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# PHASE DIAGRAMS OF A RELATIVISTIC SELF-INTERACTING BOSON SYSTEM

*Within the Canonical Ensemble, we investigate a system of interacting relativistic bosons at finite temperatures and finite isospin densities in a mean-field approach. The mean field contains both attractive and repulsive terms. Temperature and isospin density dependences of thermodynamic quantities are obtained. It is shown that, in the case of attraction between particles in a bosonic system, a liquid-gas phase transition develops against the background of the Bose–Einstein condensate. The corresponding phase diagrams are given. We explain the reasons for why the presence of a Bose condensate significantly increases the critical temperature of the liquid-gas phase transition compared to that obtained for the same system within the framework of Boltzmann statistics. Our results may have implications for the interpretation of experimental data, in particular, how sensitive the critical point of the mixed phase is to the presence of the Bose–Einstein condensate.*

*Keywords*: relativistic bosonic system, Bose–Einstein condensation, phase transition.

## 1. Introduction

It is commonly accepted that QCD exhibits a rich phase structure at finite temperatures and baryon densities, for instance, the transition from hadron gas to quark-gluon plasma, the transition from the chiral symmetry breaking to the symmetry restoration [1]. The physical motivation to study QCD at a finite isospin density and the corresponding pion system is related to the investigation of compact stars, isospin asymmetric nuclear matter, and heavy ion collisions. In early studies of dense nuclear matter and compact stars, it has been suggested that charged pions and even kaons are condensed at sufficiently high densities. The knowledge of the phase structure of the meson systems, in the regime of finite temperatures and isospin densities, is crucial for understanding a wide range of phenomena from nucleus-nucleus collisions to boson, neutron stars, and cosmology. This field is essential to investigations of the hot and dense hadronic matter, a subject of active research. Meanwhile, the investigations of the meson systems have their specifics due to the possibility of the Bose–Einstein condensation (BEC) of the bosonic particles. Formation of classical pion fields in heavy-ion collisions was discussed in Refs. [2–5]. Then the study of the QCD phase structure was extended to finite isospin densities and the systems of pions and K-mesons with a finite isospin chemical potential have been considered in more recent studies [6–11]. First-principles lattice calculations provide a solid basis for our knowledge of the finite-temperature regime. New results concerning dense pion systems have been obtained recently using lattice methods [12–14].

In the present paper, we will consider an interacting particle-antiparticle boson system at the conserved isospin (charge) density $n_I$ and finite temperature $T$. We name the bosonic particles as "pions" just conventionally. The preference is made because the charged $\pi$-mesons are the lightest hadrons that couple to the isospin chemical potential. On the other hand, the pions are the lightest nuclear boson particles,

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and, thus, the account for "temperature creation" of particle-antiparticle pairs is a relevant problem based on quantum statistics. To account for the interaction between the bosons, we introduce a phenomenological Skyrme-like mean field $U(n)$, which depends only on the total meson density $n$. We regard such a self-interacting many-particle system as a toy model that can help us understand the BEC and phase transitions over a wide range of temperatures and densities. The mean field $U(n)$ rather reflects the presence of other strongly interacting particles in the system, for instance $\rho$-mesons and nucleon-antinucleon pairs at low temperatures or gluons and quark-antiquark pairs at high temperatures, $T > T_{\rm qgp} \approx 160$ MeV.

The presented study is part of a sequel [15–18] that started with the investigation of an interacting particle-antiparticle boson system at $\mu = 0$. The next development of the subject was given in Ref. [19], where the boson system was considered within the framework of the Canonical Ensemble with the canonical variables ($T$, $n_I$), i.e., at the conserved isospin (charge) density. In this formulation in [19], we calculated the temperature characteristics of a non-ideal hot "pion" gas with a fixed isospin density $n_I = n^{(-)} - n^{(+)} > 0$, where $n^{(\mp)}$ are the particle-number densities of the $\pi^-$ and $\pi^+$ mesons, respectively. In the present study, we proceed to exploit the Canonical Ensemble. But now, we focus on the isospin-density dependencies of thermodynamic quantities when the temperature is fixed.

In Sect. 2, we very shortly remind the formalism of the thermodynamic mean-field model [20] to describe the boson system of particles and antiparticles, which will be used in the presented calculations. In Sect. 2.1, we introduce the Skyrme-like parametrization of the mean field, and the corresponding thermodynamic functions are calculated.

