}\right)^{s-1} + \sum_{n=0}^{\infty}\zeta(s-n)\frac{(\ln z)^n}{n!}, \quad (48)$$
$$s \neq 1, 2, 3, ..., \quad |\ln z| < 2\pi, \quad n = 1, 2, ...$$

as well as its integer order s counterpart, see e.g. [58] or [60, 61].

3.1. Treatment of the function $G_1(\mu;z)$

We introduce the auxiliary notation $\Xi_T \equiv \frac{V}{\lambda_T^3}$ here. Going over from the fugacity $z = e^{\beta\tilde{\mu}}$ to new reduced variable

$$\lambda = \lambda(z,\mu) = -\frac{1}{\mu}\ln z \approx \frac{1-z}{\mu}, \quad (49)$$

for z close to 1, we arrive at the desired 'small μ' approximation

$$G_1(\mu;z) = \frac{5}{2}\,\Xi_T\Bigg\{\zeta\left(\frac{5}{2}\right) + \zeta\left(\frac{3}{2}\right)(\lambda-1)\,\mu - \\ - 2\sqrt{\pi}\left[\left(\frac{4}{3}\lambda - 1\right)\lambda^{\frac{1}{2}} + (2\lambda-1)\,\mathsf{M}_{\text{erfc}}(\sqrt{\lambda})\right]\mu^{\frac{3}{2}} - \\ - \frac{1}{2}\zeta\left(\frac{1}{2}\right)(3\lambda-2)\lambda\mu^2 + \frac{1}{3!}\zeta\left(-\frac{1}{2}\right)(5\lambda-3)\lambda^2\mu^3\Bigg\} + \\ + \lambda' O(\mu^2\lambda'^2), \quad (50)$$

where $\mu \to 0$, $\lambda \ll \frac{2\pi}{\mu}$, $\lambda' \equiv \lambda + 1$ and *Mills' ratio* for the complementary error function

$$\mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt, \quad (51)$$

defined [60, 61] as

$$\mathsf{M}_{\text{erfc}}(x) = \frac{\int_x^\infty e^{-t^2}dt}{e^{-x^2}} \equiv \frac{\sqrt{\pi}}{2}e^{x^2}\,\mathrm{erfc}(x), \quad (52)$$

and $\zeta(x)$ is Riemann zeta function. The original fugacity variable z is recovered as

$$z = z(\lambda, \mu) = e^{-\mu\lambda}. \tag{53}$$

To explore the behavior of $G_1(\mu; z)$ and total $\mathcal{G}_V(\mu; z)$ we set $\Xi_T = 1$ unless the contrary is specified. For convenience and as evidence for continuous or divergent behavior at $z \to 1$, we also give the explicit expressions for $G_{1,2}(\mu; z)$ when $\mu = 0$, in terms of $\tilde{\lambda} \equiv -\ln(z)$, $|\ln z| < 2\pi$:

$$G_1(0; z) = \frac{5}{2}\left(\zeta\left(\frac{5}{2}\right) + \zeta\left(\frac{3}{2}\right)\tilde{\lambda} - \frac{8}{3}\sqrt{\pi}\,\tilde{\lambda}^{3/2} - \frac{1}{2}\tilde{\lambda}^2 \times \right.$$
$$\left. \times \sum_{r=0}^{\infty} (-1)^{\left[\frac{r+1}{2}\right]} \frac{\zeta(r+1/2)}{r+1/2} \frac{(2r+3)!!}{(r+2)!}\left(-\frac{\tilde{\lambda}}{4\pi}\right)^{r}\right), \tag{54}$$

where $[\frac{r+1}{2}]$ denotes the largest integer not exceeding $\frac{r+1}{2}$, and $n!! = n(n-2)(n-4)\cdot ...$ is double factorial. Remark that, as seen from (50), (54), in the close vicinity to $z = 1$, where function $G_2(0; z)$ is logarithmically divergent, the impact from $G_1(\mu; z) \approx \approx G_1(0; 1) = \frac{5}{2}\zeta(\frac{5}{2})$ for small μ is finite (proportional to Ξ_T). The precision of approximate formula (50) and of expansion (54), compared to numerical evaluation from the definitions, is demonstrated in Fig. 1, *a*.

