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SIGNALS FROM THE EARLY UNIVERSE[1]

It is proposed to account for the time-dependent partial thermalization of the Ly α lines emitted during cosmic recombination of electrons and protons in the early Universe based on an analytically solvable nonlinear diffusion model. The amplitude of the partially thermalized and redshifted Lyα line is found to be too low to be visible in the cosmic microwave spectrum, in accordance with previous numerical models and Planck observations. New space missions with more sensitive spectrometers are required to detect Ly α-remnants from recombination as frequency fluctuations in the cosmic microwave background. (An extended version of this article has previously appeared in Scientific Reports).

Keywords: nonlinear diffusion equation, partial thermalization of Lyα photons, cosmic microwave background.

1. Introduction

Following the discovery of the cosmic microwave background (CMB) with a temperature of $(3.5 \pm \pm 1.0)$ K at a frequency of 4080 MHz [1], its spectrum has been measured with ever increasing precision. The radiation has a Planck distribution, because the cosmic background radiation (CBR) had been thermalized essentially through Compton scattering and bremsstrahlung [2] at very early times corresponding to redshifts $z > 10^7$, and expansion retains the thermal spectrum. Ground-, balloon and rocket-based observations confirmed the low-frequency Rayleigh–Jeans branch of the CMB blackbody distribution. Using the COBE satellite's far-infrared absolute spectrophotometer (FIRAS) [3], it became possible to measure across the peak at $\nu_{\rm peak} \simeq 150$ GHz in the frequency range $60 \lesssim \nu \lesssim 600$ GHz, and determine the average temperature as $T_{\rm CMB} = (2.725 \pm \pm 0.001)$ K [4]. With COBE's differential microwave radiometer (DMR), statistically significant spatial structure was discovered and described as scale-invariant fluctuations. After subtracting the dipole anisotropy of order $\Delta T/T \simeq 10^{-3}$ that is caused by the motion relative to the CMB, primordial temperature fluctuations $\Delta T/T \simeq 6 \times 10^{-6}$ were found [5], and later measured in great details with the WMAP [6] and Planck [7] spacecraft down to very small angular scales of approximately 0.1° that correspond to the physical scale of galaxies and clusters of galaxies.

Present research emphasizes the implications of the primordial spatial temperature fluctuations for the cosmological models and, in particular, for the determination of the ΛCDM-parameters that are decisive for the fraction of the dark matter and dark energy in the Universe. However, it is also of interest to investigate how frequency perturbations of the blackbody spectrum disappear, or persist, as function of time [8, 9]. Such perturbations occur, in particular, in the course of the recombination epoch of the evolution of the Universe, as investigated and reviewed, e.g., in

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[10] with an emphasis on the associated release of photons during this epoch that can cause deviations of the CMB spectrum from a perfect blackbody, which could, in principle, be observable in precise measurements. The presently available models such as [9, 10] rely on transport equations that can only be solved numerically.

In this work, it will be proposed [11] to account for the incomplete time-dependent thermalization of the Lyα emissions following recombination through solutions of a nonlinear boson diffusion equation (NBDE) [12]. It has proven to be useful in the context of the fast gluon thermalization in relativistic heavy-ion collisions [13, 14] and Bose–Einstein condensate formation in ultracold atoms [13,15]. Now, the effect of the partially thermalized Lyα line on the CMB spectrum is investigated. Although many other spectral lines from hydrogen and helium are emitted during the recombination, the Lyα line from hydrogen is the most intense. During the recombination, about 68% of the spectral lines that are emitted by the emerging neutral hydrogen atoms are Lyman-alpha lines arising from $2p_{3/2} \to 1s_{1/2}$ and $2p_{1/2} \to 1s_{1/2}$ transitions in neutral hydrogen at frequencies of 2466.071 THz and 2466.060 THz, respectively [16, 17].

Like other photons that are emitted following the recombination, the ultraviolet Lyα emissions constitute a disturbance of the high-frequency (Wien) side of the spectrum. Most of the emissions will be reabsorbed and re-emitted by other neutral hydrogen atoms, but, partially, thermalize in the course of time through the Thomson scattering from the remaining free electrons and other processes: Photons diffuse in frequency through resonant scattering due to random kicks from the thermal velocities of hydrogen atoms, and drift toward lower frequencies due to the energy loss via the atomic recoil [18], and Raman scattering converts incident ultraviolet (UV) photons around the Lyman resonance lines into optical-infrared (IR) photons [19]. The Lyman-alpha emissions from recombination thus persist in a nonequilibrium state that could possibly be detected through frequency modulations of the CMB spectrum, or may lead to modified cosmological parameters.