## 2. The Mean-Field Model for the System of Particles and Antiparticles

Our consideration of thermodynamic properties of the system of interacting bosonic particles and antiparticles at finite temperatures is carried out within the framework of the thermodynamic mean-field model, which was introduced in Refs. [21, 22] and further developed in Ref. [20]. This approach is based on the representation of the free energy $F$ of the particle-antiparticle system as the sum of two parts: the first part $F_0$ is the free energy of the two-component system of free particles, and the second part $F_{\rm int}$ is responsible for the interaction between all particles, i.e., $F = F_0 + F_{\rm int}$. Therefore, it is assumed that, in general case, the free-energy density of the two-component system looks like

$$\phi\,(T, n_1, n_2) = \phi_1^{(0)}\,(T, n_1) + \phi_2^{(0)}\,(T, n_2) + \phi_{\rm int}\,(T, n), \tag{1}$$

where $\phi = F/V$ with $V$ as the volume of the system, $\phi_1^{(0)}$ and $\phi_2^{(0)}$ are the free energy densities for the free particles of the first and second components, respectively, whereas the density of free energy $\phi_{\rm int}$ accounts for the interaction in the system, $n_1$ and $n_2$ is the particle-number density of each component and $n = n_1 + n_2$ is the total particle-number density. Next, the chemical potential associated with each component is calculated as correspondent derivative

$$\mu_i = \left[\frac{\partial \phi(T, n_1, n_2)}{\partial n_i}\right]_T, \tag{2}$$

where $i = 1, 2$. This results in

$$\mu_i^{(0)} = \mu_i(T, n_i) - U(T, n), \tag{3}$$

where we define

$$\mu_i^{(0)} = \frac{\partial \phi_i^{(0)}(T, n_i)}{\partial n_i}, \quad U(T, n) \equiv \frac{\partial \phi_{\rm int}(T, n)}{\partial n}. \tag{4}$$

Similarly, we can write the pressure, $p = \mu_1 n_1 + \mu_2 n_2 - \phi$, dividing it into free and interacting parts

$$p\,(T, n_1, n_2) = p_1^{(0)} + p_2^{(0)} + P_{\rm ex}(T, n), \tag{5}$$

where $p_i^{(0)} = \mu_i^{(0)} n_i - \phi_i^{(0)}$ is the pressure of the ideal gas created by the $i$-th component of the system and

$$P_{\rm ex}(T, n) \equiv n \left[\frac{\partial \phi_{\rm int}(T, n)}{\partial n}\right]_T - \phi_{\rm int} \tag{6}$$

is the excess pressure. It is seen that the definitions of $U(T, n)$ and $P_{\rm ex}(T, n)$ lead to a differential correspondence between these quantities:

$$n \left[\frac{\partial U(T, n)}{\partial n}\right]_T = \left[\frac{\partial P_{\rm ex}(T, n)}{\partial n}\right]_T. \tag{7}$$

We limit our consideration to the case where, at a fixed temperature, the interacting boson particles and boson antiparticles are in dynamic equilibrium with respect to the processes of annihilation and pair-creation. Due to the opposite signs of the charge, the chemical potentials of the bosonic particles $\mu_1$ and the bosonic antiparticles $\mu_2$ have opposite signs (for details, see [20]):

$$\mu_1 = -\mu_2 \equiv \mu_I . \tag{8}$$

Therefore, the Euler relation includes only the isospin number density, $n_I = n^{(-)} - n^{(+)}$, in the following way:

$$\varepsilon + p = T\, s + \mu_I\, n_I , \tag{9}$$

where $n_1 \to n^{(-)}$ is the particle-number density of bosonic particles, and $n_2 \to n^{(+)}$ is the particle-number density of bosonic antiparticles, $\varepsilon$ is the energy density, and $s$ is the entropy density [1]. In what follows we will consider the boson particle-antiparticle system with conserved isospin number density $n_I$, whereas, in this study, the total particle-number density is a thermodynamic quantity that depends on $T$ and $n_I$, i.e., $n(T, n_I)$ [2].

Exploiting Eq. (5) with formula for the ideal gas in the Grand Canonical Ensemble the total pressure in the particle-antiparticle system reads [3]

$$\begin{aligned} p = {} & -T \int \frac{d^3k}{(2\pi)^3} \times \\ & \times \ln\left[1 - \exp\left(-\frac{\sqrt{m^2 + \mathbf{k}^2} + U(T,n) - \mu_I}{T}\right)\right] - \\ & - T \int \frac{d^3k}{(2\pi)^3} \times \\ & \times \ln\left[1 - \exp\left(-\frac{\sqrt{m^2 + \mathbf{k}^2} + U(T,n) + \mu_I}{T}\right)\right] + \\ & + P_{\rm ex}(T,n), \end{aligned} \tag{10}$$

[1] We use the negative total electric charge in the system because of the predominance of the creation of negative pions over positive ones in relativistic nucleus-nucleus collisions.