3.2. Treatment of the function $G_2(\mu; z)$

Note that it is the function G_2 that makes much larger contribution into the sum, if $1-\delta < z < 1$ and $\mu \to 0$. For G_2, to derive analogous approximate formula, we first invoke approximation of integer μ^{-1} (which is reasonable due to smallness of μ), that reads

$$g_0^{(\mu)}(z) = \mu^{-1} e^{\lambda(z,\mu)} \left(\mathrm{Li}_1(z) - \sum_{n=1}^{M} \frac{z^n}{n}\right) - O(1), \tag{55}$$

$$g_1^{(\mu)}(z) = \mathrm{Li}_1(z) - \mu\, g_0^{(\mu)}(z), \tag{56}$$

for arbitrary continuous μ, $0 < \mu < 1$, with integer part $M \equiv [\mu^{-1}]$ of its inverse ratio. Then, to calculate the sum $\sum_{n=1}^{M} \frac{z^n}{n}$ explicitly (and approximately) we apply the quadrature midpoint rule, namely

$$\sum_{n=1}^{M} \frac{1-z^n}{n} = \int\limits_{\mu/2}^{(M+1/2)\mu} \frac{1-e^{-\lambda x}}{x} dx \; - \; O_\lambda(\mu^2) =$$
$$= \mathrm{Ein}\left(\lambda + \frac{1}{2}\mu\lambda\right) - \mathrm{Ein}\left(\frac{1}{2}\mu\lambda\right) - 2\,O\left(\mu + \frac{\mu^2\lambda^3}{72}\right), \tag{57}$$