The nonlinear boson diffusion equation (NBDE) [12, 13] accounts for the time-dependent (partial or complete) thermalization toward the Bose–Einstein stationary distribution that is reached in the limit $t \to \infty$. In the general case of frequency-dependent transport coefficients, it can only be solved numerically, but analytic solutions exist for the constant drift and diffusion coefficients. It is one of the few nonlinear partial differential equations with a clear physical meaning that has analytic solutions. In this work, these solutions are applied to the partial thermalization of the Lyα line that is emitted at the recombination, corresponding to a redshift of the last-scattering surface $z_{\rm rec} \simeq 1100$, and an average recombination temperature of $T_{\rm rec} \simeq 3000$ K.

The focus is on the implementation of the model into the cosmological scenario using phenomenological values for the drift coefficient J and the associated diffusion coefficient D. As in a more general model with frequency- and time-dependent transport coefficients, these are related to the equilibrium temperature through a fluctuation–dissipation relation $T = -\lim_{t\to\infty} D(\nu, t)/J(\nu, t)$, thus constraining the value of the drift once the diffusion coefficient has been determined, and vice versa. The thermalization timescale is $\tau_{\rm eq} \propto D/J^2$, and the proportionality factor will eventually have to be derived from astrophysical input. In this work, the drift coefficient J is estimated on phenomenological grounds, and the diffusion coefficient is computed from the fluctuation-dissipation relation with the equilibrium temperature T at the end of the recombination epoch.

The nonlinear boson diffusion model is adapted to the cosmological scenario in the next section, and the analytic solution in the case of constant transport coefficients is reconsidered. In particular, the initial conditions for the specific case of Lyα thermalization in the early Universe are incorporated into the analytic solution scheme. In Sect. 3, the time-dependent results of the thermalization problem for the Lyα initial conditions are presented, and the relation of the transport coefficients to the equilibration timescale is discussed. As a model calculation that does not yet reflect the physically realistic situation in cosmology, the case of complete thermalization is investigated in Sect. 4, where it is shown that the solutions of the NBDE correctly approach the Bose–Einstein limit for large times. In Sect. 5, the case of incomplete thermalization is discussed that corresponds to the actual time evolution of the Lyα line from recombination in cosmology. An upper limit for the effect of the partially thermalized Lyα line from the recombination on the CMB is calculated. The conclusions are drawn in Sect. 6.

2. Nonlinear Boson Diffusion Model in Cosmology

Planck's equilibrium spectrum for the specific intensity (spectral radiance) as function of frequency ν at temperature T is

$$L(\nu,T)=\frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1}. \qquad (1)$$

Due to the expansion of the Universe the equilibrium temperature T decreases, but the thermal spectrum is maintained, because the both temperature and frequency are reduced with redshift as $(1+z)$ such that ν/T is unchanged. The chemical potential μ may initially be smaller than zero, but is driven toward zero in the course of time and, hence, does not appear in the thermal spectrum. The cosmic background radiation at early times as well as the CMB radiation at present are, therefore, modeled as blackbody spectra with $\mu==0$, as in Eq. (1).

At the time of recombination $\tau_\mathrm{rec}\simeq 380$ ky, the CBR intensity for an average recombination temperature of $T=T_\mathrm{rec}\simeq 3000$ K is shown in Fig. 1 together with the Lyman-α line at $\nu_\alpha=2466$ THz in the vacuum ultraviolet (VUV) region of the electromagnetic spectrum. This line is a doublet with transition frequencies 2466.071 THz and 2466.060 THz [17] corresponding to the respective $2p_{3/2}\to 1s_{1/2}$ and $2p_{1/2}\to 1s_{1/2}$ transitions in neutral hydrogen having Lorentzian line profiles. At the recombination temperature, however, the natural line widths are thermally broadened, such that a single Gaussian profile can be used to represent the occupation-number distribution of the Lyman-alpha line

$$n_\alpha(\nu)=\frac{N_\alpha}{\sqrt{2\pi}\sigma_\alpha}\exp\left[-\frac{(\nu-\nu_\alpha)^2}{2\sigma_\alpha^2}\right] \qquad (2)$$

with normalization N_α in THz, and standard deviation $\sigma_\alpha=\Gamma_\alpha/\sqrt{8\ln 2}$. The specific line intensity at recombination $L_\alpha(\nu,T_\mathrm{rec})=2h\nu^3/c^2\times n_\alpha(\nu,T_\mathrm{rec})$ normalized to Rayleigh–Jeans is shown in Fig. 1. An upper-limit estimate with a physically realistic normalization will be given in Sec. 5. In the following, we use a system of units $h/(2\pi)\equiv\hbar=c=k_\mathrm{B}=1$.