[2] The dynamical conservation of the total number of pions in a pion-enriched system created on an intermediate stage of a heavy-ion collisions was considered in Refs. [24–26].

[3] Here and below we adopt the system of units $\hbar = c = 1$, $k_{\rm B} = 1$.

where $\mu_1^{(0)}$ and $\mu_2^{(0)}$ are altered for $(\mu_I - U)$ and $(-\mu_I - U)$, respectively, in accordance with Eq. (3) and Eq. (8).

The thermodynamic consistency of the mean-field model can be obtained by comparing two expressions which must eventually coincide. These expressions, which determine the isospin density, read

$$n_I = \left(\frac{\partial p}{\partial \mu_I}\right)_T ,$$

and

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left[ f_{\rm BE}\big(E(k,n), \mu_I\big) - f_{\rm BE}\big(E(k,n), -\mu_I\big)\right], \tag{11}$$

where pressure is given by Eq. (10). Here, $E(k,n) = \omega_k + U(T,n)$ with $\omega_k = \sqrt{m^2 + \mathbf{k}^2}$ and the Bose–Einstein distribution function reads

$$f_{\rm BE}(E,\mu) = \left[\exp\left(\frac{E-\mu}{T}\right) - 1\right]^{-1} . \tag{12}$$

In order expressions (11) to coincide, as a result, the following relation between the mean field and the excess pressure arises as the necessary condition:

$$n\, \frac{\partial U(T,n)}{\partial n} = \frac{\partial P_{\rm ex}(T,n)}{\partial n} . \tag{13}$$

As we see, this relation coincides literally with relation (7), which was derived using the definitions of the mean field $U(T,n)$ and excess pressure $P_{\rm ex}(T,n)$ in Eq. (4) and in Eq. (6), respectively. Relation (13) that provides the thermodynamic consistency of the model has a natural basis, because there is only one source for both quantities $U$ and $P_{\rm ex}$, it is interaction in the system.

### *2.1. Parametrization of the mean field*

The thermodynamic mean-field model has been applied to several physically interesting systems including the hadron-resonance gas [20] and the pionic gas [23]. This approach was extended to the case of a bosonic system at $\mu_I = 0$ which can undergo the Bose condensation [15, 17]. In the present study, a generalized formalism given in Section 2 is used to describe the particle-antiparticle system of bosons, when the isospin density is finite, $n_I \neq 0$. At the same time, we assume that the interaction between particles is described by the Skyrme-like mean field which depends on the total particle-number density $n$. The latter means that we consider just a strong interaction. For further calculations we adopt the following form of the mean field

$$U(n) = -A\,n + B\,n^2, \tag{14}$$

where $A$ and $B$ are the model parameters, which should be specified. In accordance with relation (13) we calculate the excess pressure

$$P_{\rm ex}(n) = -\frac{1}{2}\,A\,n^2 + \frac{2}{3}B\,n^3. \tag{15}$$

The mean field $U(n)$ can be thought as some effective one which includes several contributions. For instance, the investigation of the properties of a dense and hot pion gas is well inspired by the formation of the medium with low baryon numbers at midrapidity what was proved in the experiments at RHIC and LHC [27, 28]. By this reason, in our calculations, we consider a general case of $A > 0$, to study a bosonic system with both attractive and repulsive contributions to the mean field (14). Our main goal is the study how the relation between repulsion and attraction in the system influences the Bose-condensation and thermodynamic properties of the bosonic system. In the present paper, to investigate these features, we keep the repulsive coefficient $B$ as a constant, whereas the coefficient $A$, which determines the intensity of attraction of the mean field (14), will be varied. To do this, it is advisable to parametrize the coefficient $A$ with the help of solutions of equation $U(n) + m = 0$, similar to parametrization adopted in Refs. [15, 17]. For the given mean field (14) there are two roots of this equation ($n_{1,2} = (A \mp \sqrt{A^2 - 4mB})/2B$)

$$\begin{aligned} n_1 &= \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right),\\ n_2 &= \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right). \end{aligned} \tag{16}$$

where

$$\kappa \equiv \frac{A}{2\sqrt{m\,B}}.$$

Then we can parameterize the attraction coefficient as $A = \kappa A_{\rm c}$ with $A_{\rm c} = 2\sqrt{mB}$. As we will show below, the dimensionless parameter $\kappa$ is the scale parameter of the model which determines the phase structure of the system. As it is seen from Eq. (16) for the values of parameter $\kappa < 1$ there are no real roots. The critical value $A_{\rm c}$ is obtained when both roots coincide, i.e. when $\kappa = 1$, then $A = A_{\rm c} = 2\sqrt{mB}$.