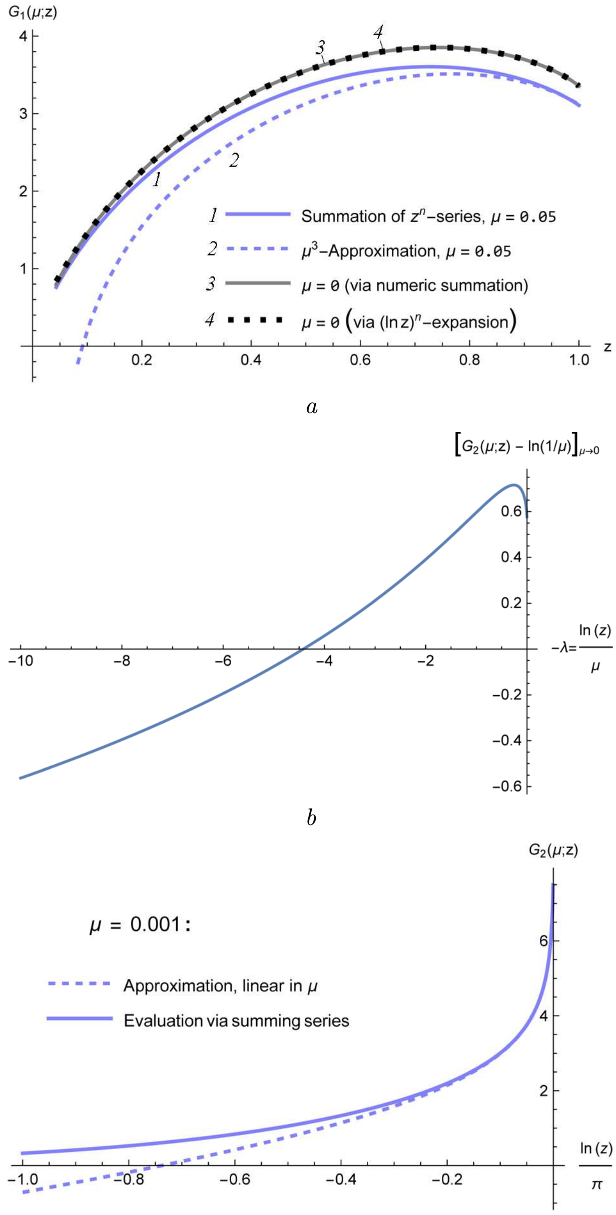

Fig. 1. Dependence of G_1 on z (*a*) and the difference $G_2 - \ln\frac{1}{\mu}$ on $-\lambda = \frac{\ln(z)}{\mu}$ (*b*). The bottom plots show a discrepancy between the approximation and numerical evaluation for G_2. Pane (*a*): solid curves *1* and *3* show the behaviour of numerically evaluated G_1 basing on the summation of defining z^n-series, resp. for $\mu = 0.05$ and $\mu = 0$; dashed *2* and dotted *4* curves are obtained via μ^3-approximation (50) and $(\ln z)^n$ expansion (54), resp. for $\mu = 0.05$ and $\mu = 0$. Pane (*c*): dashed curve corresponds to linear-in-μ approximation, see (59) and footnote 1, whilst solid one is evaluated via summing series (for both cases $\mu = 0.001$)

with [59, 60] modified exponential integral

$$\mathrm{Ein}(x) = \int\limits_0^x \frac{1-e^{-t}}{t}dt. \tag{58}$$

After some analysis we obtain [1]

$$G_2(\mu;z) = \ln\left(\frac{1}{\mu}\right) + \left[(2\lambda-1)\,e^{\lambda}\,E_1(\lambda) - \ln(\lambda)\right] + \\ + e^{\lambda+2}\left(O(\mu) + \left[12 + \left(\frac{\delta z}{\mu}\right)^2\right]O(\delta z)\right), \tag{59}$$

with $\mu \to +0$, $\delta z \to +0$ and

$$\lambda \equiv \lambda(z,\mu) = \frac{|\ln z|}{\mu} = \frac{\delta z}{\mu}\left(1 + \frac{\delta z}{2} + ...\right), \tag{60}$$

where $\delta z \equiv 1 - z$ and $E_1(\lambda)$ denotes exponential integral [59, 60]

$$E_1(x) = \int\limits_x^\infty \frac{e^{-t}}{t}dt, \quad |\arg(x)| < \pi. \tag{61}$$

Since, for the extremely small (but positive) values of μ, the function $G_1(\mu;z) \sim$ const, when z is close to 1, the prevailing behavior of $G_1(\mu;z) + G_2(\mu;z)$ is determined by the first meaningful terms of G_2. The respective dependence on λ is shown in Fig. 1, *b*. The validity of the approximation (59) close to $z = 1$ is demonstrated in Fig. 1, *c*.

The *extremum condition for* λ

$$(1+2\lambda)E_1(\lambda) = 2e^{-\lambda} \tag{62}$$

can be solved either analytically or numerically. The first method involves known power series expansion for $E_1(\lambda)$, see e.g. [59],

$$E_1(\lambda) = -\gamma - \ln(\lambda) - \sum_{n=1}^{\infty}\frac{(-1)^n\lambda^n}{nn!}, \tag{63}$$

where $\gamma = 0.577215...$ is the Euler constant, implying smallness of λ, and leads for $n \le 2$ to quadratic equation

$$7.25\lambda^2 - \left(1 + 4\ln 2 - 2\gamma\right)\lambda + \left(1 + \gamma - 2\ln 2\right) = 0,$$

with $\lambda \approx 0.2597...$. The second method provides more precise solution $\lambda = \lambda^{(0)} = 0.2589...$, found by means of *Wolfram Mathematica software*. So respective fugacity $z_0 \approx 1 - 0.26\mu$. Corresponding maximum value of $G_2(\mu;z)$ immediately follows from (59):

$$\bar{G}_2 = \ln\left(\frac{1}{\mu}\right) + \lim_{\mu\to 0}\left[G_2\left(\mu, z(\lambda^{(0)},\mu)\right) - \ln\left(\frac{1}{\mu}\right)\right] = \\ = 0.7159... + \ln\left(\frac{1}{\mu}\right). \tag{64}$$