The position of a redshifted Lyα line in the CMB spectrum would be at about 2240 GHz, in the far Wien end. It is, however, likely that the line is partially thermalized till today through the random scatterings with the remaining free electrons (few parts

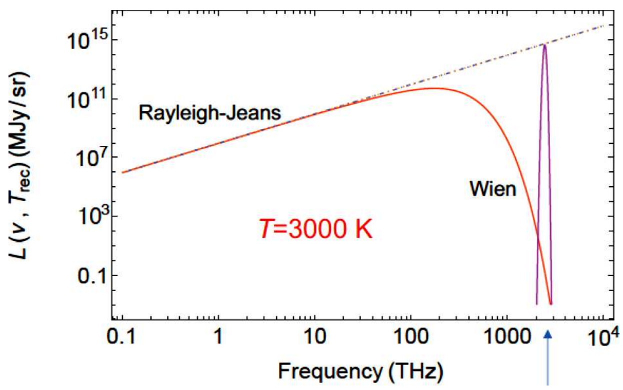

Fig. 1. Planck spectrum of blackbody radiation at the recombination with an average temperature of $T_\mathrm{rec}=3000$ K at redshift $z_\mathrm{rec}\simeq 1100$, and the thermally broadened Lyman-α line of neutral hydrogen at $\nu_\alpha=2466$ THz (arrow) in the VUV region of the spectrum. The dot-dashed line is the Rayleigh–Jeans distribution, $L_\mathrm{RJ}(\nu,T)\propto\nu^2T_\mathrm{rec}$

in 10^4 after the recombination), resonance scattering, and other processes. Since the timescale for the thermalization is larger than the inverse expansion rate, the expansion during the thermalization has to be taken into account. Lyα thermalization is likely not completed till the present time, such that remnants of the recombination lines could survive in the CMB above several hundred GHz.

Once the diffusion function $D(\nu,t)$ and the drift function $J(\nu,t)$ that account for the thermalization are known, the time-dependence of the photonic single-particle occupation number distribution $n(\nu,t)$ from its initial distribution Eq. (2) toward the equilibrium distribution $n_\infty(\nu)$ can be calculated from solutions of the nonlinear boson diffusion equation [12,13]

$$\frac{\partial n}{\partial t}=-\frac{\partial}{\partial\nu}\left[J\,n\,(1+n)+n\frac{\partial D}{\partial\nu}\right]+\frac{\partial^2}{\partial\nu^2}\,[D\,n]. \qquad (3)$$

The mean occupation number $n(\nu,t)$ is equal to the mean energy, divided by the energy per photon. This nonlinear diffusion equation has been derived for bosonic systems in [12] to account for the fast thermalization of gluons in relativistic heavy-ion collisions, but it also accounts for the thermalization in other nonequilibrium Bose systems, such as cold atoms, or photons. To obtain the NBDE, the quantum Boltzmann equation is first rewritten in form of a master equation, and the discrete transition probabilities between quantum states are expressed as integrals by introducing the corresponding densities of states. An approximation to the master equation is then obtained through a gradient expansion in the energy space, and the drift and diffusion coefficients are introduced as first and second moments of the transition probabilities, respectively, to finally arrive at the above nonlinear Eq. (3).

The drift term $J(\nu, t)$ in the NBDE is negative. It is mainly responsible for dissipative effects such as recoil [22] that drive the distribution toward lower frequencies and cause boson (photon) enhancement, the diffusion term $D(\nu, t)$ accounts via the fluctuation-dissipation theorem for the diffusion in the frequency (energy) space. The derivative-term of the diffusion coefficient is required such that the stationary solution $n_\infty(\nu)$ becomes a Bose–Einstein equilibrium distribution. This can be seen by rewriting the equation, setting the time derivative to zero and solving for n_∞.