We consider two intervals of the parameter $\kappa$. First interval corresponds to $\kappa \le 1$, there are no real roots of equation $U(n) + m = 0$. We associate these values of $\kappa$ with a "weak" attractive interaction and in the present study we consider variations in the attraction coefficient $A$ for values of $\kappa$ only from this interval. Second interval corresponds to $\kappa > 1$, there are two real roots of equation $U(n) + m = 0$. We associate this interval with a "strong" attractive interaction. This case will be considered elsewhere.

## 3. The Phase Transition to the Bose–Einstein Condensate

In the mean-field approach the behavior of the particle-antiparticle boson system, when both components in thermal (kinetic) phase, is determined by the set of two transcendental equations

$$n = \int \frac{d^3k}{(2\pi)^3}\left[f_{\rm BE}\big(E(k,n),\mu_I\big) + f_{\rm BE}\big(E(k,n),-\mu_I\big)\right], \tag{17}$$

$$n_I = \int \frac{d^3k}{(2\pi)^3}\left[f_{\rm BE}\big(E(k,n),\mu_I\big) - f_{\rm BE}\big(E(k,n),-\mu_I\big)\right], \tag{18}$$

where the Bose–Einstein distribution function $f_{\rm BE}\big(E,\mu_I\big)$ is defined in (12) and $E(k,n) = \omega_k + U(n)$, the degeneracy factor $g = 1$ because the spin of particles is zero. Equations (17)–(18) should be solved selfconsistently with respect to $n$ and $\mu_I$ for the given canonical variables ($T$, $n_I$). Remind, in the present we consider boson system in the Canonical Ensemble. In this approach the chemical potential $\mu_I$ is a thermodynamic quantity which depends on the canonical variables, i.e. $\mu_I(T, n_I)$.

In case of the cross state, when the particles, i.e. $\pi^-$-mesons, are in the condensate phase and antiparticles are still in the thermal (kinetic) phase, Eqs. (17), (18) should be generalized to include condensate component $n_{\rm cond}^{(-)}$. We should take into account also that the particles ($\pi^-$ or high-density component) can be in condensed state just under the necessary condition

$$U(n) - \mu_I = -m. \tag{19}$$

During decreasing of temperature from high values, where both $\pi^-$ and $\pi^+$ are in the thermal phase, the density of $\pi^-$-component $n^{(-)}(T, n_I)$ achieves first the critical curve at temperature $T_{\rm cd}$, where at the crossing point condition (19) is valid, but the density of the condensate is zero at this point, i.e., $n_{\rm cond} = 0$. This means that the curve $n_{\rm lim}(T)$, which is defined as

$$n_{\rm lim}(T) = \int \frac{d^3k}{(2\pi)^3}\, f_{\rm BE}\big(\omega_k, \mu_I\big)\Big|_{\mu_I = m}, \qquad (20)$$

is the critical curve for $\pi^-$-mesons or for high-density component of the gas. As we see function (20) represents the maximal density of thermal (kinetic) boson particles of the ideal gas at temperatures $T \le T_{\rm cd}$ because the chemical potential has its maximum allowed value. Hence, we obtain that the critical curve of the particle-antiparticle boson system calculated in the mean-field approach coincides with the critical curve for the ideal gas.

With account for Eqs. (19) and (20) we write the generalization of the set of Eqs. (17), (18)

$$n = n^{(-)}_{\rm cond}(T) + n_{\rm lim}(T) + \\ + \int \frac{d^3k}{(2\pi)^3}\, f_{\rm BE}\big(E(k,n), -\mu_I\big), \qquad (21)$$

$$n_I = n^{(-)}_{\rm cond}(T) + n_{\rm lim}(T) - \\ - \int \frac{d^3k}{(2\pi)^3}\, f_{\rm BE}\big(E(k,n), -\mu_I\big). \qquad (22)$$

