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EQUATIONS FOR PARTICLES WITH SPIN $S = 0$ AND $S = 1$ IN SPINOR REPRESENTATION [1]

Equations for particles with spin $S = 0$ and $S = 1$ are presented in the form of a system of two Dirac equations with additional conditions (constraints) imposed on the components of the wave functions. In case of identical masses (or at the high-energy limit, where the difference in mass is negligible), the joint system of equations is formulated having particular solutions coinciding with those for spin $S = 0$ and $S = 1$ cases, and simultaneously being the two Dirac equations for two independent particles with spin $S = 1/2$. A principle of constructing the equations for a particle with an arbitrary spin in the spinor representation is proposed.

Keywords: Dirac equation, first-order differential equations for particles with spin $S = 0$ and $S = 1$.

1. Introduction

The first-order differential equations for free elementary particles can be written in different forms. In the present paper, we derive the equations for particles with spin $S = 0$ and $S = 1$ in the form of a system of two Dirac equations with additional conditions (constraints) imposed on the components of the wave functions. Preliminary version of this article can be found in [1].

We start with the well-known equations [2–4] for the particles with zero and unitary spin and make an identical transformations with these equations to obtain a system of Dirac equations. An additional conditions arise automatically without additional assumptions.

In the next section, we start with equations for a particle with spin $S = 0$. Then, in section 3, we consider the cases $S = 0$ and $S = 1$ and formulate the systems of two Dirac equations with additional conditions for the components of the wave functions. At the end of the section, we propose a principle of formulating the equations for a particle with an arbitrary spin in the form of a system of Dirac equations with additional conditions (constraints). In section 4, we consider the case of equal masses of particles with spin $S = 0$ and $S = 1$, and we formulate for them a joint system of equations, where an appearance of degrees of freedom with spin $S = 1/2$ is observed. In section 5, a brief conclusion is given.

2. Preliminary Transformations of the Duffin–Kemmer–Petiau Equations for a Particle with Spin $S = 0$

We remind that the well-known Duffin–Kemmer–Petiau first-order differential equations [2–4] for a particle with spin $S = 0$ can be derived from the Klein–Gordon–Fock equation (here and further, $\hbar = 1$ and $c = 1$)

$$\left(\Box - m^2\right)\varphi = 0 \tag{1}$$

by introducing the new fields A_μ ($\mu = 0, 1, 2, 3$)

$$A_\mu \equiv \partial_\mu \varphi. \tag{2}$$

Then, instead of (1), one has the system of Duffin–Kemmer–Petiau equations

$$\begin{cases} \partial_t \varphi = A_0, \\ -\partial_t A_0 = (\nabla \cdot \mathbf{A}) + m^2 \varphi, \\ \mathbf{A} = -\nabla \varphi, \end{cases} \tag{3}$$

for the five-component wave function $(\varphi, A_0, \mathbf{A})$. The system of equations (3) is often written in a matrix

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form [2, 3] (in particular, we give below the corresponding relations from [3], where another notations of space-time indices are used: $1, 2, 3, 4$ instead of $0, 1, 2, 3$; and, in this case, A_4 corresponds to iA_0 from (2)),

$$\left(\hat{\beta}^{\mu}\partial_{\mu} + m\right)\hat{\Psi} = 0, \tag{4}$$

where $\hat{\Psi}$ is a five-component column of wave functions

$$\Psi_{\mu} = \frac{1}{\sqrt{m}}A_{\mu}, \quad \mu = 1, 2, 3, 4, \quad \Psi_5 = \sqrt{m}\,\varphi, \tag{5}$$

and matrices $\hat{\beta}_{\mu}$ from (4) have the form [3]:

$$\hat{\beta}_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix}, \quad \hat{\beta}_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix},$$
$$\hat{\beta}_3 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix}, \quad \hat{\beta}_4 = \begin{pmatrix} 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0 \end{pmatrix}. \tag{6}$$