With the condition that the ratio $J(\nu, t)/D(\nu, t)$ must have no frequency (energy) dependence for $t \to \infty$ such that $\lim_{t\to\infty}[-J(\nu, t)/D(\nu, t)] \equiv 1/T$, it can be shown [14] that the stationary distribution n_∞ equals the Bose–Einstein equilibrium distribution n_{eq}, respectively,

$$n_\infty(\nu) = n_{\mathrm{eq}}(\nu) = \frac{1}{e^{(2\pi\nu-\mu)/T} - 1}. \quad (4)$$

Here, the chemical potential $\mu \le 0$ appears as a parameter.

The nonlinear diffusion equation for the occupation-number distribution $n(\nu, t)$ becomes particularly simple for frequency-independent transport coefficients

$$\frac{\partial n}{\partial t} = -J\,\frac{\partial}{\partial \nu}\Big[n\,(1+n)\Big] + D\,\frac{\partial^2 n}{\partial \nu^2}, \quad (5)$$

where the derivative-term of the diffusion coefficient is now absent, and the transport coefficients have been pulled in front of the derivatives.

The equation with constant transport coefficients differs significantly from a linear Fokker–Planck equation – which has the Maxwell–Boltzmann distribution as stationary solution – due to the nonlinear term: It preserves the essential features of Bose–Einstein statistics that are contained in the quantum Boltzmann equation. This refers especially to the Bose enhancement in the low-frequency region that increases rapidly with time. Indeed, for ultracold bosonic atoms, it has been shown in [13, 15] that the simplified equation with constant transport coefficients – together with the requirement of particle-number conservation – already accounts for time-dependent condensate formation in agreement with recent Cambridge data [23]. At much higher energies and temperatures, the NBDE has been used in [13, 14] to account for the fast thermalization of gluons in relativistic heavy-ion collisions at energies reached at CERN's large hadron collider (LHC).

The diffusion equation with constant coefficients can be solved in closed form for any given initial condition $n_0(\nu)$ using the nonlinear transformation outlined in [12, 13]. The resulting exact solution of the NBDE can be expressed as

$$n\,(\nu, t) = T\,\partial_\nu \ln \mathcal{Z}(\nu, t) - \frac{1}{2} = \frac{T}{\mathcal{Z}}\,\partial_\nu \mathcal{Z} - \frac{1}{2}, \quad (6)$$

where the generalized (time-dependent) partition function $\mathcal{Z}(\nu, t)$ obeys a linear diffusion (heat) equation

$$\frac{\partial}{\partial t}\mathcal{Z}(\nu, t) = D\,\frac{\partial^2}{\partial \nu^2}\mathcal{Z}(\nu, t). \quad (7)$$

The time-dependent partition function can be written as an integral over Green's function of the above Eq. (7) and an exponential function $F(x)$ which depends on the initial occupation-number distribution n_0 according to

$$F(x) = \exp\left[-\frac{1}{2D}\left(Jx + 2J\int^{x} n_0(y)\,dy\right)\right]. \quad (8)$$

Here, the integration constant can be omitted, because it will drop out once the logarithmic derivative of the partition function is taken. The time-dependent partition function with boundary conditions at the singularity becomes

$$\mathcal{Z}_{\mathrm{bound}}(\nu, t) = \int_0^{+\infty} G_{\mathrm{bound}}(\nu, x, t)\,F(x)\,dx. \quad (9)$$

For sufficiently simple initial conditions, it can be calculated analytically, as has been done in [24] for ultracold atoms. In Eq. (9), Green's function G_{bound} accounts for the boundary conditions at the singularity $2\pi\nu = \mu = 0$. They can be expressed as $\lim_{\nu\downarrow 0} n\,(\nu, t) = \infty\,\forall\, t$. One obtains a vanishing partition function – corresponding to an infinite occupation-number distribution – at the boundary, $\mathcal{Z}_{\rm bound}(\nu = 0, t) = 0$, and the energy range is restricted to $\nu \geq 0$. This requires a Green's function that equals zero at $\nu = 0\,\forall\, t$. It can be written as

$$G_{\rm bound}(\nu, x, t) = G_{\rm free}(\nu, x, t) - G_{\rm free}(\nu, -x, t), \quad (10)$$

with the free Green's function $G \equiv G_{\rm free}$ of the linear diffusion Eq. (7),

$$G_{\rm free}(\nu, x, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left[-\frac{(\nu - x)^2}{4Dt}\right]. \quad (11)$$

Finally, the occupation-number distribution is obtained via the basic nonlinear transformation, Eq. (6).