Here $\mu_I = U(n) + m$. One can see from Eqs. (21), (22) that the particle-number density $n^{(+)}$ is provided only by thermal $\pi^+$ mesons. Whereas, the density $n^{(-)}$ is provided by two fractions: the condensed particles ($\pi^-$ mesons at $\boldsymbol{k} = 0$) with the particle-number density $n^{(-)}_{\rm cond}(T)$, and thermal $\pi^-$ mesons at $|\boldsymbol{k}| > 0$ with the particle-number density $n_{\rm lim}(T)$. Hence, the particle-density sum rule for the phase of $\pi^-$ mesons only in the interval $T < T_{\rm cd}$ reads: $n^{(-)} = n^{(-)}_{\rm cond}(T) + n_{\rm lim}(T)$.

It is necessary to note that expression "particles are in the condensate phase" is, of course, a conventional one, because, in the essence, it is a mixture phase, where, at a fixed temperature, some fraction of particles, i.e., a fraction of $\pi^-$-mesons, belongs to the thermal phase with momentum $|\boldsymbol{k}| > 0$, and the other fraction of $\pi^-$-mesons belongs to the Bose–Einstein condensate, where all $\pi^-$-mesons have zero momentum, $\boldsymbol{k} = 0$.

For the further evaluation of the phase diagram, one needs the value of the pressure. First, we give a pressure for the state of the system, when both $\pi^-$ and $\pi^+$ mesons are in the thermal phase. When both components of the $\pi^-$-$\pi^+$ system are in the thermal (kinetic) phase, the pressure reads

$$p = \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\,\frac{\boldsymbol{k}^2}{\omega_k}\big[f_{\rm BE}\big(E(k,n), \mu_I\big) + \\ + f_{\rm BE}\big(E(k,n), -\mu_I\big)\big] \,+\, P_{\rm ex}(n), \qquad (23)$$

where $n(T, n_I)$ and $\mu_I(T, n_I)$ are a solution of Eqs. (17), (18).

For the cross state of the system, i.e., when $\pi^-$ mesons are in the condensate phase, but $\pi^+$ mesons are in the thermal phase, with regard for Eqs. (19) and (20), we write

$$p = \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\,\frac{\boldsymbol{k}^2}{\omega_k}\big[f_{\rm BE}\big(E(k,n), \mu_I\big) + \\ + f_{\rm BE}\big(E(k,n), -\mu_I\big)\big]_{\mu_I = U(n)+m} \,+\, P_{\rm ex}(n), \qquad (24)$$

where $\mu_I = U(n) + m$ and, at this value of the chemical potential, we get $E(k,n) - \mu_I = \sqrt{m^2 + \boldsymbol{k}^2} - m$. Here, we account for that the total particle-number density $n$ consists of three pieces, $n = n^{(-)}_{\rm cond} + n^{(-)}_{\rm th} + n^{(+)}$. Because, in the condensate phase, $n^{(-)}_{\rm th} = n_{\rm lim}$, we can calculate the density of the condensate as $n^{(-)}_{\rm cond} = n^{(-)} - n_{\rm lim}$. In view of the chemical potential in the distribution function of $\pi^-$ mesons, we get

$$p = \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\,\frac{\boldsymbol{k}^2}{\omega_k}\big[f_{\rm BE}\big(\omega_k, m\big) + \\ + f_{\rm BE}\big(E(k,n), -\mu_I\big)\big] + P_{\rm ex}(n), \qquad (25)$$

where $\mu_I = U(n) + m$.

### *3.1. Thermodynamic quantities*

In what follows, a value of the repulsive coefficient $B$ of the mean field (14), is fixed. At the same time the coefficient $A$, which determines the intensity of attraction of the mean field (14), will be varied. The coefficient $B$ is obtained from the estimate based on the virial expansion [29], $B = 10\, m v_0^2$ with $v_0$ equal to four times the proper volume of a particle, i.e., $v_0 = 16\pi r_0^3/3$. We take $v_0 = 0.45$ fm$^3$ that corresponds to a "particle radius" $r_0 \approx 0.3$ fm. The numerical calculations will be done for bosons with mass

$m = 139$ MeV, which we call "pions". In this case, the repulsive coefficient is $B/m = 2.025$ fm$^6$, and it is kept constant through all present calculations.

At high temperatures, i.e. $T \geq T_{\rm cd}$, both components of the bosonic particle-antiparticle system are in the thermal phase, and thermodynamic properties of the system are determined by the set of Eqs. (17) and (18). Solving this set for given values $T$ and $n_I$, we obtain the functions $\mu_I(T, n_I)$ and $n(T, n_I)$ and then other thermodynamic quantities.