The above-presented Duffin–Kemmer–Petiau equations (3) contain the second power of the mass of particle which is not natural for the first-order differential equations. The number of the wave function components is "excessive", since the number of independent (free) components should be $2\,(2S+1)$ [4], i.e., two for $S = 0$. Moreover, in the limit $m \to 0$, the system of equations (3) (after differentiating the last equation by time variable) splits into the system of four closed equations

$$\begin{cases} -\partial_t A_0 = (\nabla \cdot \mathbf{A}), \\ \partial_t \mathbf{A} \;=\; -\nabla A_0, \end{cases} \tag{7}$$

with an additional condition

$$[\nabla \times \mathbf{A}] = 0, \tag{8}$$

following from the last equality of the system (3) due to identity $[\nabla \times \nabla\varphi] \equiv 0$, and a separate equation $\partial_t\varphi = A_0$, which becomes rather a definition of an additional field φ.

Thus, in the limit $m \to 0$, the structure of the system of the Duffin–Kemmer–Petiau equations is changed essentially: a five-component field becomes a four-component one, and a system of equations (3) is reduced to the system (7) with an additional condition (8). It might seem that the equations in the form (4) with linear mass overcome this problem. But the limit $m \to 0$ for system (4) is even less regular. Really, in this limit, the system of equations (4) is reduced to only one equation

$$\partial_t\Psi_4 + (\nabla \cdot \mathbf{\Psi}) = 0 \tag{9}$$

and trivial result for Ψ_5, namely, $\Psi_5 = \text{Const}$. Equation (9) is insufficient for determining the solutions for all the components of $\hat{\Psi}$.

Now, we are going to construct another system of equations having the regular limit at $m \to 0$. Let us introduce, instead of $(A_0, \mathbf{A})$ (2), the following four components $(B_0, \mathbf{B})$:

$$\begin{cases} B_0 \equiv (-\partial_t + im)\,\varphi, \\ \mathbf{B} \;\equiv\; -\nabla\varphi, \end{cases} \tag{10}$$

where φ obeys Eq. (1). Then, one has

$$(\partial_t + im)\,B_0 = -\,(\partial_t + im)\,(\partial_t - im)\,\varphi =$$
$$= -\left(\partial_t^2 + m^2\right)\varphi. \tag{11}$$

Due to the main equation (1), the last expression equals $-\left(\partial_t^2 + m^2\right)\varphi = -\triangle\varphi$. Taking into account the identity $\triangle\varphi \equiv (\nabla \cdot \nabla\varphi)$ and the definition of $\mathbf{B}$ from (10), one has, instead of (11):

$$(\partial_t + im)\,B_0 = (\nabla \cdot \mathbf{B}). \tag{12}$$

Taking the derivative ∂_t from the second equality from (10) and using the first definition from (10) to remove the field φ from consideration, we obtain:

$$\partial_t\mathbf{B} = -\nabla\partial_t\varphi = \nabla B_0 - im\nabla\varphi =$$
$$= \nabla B_0 + im\mathbf{B}. \tag{13}$$

Thus the obtained equations (12) and (13) may serve to be the closed system of equations for $(B_0, \mathbf{B})$:

$$\begin{cases} \partial_t B_0 = (\nabla \cdot \mathbf{B}) - imB_0, \\ \partial_t \mathbf{B} = \nabla B_0 + im\mathbf{B}, \end{cases} \tag{14}$$

with the additional condition

$$[\nabla \times \mathbf{B}] = 0. \tag{15}$$

The latter follows from the definition of $\mathbf{B}$ (the last equality from (10)) and the known identity $[\nabla \times \nabla\varphi] \equiv 0$.

It is important to note that the Klein–Gordon–Fock equation for B_0 immediately follows from the system of equations (14). But to obtain the Klein–Gordon–Fock equation for $\mathbf{B}$, one has to use the additional condition (15) together with the system of equations (14). Thus, the additional condition (15) is an obligatory part of the obtained system of equations. It should be noted that the mass m is present in (14) in the first degree on par with derivatives. The limit $m \to 0$ for the system of equations is trivial and does not change the number of equations and components of the field.