3. Time-Dependent Calculations

To account for the time-dependent – partial or complete – thermalization of the Lyα energy content subsequent to recombination, the integral over the initial distribution of the Lyα line is obtained as

$$\int^{x} n_0(y)\, dy = \int^{x} \frac{N_\alpha}{\sqrt{2\pi}\sigma_\alpha} \exp\left[\frac{(y - \nu_\alpha)^2}{2\sigma_\alpha^2}\right] dy = \\ = \frac{N_\alpha}{2\sigma_\alpha} \operatorname{erf}\left[\frac{x - \nu_\alpha}{\sqrt{2}\sigma_\alpha}\right], \quad (12)$$

which is inserted into the exponential function $F(x)$ in Eq. (8). With Green's function from Eq. (10), the partition function is obtained from Eq. (9), and the time-dependent occupation-number distributions can be computed from the nonlinear transformation, Eq. (6).

For a realistic calculation of the time-dependent thermalization, the values of the transport coefficients are decisive. They are related to the equilibrium temperature through a fluctuation–dissipation relation $T = -D/J$. Moreover, the equilibration time scale is related to the transport coefficients [13] according to $\tau_{\rm eq} = a_\tau D/J^2$ with a proportionality constant a_τ. Hence, the transport coefficients are obtained as

$$D = a_\tau\, T^2/\tau_{\rm eq}, \quad (13)$$

$$J = -a_\tau\, T/\tau_{\rm eq}. \quad (14)$$

The proportionality factor a_τ for thermalization in a Bose system depends significantly on the initial condition. So far, it has been calculated analytically only for a θ-function initial distribution that overlaps with the low-frequency branch of the thermal distribution [12]. No derivation is available for the present case, where a narrow initial distribution at the UV side of the spectrum is far away from the thermal Bose enhancement at the opposite side of the spectrum, and hence, a_τ and $\tau_{\rm eq}$ are treated as parameters that have to be determined in the cosmological context. Alternatively, the drift coefficient is taken as a parameter – see the next section – and the diffusion coefficient is computed from the fluctuation-dissipation relation at temperature T.

4. Complete Thermalization

The analytic solutions of the nonlinear diffusion equation are first applied to an idealized situation of complete thermalization of the Lyα line during the time evolution. This is, of course, not realistic when accounting for the physics of recombination and the subsequent partial frequency redistribution [10] of the photons that are emitted following hydrogen and helium recombination: Scattering is known to hardly be able to thermalize the Ly-alpha distortion even at earlier times. Nevertheless, this calculation serves to demonstrate the method, and it will subsequently be adapted to the actual physical situation of incomplete thermalization of the Lyα-line in the next section.

The value of the drift coefficient in this schematic calculation is chosen as $J = -1$ THz/ky. The frequency shift of the initial Lyα line at short time intervals $\Delta t \ll \tau_{\rm eq}$ is approximately $\Delta\nu_\alpha \simeq J\Delta t$ THz. It later becomes a nonlinear function of time, especially when Bose enhancement sets in at smaller frequencies. Physically, the drift is a consequence of several effects that cause a frequency redistribution toward lower energies such as atomic recoil [22] and electron scattering during recombination. Neglecting expansion and cooling for the moment (it will be discussed in the next section), the equilibrium temperature is kept at $T = T_{\rm rec}$, and, due to the fluctuation-dissipation relation $D = -T/J$, the corresponding diffusion coefficient becomes $D \simeq 63$ THz^2/ky.

Results of the time-dependent thermalization with the above parameters are shown for eight timesteps in Fig. 2. The Lyα emissions are first broadened and shifted to the low-frequency region, until Bose's enhancement sets in. Equilibrium is reached at $t = \tau_{\rm eq} \simeq 4 \times 10^3$ ky in this particular example. In case of complete thermalization, no remainder of the Lyα emissions from recombination would survive in