Remark. From the definition of parameter $\lambda = = \lambda(z,\mu)$, see (49), and the known one for fugacity $z = e^{\beta\tilde{\mu}}$ we directly obtain relation between chemical potential $\tilde{\mu}$ and deformation parameter μ:

$$\tilde{\mu} \equiv \beta^{-1}\ln z = -\mu\frac{\lambda(z,\mu)}{\beta}. \tag{65}$$

So, for the case when $G_2(\mu;z)$ is prevailing over $G_1(\mu;z)$ in the neighbourhood of $z = 1$, we estimate chemical potential for maximum point of $G_1 + G_2$ (where $\lambda(z_0,\mu) \equiv \lambda^{(0)}$) as

$$\tilde{\mu}_0 = -\lambda^{(0)}\mu k_{\mathrm{B}}T \approx -0.26\mu \times k_{\mathrm{B}}T \tag{66}$$

with μ being small enough.

Explicit expression for $G_2(\mu;z)$ at $\mu = 0$ and $|\ln z| < 2\pi$ is given as

$$G_2(0;z) = -\ln|\ln z| + 2 - \frac{1}{2}\tilde{\lambda} + \frac{1}{6}\tilde{\lambda}^2\frac{{}_1F_1(2;4;-\tilde{\lambda})}{{}_1F_1(1;2;-\tilde{\lambda})} - \\ - \sum_{m=2}^{\infty} B_m\frac{\tilde{\lambda}^m}{m\,m!}, \quad \tilde{\lambda} \equiv -\ln(z). \tag{67}$$

Here ${}_1F_1(...)$ is Kummer hypergeometric function and B_m are Bernoulli numbers, see e.g. [59] for their definitions.

3.3. The sum $G_1 + G_2$ at z close to unity

Let us also observe certain "universality" in dependence of $G_1(\mu;z) + G_2(\mu;z)$ on z: in the neighborhood of $z = 1$ at small μ it manifests itself through just combined variable $\lambda(z,\mu)$ (as seen from (50) and (59)). Using this fact and the above approximations for $G_1(\mu;z)$ and $G_2(\mu;z)$ we establish the following.

Statement. The narrow, convex upwards, maximum $\bar{\mathcal{G}}_V(\mu)$ *exists* for arbitrarily small μ, $0 < \mu < \delta$, and is *continuously* tending to infinity ($+\infty$) when μ tends to zero, $\mu \to +0$. At the same time, the respective fugacity, given by

$$z_0 \simeq 1 - \mu\lambda^{(0)} \approx 1 - \frac{\mu}{4}, \tag{68}$$

[1] More detailed treatment based on inverse-integer representation for μ yields formula with linear-in-μ correction: $G_2(\mu;z) = \ln\frac{1}{\mu} + \left[(2\lambda-1)\,e^{\lambda}\,E_1(\lambda) - \ln(\lambda)\right] - \frac{1}{2}(\lambda-1)\,\mu + + e^{\lambda}O(\mu^2\lambda'^4)$.

continuously tends from the left to unity ($\delta z_0 \equiv 1 - - z_0 \simeq \frac{\mu}{4} \to +0$). The height of the maximum, at small μ, is given with 'order of μ' error by

$$\bar{\mathcal{G}}_V(\mu) \approx 0.7159... + \frac{5}{2}\,\zeta\left(\frac{5}{2}\right)\Xi_T + \ln\left(\frac{1}{\mu}\right) \tag{69}$$

and *is logarithmically increasing with* $\frac{1}{\mu}$.

Corollary. Taking the expression for cosmological constant involved in μ-deformed Einstein equation (47) as

$$\Lambda_\mu = \Lambda_{\text{eff}}(\mu; z) = \frac{1}{G_1(\mu; z) + G_2(\mu; z)}\, l_{\text{P}}^{-2}, \tag{70}$$

where l_{P} is the Planck length, we find the appropriate value $\mu = \mu^*$ of deformation parameter to fit the presently known value of cosmological constant

$$\Lambda_{\text{ex}} = 3\left(\frac{H_0}{c}\right)^2 \Omega_\Lambda = 1.466 \times 10^{-52} m^{-2} = \\ = \underbrace{3.827 \times 10^{-123}}_{\Lambda_{\text{ex}}^{(\text{P})}}\, l_{\text{P}}^{-2}, \tag{71}$$