A connection of Eqs. (14), (15) with the Duffin–Kemmer–Petiau ones (3) becomes obvious, if one accounts for the relations $\mathbf{B} \equiv \mathbf{A}$ and $B_0 \equiv -A_0 + im\varphi$.

Now, let us write down the obtained equations in matrix form. We introduce the notations:

$$\hat{\Phi} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{16}$$

$$\hat{a}_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \hat{a}_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \hat{a}_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}, \tag{17}$$

$$\hat{b} = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}, \tag{18}$$

which enable us to rewrite the system of equations (14) as

$$i\partial_t \hat{\Phi} = i \left(\hat{\mathbf{a}} \cdot \nabla \right) \hat{\Phi} + m\hat{b}\hat{\Phi}. \tag{19}$$

An additional condition (15) can be rewritten in the form:

$$\left(\hat{\boldsymbol{\omega}} \cdot \nabla \right) \hat{\Phi} = 0, \tag{20}$$

where a three-dimensional vector $\hat{\boldsymbol{\omega}}$ has the following components:

$$\begin{aligned} &\hat{\omega}_1 \equiv -i \left(\hat{a}_2 \hat{a}_3 - \hat{a}_3 \hat{a}_2 \right), \\ &\hat{\omega}_2 \equiv -i \left(\hat{a}_3 \hat{a}_1 - \hat{a}_1 \hat{a}_3 \right), \\ &\hat{\omega}_3 \equiv -i \left(\hat{a}_1 \hat{a}_2 - \hat{a}_2 \hat{a}_1 \right). \end{aligned} \tag{21}$$

If matrices $\hat{a}_k$ are presented as (17), then, for matrices $\hat{\omega}_k$, one has

$$\hat{\omega}_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix}, \quad \hat{\omega}_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix}. \tag{22}$$

Matrices (17) can be changed by nondegenerate ones. For this purpose, we can add the expression in the left-hand side of (20) (which is equal to zero), multiplied by i, to the right-hand side of equation (19). Then, instead of (19), one has an equivalent equation:

$$i\partial_t \hat{\Phi} = i \left(\hat{\hat{\mathbf{a}}} \cdot \nabla \right) \hat{\Phi} + m\hat{b}\hat{\Phi}, \tag{23}$$

where

$$\hat{\hat{\mathbf{a}}} \equiv \hat{\mathbf{a}} + \hat{\boldsymbol{\omega}}. \tag{24}$$

The explicit form of matrices $\hat{\hat{a}}_k$ is the following:

$$\hat{\hat{a}}_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix}, \quad \hat{\hat{a}}_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&i \\ 1&0&0&0 \\ 0&-i&0&0 \end{pmatrix}, \quad \hat{\hat{a}}_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 1&0&0&0 \end{pmatrix}. \tag{25}$$

It can be directly verified that these matrices with $\det \hat{\hat{a}}_k = 1$ for $k = 1, 2, 3$ obey the relations:

$$\begin{aligned} &\hat{\hat{a}}_k^\dagger = \hat{\hat{a}}_k, \quad \hat{\hat{a}}_k^2 = \hat{I}, \\ &\hat{\hat{a}}_k \hat{\hat{a}}_n + \hat{\hat{a}}_n \hat{\hat{a}}_k = 2\delta_{kn} \hat{I}, \\ &\hat{\hat{a}}_k \hat{\hat{a}}_n = i\hat{\hat{a}}_m, \end{aligned} \tag{26}$$

where $\hat{I}$ is a unitary matrix, and indices in the last equality are $(k, n, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

Relations (26) are known to define the Pauli matrices. Thus, in another representation, matrices $\hat{\hat{a}}_k$ could be reduced to $\begin{pmatrix} \hat{\sigma}_k & 0 \\ 0 & \hat{\sigma}_k \end{pmatrix}$, where $\hat{\sigma}_k$ are the well-known Pauli matrices.