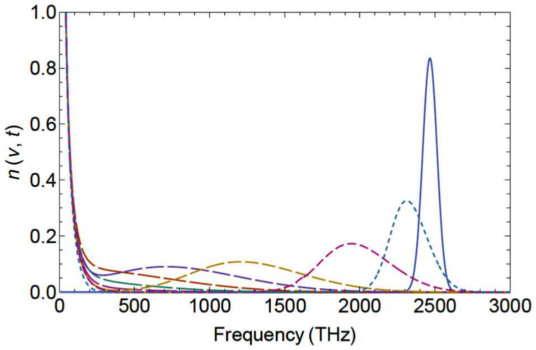

Fig. 2. Time-dependent thermalization of the Lyα line (blue, right at 2466 THz) toward the Bose–Einstein occupation-number distribution (solid, left) as calculated from solutions of the nonlinear boson diffusion equation (NBDE). Results are shown for $t/10^3$ ky $= 0.1; 0.4; 1.1; 1.5; 2; 2.5; 3; 4$ (increasing dash lengths). Here, complete thermalization is assumed, with $D = 63$ THz^2/ky and $J = -1$ THz/ky. From Ref. [11]

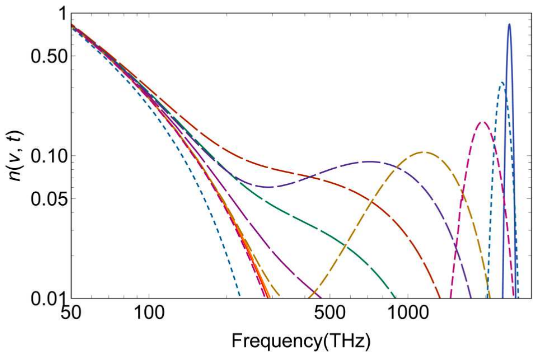

Fig. 3. Time-dependent evolution of the Lyα line in the ultraviolet part of the spectrum (blue, right at 2466 THz) toward equilibrium (solid, left) assuming complete thermalization toward $T = T_{\rm rec}$ and time-independent D, J. Timesteps as in Fig. 2, but in a double-log plot to illustrate the approach toward the thermal occupation-number distribution in the visible and infrared region of the spectrum. From Ref. [11]

today's CMB spectrum. With the consideration of expansion and cooling, and more realistic values for the transport parameters from astrophysical arguments as will be discussed below, the time dependence will differ, and the occupation-number distribution will remain far from equilibrium, but the principal effects of the approach to equilibrium shall persevere.

The results of Fig. 2 are shown again for eight timesteps in Fig. 3 in a double-log plot, which emphasizes the approach to Bose–Einstein equilibrium in the near-infrared region. It can also be seen that even at short times, the solutions of the NBDE generate a low-frequency branch (short-dashed curves on the left) that thermalizes very quickly. It is due to the Bose enhancement that is contained in the NBDE, and leads to a Rayleigh–Jeans-like slope in the infrared when calculating the specific intensity, see the next section. Beyond the equilibration time scale, the full distribution function of the specific intensity becomes again a Planck spectrum, Eq. (1).

5. Incomplete Thermalization

Whereas the above results show the general viability of the nonlinear diffusion model to account for the thermalization in a bosonic (here: photonic) system, the actual Lyα physics in the course of recombination remains far from equilibrium. One important reason is the rapidly falling free-electron number density during the recombination, thus diminishing scattering processes that are required for the thermalization: At the end of the recombination era at redshift $z \simeq 500$ corresponding to an equilibrium temperature of $T \simeq 0.45\,T_{\rm rec}$, the relative free-electron density has dropped below 10^{-3} [10], the values of the transport coefficients in the NBDE must be reduced accordingly, and their time dependence during recombination has to be considered.

The partial frequency redistribution of Lyα photons has been treated in [25] based on a Fokker–Planck approximation – which may, however, not be sufficient toward the end of hydrogen recombination. That numerical approach had been proposed by Rybicki [26], who had also discussed a correspondence to Kompaneets' equation [27] when written in terms of the photon occupation number. This equation concentrates, in particular, on the role of the Compton effect in the establishment of equilibrium between quanta and electrons in a nonrelativistic approximation.

As a complement, the nonlinear diffusion equation offers a related solution to the problem of the partial thermalization that properly accounts for Bose statistics, involves the boundary conditions, and provides a transparent analytic solution through a suitable nonlinear transformation.