With decreasing the temperature, the particle density $n^{(-)}(T)$ crosses the critical curve at the point, which corresponds to the value $T = T_{\rm cd}$, see Fig. 1 (on the graph, $T_{\rm cd}$ is denoted as $T_{\rm c}$). The dependence of $\pi^-$-meson density on the temperature is depicted as blue solid line *1*, the dependence of $\pi^+$-meson density as blue dashed line *2*, and the total density $n = n_{\rm tot}$ as black solid line.

During a further decrease in the temperature in the interval $T < T_{\rm cd}$, the $\pi^-$-mesons start to "drop down" into the condensate state, which is characterized by the value of momentum $\boldsymbol{k} = 0$ [4]. In the limit, when $T = 0$, all particles of the high-density component, i.e., $\pi^-$-mesons, will be in the condensate phase. At the same time, the density of the low-density component or $\pi^+$ mesons, which are in the thermal phase, decreases with decreasing the temperature, and it becomes zero at $T = 0$. For the temperatures less than the critical one, i.e., $T < T_{\rm cd}$, the thermodynamic properties of the system are determined by Eqs. (21), (22), where we take into account that $\mu_I = U(n) + m$ for all temperatures of this interval unless the high-density component $n^{(-)}$ is in condensed state.

Equation (21) can be used to determine the critical temperature $T_{\rm cd}$. Indeed, let us consider that, at the crossing point with the critical curve, the density of the condensate is zero, so far, $n^{(-)}_{\rm cond}(T_{\rm cd}) = 0$, and the density of thermal $\pi^-$ particles becomes equal to $n^{(-)}(T_{\rm cd}) = n_{\rm lim}(T_{\rm cd})$. Then, at this temperature $T = T_{\rm cd}$ on the l.h.s. of Eq. (21), we have $n = 2n_{\rm lim}(T_{\rm cd}) - n_I$, and now, at this temperature point on the critical curve, Eq. (21) with respect to $T$ reads as:

$$n_{\rm lim}(T) - n_I = \\ = \int \frac{d^3k}{(2\pi)^3}\, f\big(E(k,n), -\mu_I\big)\Big|_{\mu_I = U(n)+m} \qquad (26)$$

[4] We apply our consideration to the pion gas with $n_I = n^{(-)} - n^{(+)} > 0$.

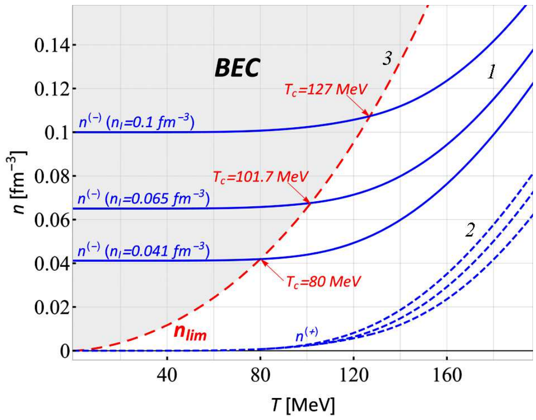



***Fig. 1.*** Temperature dependence of the particle-number densities of $\pi^-$ mesons, $n^{(-)}$ (blue solid lines *1*), and of $\pi^+$ mesons, $n^{(+)}$ (blue dashed lines *2*), for the interacting $\pi^+$-$\pi^-$ pion gas at $\kappa = 0.1$ and $n_I = 0.041, 0.065, 0.1$ fm$^{-3}$. The red dashed line *3* is the critical curve that fixes the particle density of a single-component ideal gas at $\mu = m_\pi$. The shaded area, labeled BEC, indicates the region, where the $\pi^-$ mesons develop the Bose–Einstein condensate