Equation (23) looks very similar to the Dirac equation written in the Heisenberg form

$$i\partial_t \hat{\Phi} = \hat{H} \hat{\Phi}, \tag{27}$$

but with constraint (20). Unlike the veritable Dirac equation for a particle with spin $S = 1/2$, Eq. (23) can be reduced to the Klein–Gordon–Fock equation for each component of the wave function $\hat{\Phi}$ only with the additional condition (20) taken into account. A very similar situation is observed in the case of equations [5, 6] for a particle with spin $S = 1$. This seems to be almost common situation [4] for the first-order differential equations for particles with definite spin.

In the next section, we demonstrate that the first-order differential equations for a particle with spin $S = 0$ and those for a particle with spin $S = 1$ can be written in the form of two Dirac equations, but with additional constraints (different for the cases $S = 0$ and $S = 1$).

3. A System of two Dirac Equations for Particles with Spin $S = 0$ and $S = 1$

We recall that quaternions, or hypercomplex numbers of the form

$$\mathbf{q} = \mu_0 + \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \mu_3\mathbf{e}_3, \quad (28)$$

with real numbers μ_k, are elements of four-dimensional linear space with a certain multiplication rule, which is suitable to define by the following table of mutiplication for $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1,$$
$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2, \quad (29)$$
$$\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_3, \quad \mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_2.$$

For each quaternion $\mathbf{q}$, one may consider the conjugate quantity $\bar{\mathbf{q}}$:

$$\bar{\mathbf{q}} = \mu_0 - \mu_1\mathbf{e}_1 - \mu_2\mathbf{e}_2 - \mu_3\mathbf{e}_3. \quad (30)$$

Then the absolute value squared of quaternion (28) is determined as

$$|\mathbf{q}|^2 = \bar{\mathbf{q}}\mathbf{q} = \mathbf{q}\bar{\mathbf{q}} = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^3. \quad (31)$$

Quaternions (28) can be represented by means of matrices 2×2. In particular, let us represent the quaternions $\hat{\mathbf{q}}$ with the use of Pauli matrices $\hat{\sigma}_k$ as follows:

$$\hat{\mathbf{q}} = \mu_0\hat{I} - i\mu_1\hat{\sigma}_1 - i\mu_2\hat{\sigma}_2 - i\mu_3\hat{\sigma}_3, \quad (32)$$

where the Pauli matrices are commonly used in the form

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (33)$$

In representation (32), the formally defined rules of multiplication (29) for quaternions (28) are valid due to ordinary general properties of matrices and well-known properties of the Pauli matrices. In matrix representation of quaternions, the absolute value squared (31) can be calculated as

$$|\hat{\mathbf{q}}|^2 = \frac{1}{2}\mathrm{tr}\left(\bar{\hat{\mathbf{q}}}\hat{\mathbf{q}}\right) = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^3, \quad (34)$$

where $\bar{\hat{\mathbf{q}}}$ is a conjugate matrix to $\hat{\mathbf{q}}$ (compare with (32)):

$$\bar{\hat{\mathbf{q}}} = \mu_0\hat{I} + i\mu_1\hat{\sigma}_1 + i\mu_2\hat{\sigma}_2 + i\mu_3\hat{\sigma}_3. \quad (35)$$

Since μ_k are assumed to be real numbers, the value $\bar{\hat{\mathbf{q}}}$ (35) coincides with the Hermitian conjugate matrix, $\bar{\hat{\mathbf{q}}} = \hat{\mathbf{q}}^\dagger$.

Now, we are going to generalize (32) and assume μ_k to be complex numbers. This means that, instead of quaternions with real μ_k, we now can consider general complex matrices $\hat{\mathbf{q}}$ of 2×2 dimension parameterized by complex numbers μ_k according to (32) with the use of the Pauli matrices expansion. The absolute value squared for the new "numbers" is $\frac{1}{2}tr\left(\hat{\mathbf{q}}^\dagger\hat{\mathbf{q}}\right) = = |\mu_0|^2 + |\mu_1|^2 + |\mu_2|^2 + |\mu_3|^2$, where

$$\hat{\mathbf{q}}^\dagger = \mu_0^*\hat{I} + i\mu_1^*\hat{\sigma}_1 + i\mu_2^*\hat{\sigma}_2 + i\mu_3^*\hat{\sigma}_3. \quad (36)$$