First-principles calculations of the drift and diffusion coefficients in the NBDE based on the relevant physical processes electron scattering, and resonance scattering off moving atoms – both including recoil, Doppler broadening and induced scatterings – are not yet available in the cosmological context for the proposed NBDE-model. The transport coefficients are instead estimated here on phenomenological grounds for an equilibrium temperature at the end of the recombination era $T \simeq 0.45\,T_{\rm rec} \simeq 1350$ K, redshift $z \simeq 500$ corresponding to an equilibration time scale $\tau_{\rm eq} \simeq 1.5 \times 10^3$ ky in the ΛCDM model, and exponential time dependencies,

$$D(t) = D_0 \exp(-t/\tau_{\rm eq}), \tag{15}$$

$$J(t) = J_0 \exp(-t/\tau_{\rm eq}), \tag{16}$$

with $J_0 = -1$ THz/ky as in the previous model calculation, and $D_0 = -0.45\,T_{\rm rec}/J_0 = 28$ THz^2/ky. The initial value J_0 of the drift may turn out to be somewhat overestimated when compared with detailed numerical simulations of the frequency redistribution such as in [25], but could be adapted accordingly. The value of D_0 is, however, computed from the fluctuation-dissipation relation, which is inherent to the present model. It could only be modified by permitting energy-dependent transport coefficients, as in Eq. (3).

The result of the time-dependent calculation with the above parameter set is shown in Fig. 4 in four timesteps up to $t = 1.5 \times 10^3$ ky, corresponding to redshift $z = 500$. The distribution functions become already asymmetric (note the log-scale), indicating that a linear Fokker–Planck approximation may not be justified. They remain, however, far from equilibrium – except for the low-frequency branch, which thermalizes quickly even for short times, and leads to a Rayleigh–Jeans slope in the specific intensity of the partially thermalized Lyα line, see below.

Assuming that the thermalization indeed terminates at the end of the recombination epoch, the effect of the Lyα photons from hydrogen recombination – which represent most of the photons from both, hydrogen and helium ($\simeq 24\%$), the recombination – on the CMB distribution based on the NBDE evolution can be estimated by propagating the distribution function at $z = 500$ taken from Fig. 4 (solid yellow curve) to $z = 0$. The normalization is taken according for the ratio of photons in the isotropic blackbody radiation to baryons – mostly protons – which is approximately 1.6×10^9. This yields an upper limit to the effect of the partially thermalized hydrogen Lyα line on distorting today's CMB spectrum.

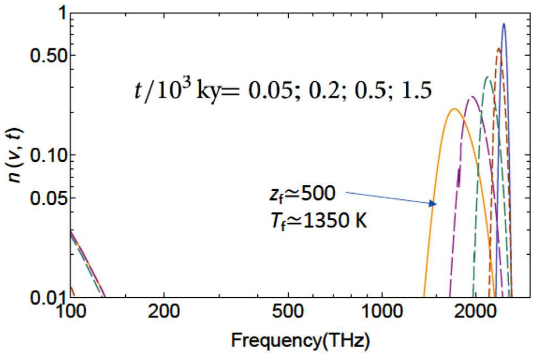

Fig. 4. Nonequilibrium evolution of the Lyα line (blue, right at 2466 THz): Incomplete thermalization with time-dependent transport coefficients $D(t)$, $J(t)$, see text. Timesteps $t/10^3$ ky= = 0.05; 0.2; 0.5; 1.5 are shown, with no significant thermalization expected after the last step, which corresponds to redshift $z = 500$. The curve at the lower left is the equilibrium distribution for $T = 0.45\,T_{\rm rec} \simeq 1350$ K, which is reached only at low frequencies

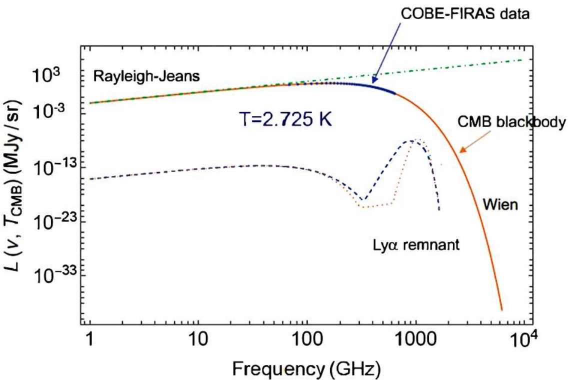

Fig. 5. Today's CMB spectrum of blackbody radiation with $T = 2.725$ K (solid curve), the Rayleigh–Jeans distribution (green dot-dashed line), and the partially thermalized NBDE-solution for the redistributed, and redshifted Lyα line featuring a low-frequency R-J branch due to Bose enhancement (dashed curve). The dotted curve is the NBDE-result for a 50% reduction in the drift and diffusion coefficients. COBE-FIRAS data [4] are shown as black dots, error bars are smaller than the symbol size