where "ex" means observable. In view of extraordinary smallness of $\Lambda_{\text{ex}}^{(\text{P})}$ we expect to achieve the proper value (71) of the effective cosmological constant (70) at the minimum point on the curve $\Lambda_\mu(z)$. So, to obtain this proper value for Λ_μ, we put

$$\Lambda_\mu(z_0) = \frac{1}{\mathcal{G}_V(\mu; z_0)}\, l_{\text{P}}^{-2} = \Lambda_{\text{ex}}^{(\text{P})}\, l_{\text{P}}^{-2}$$

that is equivalent, see (69), to

$$\mathcal{G}_V(\mu; z_0) \approx 0.72 + \frac{5}{2}\zeta\left(\frac{5}{2}\right) + \ln\left(\frac{1}{\mu}\right)\bigg|_{\mu=\mu^*} = \frac{1}{\Lambda_{\text{ex}}^{(\text{P})}}. \tag{72}$$

This yields the corresponding value of deformation parameter μ^*, which provides *the above value* (71) *in the extremum*,

$$\mu^* \approx e^{-1/\Lambda_{\text{ex}}^{(\text{P})} + 0.72 + 5/2\,\zeta(5/2)} \sim e^{-2.61 \times 10^{122}}. \tag{73}$$

In fact, the extremum at $z = z_0$ is so close to $z = 1$ point on the $\mu = \mu^*$ curve that it is inessential what fugacity from the $[z_0, 1]$-interval is taken, resulting in almost equal values $\Lambda_{\text{eff}}(\mu; z) \approx \Lambda_{\text{ex}}$, up to very high precision [2]. The fugacity z_0 for this extremum stems from (59):

$$z_0 = e^{-\mu\lambda^{(0)}} \approx 1 - \lambda^{(0)}\mu, \tag{74}$$

[2] Let us observe, using (50) and (59), that the factor $\mathcal{G}_V(\mu^*; z)$ taken at $z = 1$ is lesser than $\mathcal{G}_V(\mu^*; z_0) \sim 10^{122}$ in the extremum just by $\Delta\mathcal{G}_V(\mu^*) \approx 0.7159... - \gamma \approx 0.14 \ll 10^{122}$. So, due to (70), the effective cosmological constant remains almost unchanged.

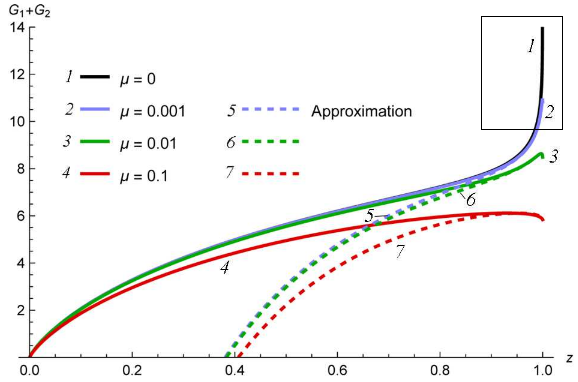

Fig. 2. "Full-range" behavior of $G_1(\mu; z) + G_2(\mu; z)$ given by solid curves *1–4*, along with the small μ approximation based on (50) and (59), presented by dashed curves *5–7*. For curve *1* – $\mu = 0$, for curves *2* and *5* – $\mu = 0.001$, for curves *3* and *6* – $\mu = 0.01$, and for curves *4*, *7* – $\mu = 0.1$

that for the fitting curve ($\mu = \mu^*$) yields numerically

$$1 - z_0 \approx \lambda^{(0)} \cdot \mu^* \sim \exp\left(-2.61 \times 10^{122}\right). \tag{75}$$

The respective chemical potential $\tilde{\mu}$ stems from (66):

$$\tilde{\mu}_0 = -\lambda^{(0)}\mu \cdot k_{\text{B}}T\big|_{\mu=\mu^*} \sim -\exp\left(-2.61 \times 10^{122}\right) \cdot k_{\text{B}}T. \tag{76}$$