Instead of $\hat{\Phi}$ in the form (16), we now consider the wave function for a particle in the form:

$$\hat{\Phi} = B_0\hat{I} - iB_1\hat{\sigma}_1 - iB_2\hat{\sigma}_2 - iB_3\hat{\sigma}_3 \equiv B_0\hat{I} - i\left(\mathbf{B}\cdot\hat{\boldsymbol{\sigma}}\right), \quad (37)$$

where B_0 and B_k $(k = 1, 2, 3)$ are the same values as in (10) or (16). In these notations, the system of equations (14) can be written in the form:

$$i\partial_t\hat{\Phi} = \left(i\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{p}\right) + m\right)\hat{\Phi}^\dagger, \quad (38)$$

where $\mathbf{p} \equiv -i\nabla$ is the momentum operator, and $\hat{\Phi}^\dagger$ is a Hermitian conjugate matrix to $\hat{\Phi}$. It is suitable to accomplish (38) with the equation for $\hat{\Phi}^\dagger$:

$$i\partial_t\hat{\Phi}^\dagger = \left(-i\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{p}\right) + m\right)\hat{\Phi}. \quad (39)$$

Each of the two equations (38), (39) is equivalent to the system of equations (14). Now, we unite the functions $\hat{\Phi}$ and $\hat{\Phi}^\dagger$ into one matrix $\hat{\psi}$ (having four rows and two columns)

$$\hat{\psi} \equiv \begin{pmatrix} \hat{\Phi}^\dagger \\ \hat{\Phi} \end{pmatrix} = \begin{pmatrix} B_0\hat{I} + i\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{B}\right) \\ B_0\hat{I} - i\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{B}\right) \end{pmatrix} = \begin{pmatrix} B_0 + iB_3 & iB_1 + B_2 \\ iB_1 - B_2 & B_0 - iB_3 \\ B_0 - iB_3 & -iB_1 - B_2 \\ -iB_1 + B_2 & B_0 + iB_3 \end{pmatrix} \quad (40)$$

with the absolute value squared

$$|B_0|^2 + |B_1|^2 + |B_2|^2 + |B_3|^2 = \frac{1}{4}\,tr\left(\hat{\psi}^\dagger\hat{\psi}\right). \quad (41)$$

It is obvious that the equation for $\hat{\psi}$ has the form:

$$i\partial_t\hat{\psi} = \left(\hat{\boldsymbol{\alpha}}\cdot\mathbf{p}\right)\hat{\psi} + m\hat{\beta}\hat{\psi}, \quad (42)$$

where $\hat{\alpha}_k$ and $\hat{\beta}$ are the Dirac matrices in the following representation:

$$\hat{\alpha}_k \equiv \hat{\sigma}_2 \otimes \hat{\sigma}_k, \quad \hat{\beta} \equiv \hat{\sigma}_1 \otimes \hat{I}. \quad (43)$$

Thus, we have the Dirac equation (42) for two four-component columns, or (written for each column separately) the system of two Dirac equations for ordinary four-component wave functions. But it is essential to consider the additional condition (15) or (20). It is also important to account for that the both four-component wave functions are bounded between themselves (see (40)), since we have only four components B_μ, where $\mu = 0, 1, 2, 3$. And due to additional condition ((15) or (20)), one has only two independent components of the wave function in the considered case where $S = 0$.

Now, let us consider the case of a particle with spin $S = 1$. We omit a detailed discussion of the necessity to transform the Proca–Duffin–Kemmer–Petiau equations for a particle with spin $S = 1$ into the system of equations [5, 6] having regular limit at $m \to 0$ (which is nothing but Maxwell set of equations for massless photons). Similar discussion and transformations were carried out above for a particle with spin $S = 0$. Here, we simply use the results of reference [5, 6] and formulate the system of two Dirac equations for a particle with spin $S = 1$.