The result can be seen in Fig. 5, where the blackbody CMB (solid curve) including the COBE-FIRAS data [4] is shown together with the renormalized solution of the NBDE at $z = 500$ from Fig. 4, propagated to $z = 0$ (dashed curve). Its peak resides on the Wien side of the CMB, and the amplitude is more than seven orders of magnitude below the one of the CMB. To test the sensitivity of the nonlinear model, I have reduced the amplitudes J_0 of the drift coefficient and D_0 of the diffusion coefficient by factors of two, thus keeping the equilibrium temperature at the same value as before. The result is shown as a dotted curve in Fig. 5, which is still below the CMB signal by almost seven orders of magnitude.

This largely analytic calculation gives a clear hint why no Lyα signal from recombination – or oscillatory signal when accounting for all other radiative transitions in hydrogen and helium – is visible at the present level of precision in the CMB, although the distortions can lead to biases to several cosmological parameters [7]. As shown in Fig. 5, the NBDE result yields a Rayleigh–Jeans slope in the intensity of the Lyα line at low frequencies. This is a consequence of the Bose enhancement, which would not occur in a linear Fokker–Planck type approach to a partial thermalization. To reach the proper Wien limit also at large frequencies $\nu > 2000$ GHz, the energy-dependent transport coefficients would be required – which is beyond the scope of an analytic model.

6. Conclusions

The time-dependent incomplete thermalization of the Lyα line that is emitted from neutral hydrogen atoms during the recombination has been accounted for in a nonlinear diffusion model. This approach is complementary to the available detailed numerical treatments of the release of photons during the recombination epoch, their partial frequency redistribution, its effect on the recombination history, and possible observable distortions of the CMB.

In the analytic model, the thermally broadened Lyα emission line at an average recombination temperature of 3000 K provides the initial condition. With the proper boundary condition that causes the low-frequency Bose enhancement, the diffusion equation is solved through a nonlinear transformation in the limit of constant transport coefficients.

As is well known, the system remains far from equilibrium, because the interaction of the radiation field with the electrons can not transform a non-Planckian spectrum into a Planckian one in the course of, or after, the recombination. However, the low-frequency Rayleigh–Jeans slope in the specific intensity indicative for thermalization correctly emerges already at very short times from the analytic solutions of the nonlinear diffusion equation – which would not be the case in a linear theory for the frequency redistribution of Lyα, or other recombination lines. Additional thermalization may result at later times, in particular, during the epoch of reionisation, when the ultraviolet light from the first stars at $t \simeq 370$ My and redshift $z \simeq 12$ re-ionized hydrogen and helium till about $z \simeq 6$.

Regarding possible signatures of the redshifted Lyα recombination line in today's CMB, an upper limit of the specific intensity following partial frequency redistribution as calculated from the nonlinear diffusion model is estimated to be about seven orders of magnitude smaller than the CMB signal, and, therefore, unlikely to be directly detectable at present. This is in line with other numerical calculations, and also with Planck observations of the CMB, which at the present accuracy do not exhibit a frequency-modulated signal from the recombination spectrum. Here, this result has been obtained in a novel nonlinear diffusion model that is analytically solvable and offers a transparent description of the partial thermalization process.

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СИГНАЛИ ВІД РАННЬОГО ВСЕСВІТУ

Пропонується врахувати залежну від часу часткову термалізацію ліній Ly α, випромінюваних під час рекомбінації електронів і протонів у ранньому Всесвіті, на основі аналітично розв'язуваної моделі нелінійної дифузії. Згідно з попередніми числовими моделями та спостереженнями за допомогою космічного телескопу "Планк", амплітуда частково термалізованої та зміщеної в бік червоної частини спектру лінії Lyα надто мала, щоб її можна було побачити в космічному мікрохвильовому спектрі. Потрібні нові космічні апарати з більш чутливими спектрометрами, щоб виявити залишки ліній Ly α від рекомбінації як флуктуації частоти в космічному мікрохвильовому фоні. (Розширена версія цієї статті раніше опублікована в Scientific Reports).

Ключові слова: рівняння нелінійної дифузії, часткова термалізація фотонів Lyα, космічний мікрохвильовий фон.