Besides, from (75), (76) we observe

$$1 - z_0 \approx -\frac{\tilde{\mu}_0}{k_{\text{B}}T}. \tag{77}$$

Remark. Let us estimate the ratio $G_2(\mu; z)/G_1(\mu; z)$ for the deformation parameter value $\mu = \mu^*$ in some $\varepsilon\mu$-neighborhood of the maximum. That is, we have $(\lambda^{(0)} - \varepsilon)\mu^* < 1 - z < (\lambda^{(0)} + \varepsilon)\mu^*$. In addition, assume $\varepsilon < 1/8$. Then, making typical estimates we obtain:

$$G_1(\mu^*; z) = \\ = \frac{5}{2}\left\{\zeta\left(\frac{5}{2}\right) - \zeta\left(\frac{3}{2}\right)(1 - \lambda^{(0)} + \varepsilon)\, O(\mu^*) + 2\pi O(\mu^{*3/2})\right\},$$

$$G_2(\mu^*; z) = G_2(\mu^*; z_0) \pm 18 O(\varepsilon) + 12 e^3 O(\mu^*),$$

where $\zeta\left(\frac{3}{2}\right) = 2.61238\ldots$, $\zeta\left(\frac{5}{2}\right) = 1.34149\ldots$. Evaluating the ratio within given neighborhood,

$$\frac{G_2(\mu^*; z)}{G_1(\mu^*; z)} = \frac{2}{5}\zeta^{-1}\left(\frac{5}{2}\right)\left\{\bar{G}_2 \pm 18 O(\varepsilon) + \zeta\left(\frac{3}{2}\right) \times \\ \times 2.05 \times \overbrace{\bar{G}_2 e^{-\bar{G}_2} O(1)}^{O(\mu^{*1/2})}\right\} \approx \frac{2}{5}\zeta^{-1}\left(\frac{5}{2}\right)\bar{G}_2 \pm \\ \pm 8 O(\varepsilon) = 7.79 \times 10^{121} \pm 1 \gg 1, \tag{78}$$