The first-order differential equations for a particle with $S = 1$ can be written in the form [5, 6]:

$$\begin{cases} \partial_t \mathbf{u} = -[\nabla \times \mathbf{v}] + im\mathbf{u}, \\ \partial_t \mathbf{v} = [\nabla \times \mathbf{u}] - im\mathbf{v}, \end{cases} \tag{44}$$

with the additional conditions

$$(\nabla \cdot \mathbf{u}) = 0, \quad (\nabla \cdot \mathbf{v}) = 0. \tag{45}$$

Let us introduce the wave function $\hat{\varphi}$ in the form of a matrix (having two four-component columns):

$$\hat{\varphi} \equiv \begin{pmatrix} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{u}) \\ (\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}) \end{pmatrix} = \begin{pmatrix} u_3 & u_1 - iu_2 \\ u_1 + iu_2 & -u_3 \\ v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \tag{46}$$

with the absolute value squared

$$\sum_{k=1}^{3} \left(|v|_k^2 + |u|_k^2 \right) = \frac{1}{2} tr \left(\hat{\varphi}^\dagger \hat{\varphi} \right). \tag{47}$$

In these notations, the system of equations (44) takes the form of the Dirac equation for the wave function (46)

$$i\partial_t \hat{\varphi} = (\hat{\boldsymbol{\alpha}} \cdot \mathbf{p})\, \hat{\varphi} + m\tilde{\hat{\beta}}\hat{\varphi} \tag{48}$$

(or the system of two Dirac equations for each of the two columns of the wave function $\hat{\varphi}$). In Eq. (48), matrices $\hat{\alpha}_k$ identically coincide with the Dirac matrices given in representation (43), while $\tilde{\hat{\beta}}$ has another representation:

$$\tilde{\hat{\beta}} = -\hat{\sigma}_3 \otimes \hat{I}. \tag{49}$$

Obviously, one can carry out a similarity transformation for matrices and obtain Eq. (48) in the same representation as the one of (42). It is necessary to keep in mind that the components of the wave function (46) obey conditions (45).

Thus, we have an important conclusion that the wave functions of the both particles with spin $S = 0$ and $S = 1$ formally obey the same system of two Dirac equations, but also are constrained with different additional conditions: (15) for $S = 0$, and (45) for $S = 1$. In essence, the additional conditions "construct" particles with even spin ($S = 0$ or $S = 1$) from two particles with spin $S = \frac{1}{2}$.

We omit here a detailed discussion concerning the system of equations for a particle with spin $S = \frac{3}{2}$, for which it is well-known [3, 4, 7] that it can be written as a system of a few Dirac equations with additional conditions (constraints). This system of equations for a particle with spin $S = \frac{3}{2}$, and the above-considered examples of the equations for particles with spin $S = 0$ and $S = 1$ lead us to an important generalization – an assumption that the first-order differential equations for a particle with a given spin S can be formulated in the form of the system of a few Dirac equations with some additional conditions (constraints) imposed on the components of the wave function in order to fix the total spin S.

4. The High Energy Limit

Consider the case, where the masses of particles with spin $S = 0$ and $S = 1$ coincide. This case may be observed in the limit of high energies, where the difference between the experimental values of masses of particles is negligible. Thus, let us assume that the both particles (with spin $S = 0$ and $S = 1$) have the same mass m. If we introduce the wave function

$$\hat{F} \equiv \begin{pmatrix} \hat{I} \cdot f + i\,(\hat{\boldsymbol{\sigma}} \cdot \mathbf{u}) \\ \hat{I} \cdot g + i\,(\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}) \end{pmatrix} = \begin{pmatrix} f + iu_3 & iu_1 + u_2 \\ iu_1 - u_2 & f - iu_3 \\ g + iv_3 & iv_1 + v_2 \\ iv_1 - v_2 & g - iv_3 \end{pmatrix}, \tag{50}$$

then the Dirac equation

$$i\partial_t \hat{F} = (\hat{\boldsymbol{\alpha}} \cdot \mathbf{p})\,\hat{F} + m\tilde{\hat{\beta}}\hat{F} \tag{51}$$