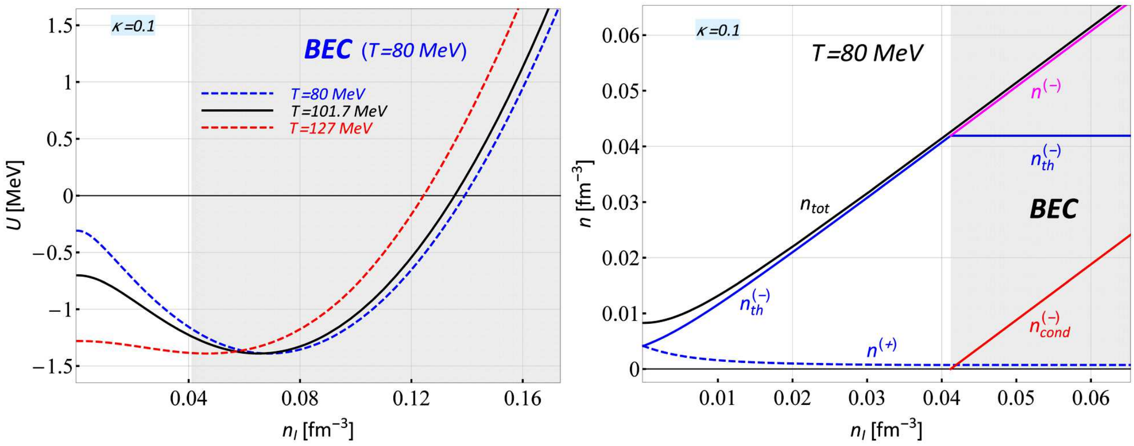
with

$$E(k,n) = \omega_k + U(2n_{\rm lim} - n_I).$$

Solving Eq. (26) at $n_I = 0.1$ fm$^{-3}$, for $\kappa = 0.5$ and $\kappa = 1.0$, we obtained $T_{\rm cd} = 128.8$ MeV and $T_{\rm cd} = 251$ MeV, respectively.

Hence, it turns out that the temperature $T^{(-)}_{\rm cd}$ determines the phase transition to BEC for whole pion system, because the antiparticles ($\pi^+$-mesons) are completely in the thermal state for all temperatures and, thus, the condensate is created just by the particles of high-density component $n^{(-)}(T)$. Then, the total density of the condensate in the two-component pion system at "weak" attraction, i.e., at $\kappa \leq 1$, is created by $\pi^-$-mesons only, i.e., $n_{\rm cond} = n^{(-)}_{\rm cond}$, and this particle-number density plays the role of the order parameter.

In what follows, we will investigate the phase structure of the particle-antiparticle system with respect to the canonical variable $n_I$. As the first step, it is reasonable to obtain the dependencies of the densities $n^{(-)}$, $n^{(+)}$ and mean field $U$ with respect to $n_I$, when we fix $T$. Every isotherm crosses two different phases; thus, for each particular phase, we have to solve the relevant set of equations: 1) when both components of the pion gas are in the thermal (kinetic) states, it is necessary to solve Eqs. (17), (18); 2) when




*a* *b*

***Fig. 2.*** On both panels the dashed area marked as BEC represents the condensate states of the interacting $\pi^+$-$\pi^-$ pion gas in the mean-field model. The mean field $U(n)$ versus isospin density $n_I$ at $\kappa = 0.1$ and $T = 80$, 101.7, 127 MeV (*a*). The particle-number densities $n^{(+)}$, $n^{(-)}$ and $n_{\rm tot} = n^{(+)} + n^{(-)}$ versus isospin density $n_I$ at $T = 80$ MeV and $\kappa = 0.1$ (*b*). Here $n^{(-)}_{\rm th}$ and $n^{(-)}_{\rm cond}$ is the particle density of the thermal and condenssed $\pi^-$ mesons, respectively

$\pi^-$-component of the pion gas has a condensate contribution we solve Eqs. (21), (22).

For the chosen isotherm $T$, the point $n_I = n_{\rm Ic}$ divides $n_I$-axis into two pieces. When $n_I \le n_{\rm Ic}$, $\pi^-$- and $\pi^+$-mesons are in the thermal phase. While, for $n_I > n_{\rm Ic}$, $\pi^-$-mesons have condensate contribution, but $\pi^+$-mesons are still completely in the thermal phase.