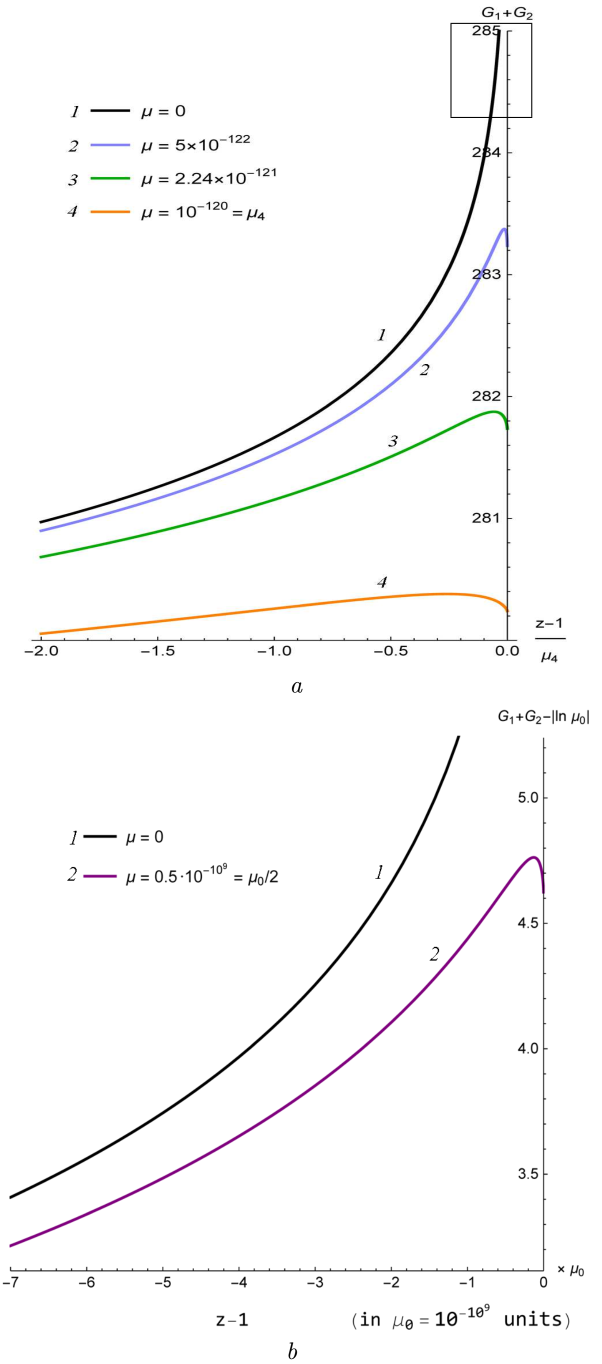

Fig. 3. Dependence of $G_1(\mu;z) + G_2(\mu;z)$ on $\delta z \equiv z - 1$ measured in the units of largest value μ_4, close to $z = 1$ for $\mu \sim 10^{-120}$ (a). Dependence of $G_1(\mu;z) + G_2(\mu;z)$ on δz, shifted by $|\ln(\mu_0)| \approx 2.30 \times 10^9$, for the *extremely* small μ-scale given by $\mu_0 = 10^{-10^9}$, however, with focus on the extremum position (b). This figure roughly corresponds to the box in the up panel

we confirm[3] the mentioned possibility to neglect $G_1(\mu;z)$ at $\Xi_T \sim 1$.

Now, let us demonstrate our treatment of $\mathcal{G}_V(\mu;z)$ by a few illustrative plots, especially for extremely small μ. The typical picture of $G_1(\mu;z) + G_2(\mu;z)$ dependence on z for a few values of μ is shown in Fig. 2. In addition, the behavior of small-μ approximation beyond its validity region (i.e., $|\ln z| \ll 1$), is given for comparison. The detailed dependencies from the box in Fig. 2, i.e., for much more decreased μ, with z approaching unity, are visualized by the magnified graphs in Fig. 3, a, and Fig. 3, b (extremely large magnification).

To summarize, the truly deformed case of $\mu > 0$, unlike the $\mu = 0$ (pure Bose gas) case, can produce realistic value of CC, due to very existence of (extra-tiny) minimum for CC at definite z_0 very close to 1, or due to (very close to the minimum) finite nonzero value, when we set $z = 1$ exactly.

4. Discussion and Conclusions

In this paper, we have derived the μ-deformed extension of Einstein (gravitational) field equations and studied, in detail, the dependence of its term which involves the effective (i.e., μ-dependent) cosmological constant as a function of the deformation parameter μ and fugacity z. In effect, we have shown that, by varying the deformation parameter μ toward its extremely small values, it is possible to find the appropriate value μ^* of μ that allows us to achieve the "realistic value" of the maximum of the function $\mathcal{G}_V(\mu;z) \equiv G_1(z,\mu) + G_2(z,\mu)$ which (through its inverse) provides the necessary observable value $\Lambda/\mathcal{G}_V(\mu^*;z_0) \sim 3.8 \times 10^{-123} l_{\rm P}^{-2}$ of the effective cosmological constant $\Lambda/\mathcal{G}_V(\mu;z)$. Certain important points are to be emphasized.

First of all, we stress the very fact of existence of the extremum of the ECC within the μ-deformed Bose-gas model based approach to realizing induced gravity. Clearly this is in contrast to the usage of conventional (undeformed) Bose gas model. Indeed, within the latter there is no extremum in the corresponding combination $\Lambda/\mathcal{G}_V(0;z_0)$ of usual $\mu = 0$

[3] Strictly speaking we have to compare also the increments: the one for $G_2(\mu;z)$ in μ-neighbourhood of $z = 1$ is of the order of 1, whilst $G_1(\mu;z) \approx$ const up to $O(\mu)$. So, this should not cause noticeable z_0's shift.

polylogarithms, when $z \to 1$ and thus no way to get the value of CC different from zero.