combines the both systems of equations for the case $S = 0$ (see (14), (15)) and for $S = 1$ (see (44), (45)). In order to make this fact obvious, we rewrite system (51) in an explicit form (without use of matrices):

$$\begin{cases} \partial_t f &= -(\nabla \cdot \mathbf{v}) + imf, \\ \partial_t \mathbf{v} &= -\nabla f + [\nabla \times \mathbf{u}] - im\mathbf{v}, \\ \partial_t g &= (\nabla \cdot \mathbf{u}) - img, \\ \partial_t \mathbf{u} &= \nabla g - [\nabla \times \mathbf{v}] + im\mathbf{u}. \end{cases} \tag{52}$$

One can immediately see that, at $f = 0$ and $g = 0$, we obtain, from (52), the system of equations (44), (45) for a particle with spin $S = 1$, while, at $f = 0$ and $\mathbf{v} = 0$ (or $g = 0$ and $\mathbf{u} = 0$), we obtain the system of equations (14), (15) for a particle with spin $S = 0$ (in other notations for the field components). Thus, the wave functions for particles with spin $S = 0$ or $S = 1$ are partial solutions of system (52).

At the same time, if we make a transition from eight independent components f, g, $\mathbf{u}$, and $\mathbf{v}$ of the wave function $\hat{F}$ (50) to eight components $\xi_\varkappa$ and $\eta_\varkappa$ ($\varkappa = 1, 2, 3, 4$) according to

$$\hat{F} = \begin{pmatrix} f+iu_3 & iu_1+u_2 \\ iu_1-u_2 & f-iu_3 \\ g+iv_3 & iv_1+v_2 \\ iv_1-v_2 & g-iv_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \\ \xi_3 & \eta_3 \\ \xi_4 & \eta_4 \end{pmatrix}, \tag{53}$$

we find that the system of equations (51) is nothing else but the set of two Dirac equations for two independent particles with spin $S = \frac{1}{2}$.

Thus, at high energies, there arise new degrees of freedom (with spin $S = \frac{1}{2}$) instead of original ones (with spin $S = 0$ and spin $S = 1$). But, at low energies, where the difference in masses of real particles can not be neglected, these new degrees of freedom (with spin $S = \frac{1}{2}$) disappear together with the validity of Eq. (51). This seems to be very similar to quark degrees of freedom explicitly present at high energies, but "disappearing" at low energies.

5. Conclusions

To summarize, we would like to mention the following conclusions and make the following generalizations.

Equations for a particle with spin S can be formulated in the form of a system of Dirac equations with additional conditions (constraints) imposed on the components of the wave function. In particular, this is demonstrated for equations for particles with spin $S = 0$ and $S = 1$.

The equations for particles with spin $S = 0$ and $S = 1$ in the limit of high energies can be united into the system which reveals new degrees of freedom being particles with spin $S = \frac{1}{2}$ without any preliminary assumptions about their existence.

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РІВНЯННЯ ДЛЯ ЧАСТИНОК ЗІ СПІНОМ $S = 0$ І $S = 1$ У СПІНОРНОМУ ПРЕДСТАВЛЕННІ

Рівняння для частинок зі спіном $S = 0$ і $S = 1$ представлено у формі системи двох рівнянь Дірака із додатковими умовами (в'язями), які накладаються на компоненти хвильових функцій. У випадку тотожних мас (або в границі високих енергій, коли різницею мас можна нехтувати), сформульовано об'єднану систему рівнянь, частинні розв'язки якої співпадають із тими, що випливають з рівнянь для спіну $S = 0$ і $S = 1$, і одночасно є двома рівняннями Дірака для двох незалежних частинок зі спіном $S = 1/2$. Запропоновано принцип побудови рівнянь для частинок із довільним спіном у спінорному представленні.

Ключові слова: рівняння Дірака, рівняння першого порядку відносно похідних для частинок зі спіном $S = 0$ і $S = 1$.