Second, when analyzing the behavior of ECC, we adopted, without special justification, that the ratio is $V/\lambda_T^3 = 1$. This value of the ratio is well-known marker that indicates the existence of truely quantum effects [62] and also relates to Bose-like condensate. Let us consider this issue in more details, coming unexpectedly to the two distinct possible situations (and interpretation) of the system which underlies the induced (entropic) gravity. These two cases of inducing system correspond to principally different scales, namely (A) the scale (size) of dwarf galaxies and (B) the scale of observable Universe.

(A) In this case, the ratio V/λ_T^3 is of the order of unity due to the fact that both the scale of dwarf galaxies and the thermal wavelength scale of the dark matter particles are few kpc (see, e.g., [19]) if the μ-deformed fuzzy dark matter (DM) model is adopted, with the mass of (Bose-like condensate) ultralight DM particles being $m = 10^{-22}$ eV.

(B) In this case, the ratio V/λ_T^3 is of the order of unity due to the fact that both the scale of observable Universe and the thermal wavelength scale are of the order $\sim 2.8 \times 10^7$ kpc if the particles, forming [64] Bose-like condensate DM, are adopted to be "massive gravitons" (it is reasonable to term this *pre-gravitons* in view of their role in generating induced gravity) with mass $m = 10^{-36}$–10^{-33} eV, see [63]. Note that, in the context of DM of dwarf galaxies, the Bose condensate of massive gravitons was considered in [64]. In our treatment, we deal with condensate of massive pre-gravitons, i.e. viewed as the system which induces real gravity along with effective cosmological constant. In connection with this picture of inducing CC, it is worth to mention the work of R. Garattini and M. Faizal in which it was shown that the appearance of cosmological constant is strictly related with just the deformation of Generalized Uncertainty Principle being behind the deformed Wheeler–DeWitt equation, see [65].

The real physical nature of the parameter μ of μ-deformed inducing system, like that of dark matter, is unknown. However the significance or meaning of its use can be justified: the deformation enables to take into account (possible) compositeness of particles and/or their interactions – natural reasons to deal with quantum system strongly deviating from ideal Bose gas. This reason is quite similar to the usage of the deformed Bose gas model in another context – for (successful) modelling the properties of quantum correlations of pions occurring in heavy-ion collisions at RHIC, see Ref. [16] and also the paragraph after Eq. (25). However, explicit physical content of μ in the both two cases, of ultralight dark matter (as Bose-like condensate) i.e. the galactic one, and the observed Universe case, may be different since the particles (and their constituents) do differ – ultralight "bosons" versus "pre-gravitons".

Let us stress that our approach to "explain" the issue of observable value of CC grounds on two pillars: deformation and (macroscopic) quantumness of underlying or inducing system. It is also worth to note the following: the fact that the issue of CC in our treatment turns out to be related with the concept of DM is quite alike the joint treatment of DM and dark energy, appearing in another contexts in some papers, see e.g. [66, 67].

As our final remark let us mention that the analysis of μ-deformed Friedmann equations stemming from the μ-deformed Einstein equations derived and studied herein, will be published in a separate paper.

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μ-ДЕФОРМОВАНІ РІВНЯННЯ ГРАВІТАЦІЙНОГО ПОЛЯ АЙНШТАЙНА ІЗ ЗАЛЕЖНОЮ ВІД μ ЕФЕКТИВНОЮ КОСМОЛОГІЧНОЮ СТАЛОЮ

Застосовуючи адаптований підхід Верлінде до узагальнених термодинамічних функцій μ-деформованого аналога моделі бозе-газу, ми отримали μ-деформовані рівняння Айнштайна. Показана основна роль параметра деформації, що забезпечує можливість варіювати значення космологічної сталої. Завдяки цьому запропоновано цікаве трактування проблеми космологічної сталої (КС) у рамках μ-деформаційного підходу. А саме: розглядаючи виведену μ-деформовану КС як ефективну та відповідно змінюючи параметр μ, ми отримали можливість радикально зменшити КС, з тим щоб наблизити її до реалістичного значення. Також обговорюється зв'язок із темною матерією.

Ключові слова: ентропійна сила, гравітація, μ-модель бозе-газу, ефективна космологічна стала, темна матерія.