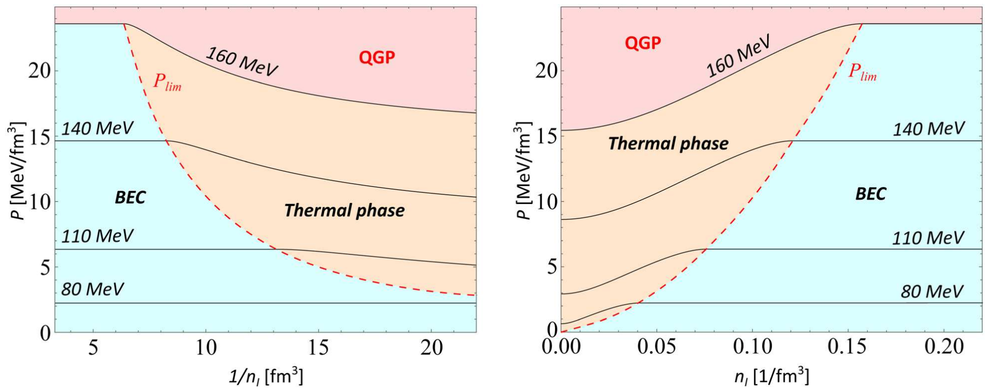
The dependence of the mean field on $n_I$ for three values of temperature, $T = 40$, 80, 100 MeV, is shown in Fig. 2 on the left panel. It is seen that the difference in the curves associated with different temperatures is very weak after the minimum of the function $U\big(n(T, n_I)\big)$. At the point $n_I = n_{\rm I0}$, where $U(n_{\rm I0}) = 0$, the mean field changes its sign and becomes completely repulsive, at $\kappa = 0.1$. We obtained $n_{\rm I0} \approx 0.14$ fm$^{-3}$.

The results of calculations of the particle-number densities $n^{(-)}$, $n^{(+)}$ and $n \equiv n_{\rm tot} = n^{(-)} + n^{(+)}$ as functions of the isospin density $n_I$ at $T = 101.7$ MeV, $\kappa = 0.1$ are depicted in Fig. 2 on the right panel.

It is seen that the density of $\pi^+$ mesons decreases for small $n_I$ with increasing $n_I$ and then becomes approximately constant. In fact, this behavior is quite understood. Indeed, in accordance with self-consistent Eqs. (17), (18) or Eqs. (21), (22) (it does not matter what pair of equations we take) we have $n^{(+)} = (n - n_I)/2$. But, as we see, the raise of the total particle density $n$ at the beginning is much lower than $n_I$. That is why, with increasing $n_I$, the particle density of $\pi^+$ mesons goes down at the beginning and then becomes approximately constant.

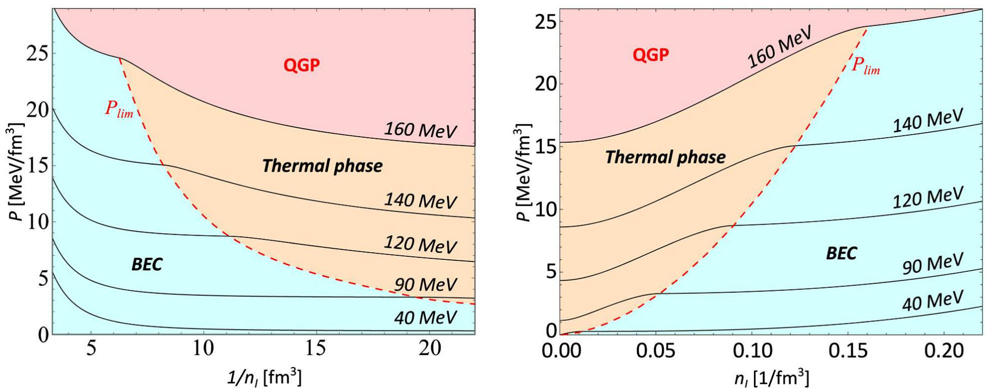
## 4. The Phase Diagram of the Boson Particle-Antiparticle System

In order to analyze the phase structure of the particle-antiparticle system, we look for the dependence of the pressure $p(T, n_I)$ on the isospin density $n_I$ at a fixed temperature $T$. That is, we study the behavior of the pressure on the isotherm. At the beginning, as the first step, we test an ideal gas of $\pi^-$-$\pi^+$ mesons as a reference point. The phase structure of the system in this case is depicted in the two upper panels in Fig. 4 for the dependencies $p = p(v_I)$ and $p = p(n_I)$, where $v_I = 1/n_I$. In this figure, the isotherm $T_{\rm qgp} = 160$ MeV separates the states of quark-gluon plasma (QGP), and the blue shaded area marked BEC represents the states of the Bose–Einstein condensate. We see that there is no liquid-gas phase transition, and the pressure of an ideal gas naturally increases with $n_I$ in the thermal phase, but becomes constant in the condensed phase. This effect arises because, in a multiparticle system without interaction, pressure exists only due to the kinetic movement of thermal particles with nonzero momentum, $\boldsymbol{k} \ne 0$. When we increase the particle density




*a* *b*

***Fig. 3.*** Phase diagrams in the ideal $\pi^-$-$\pi^+$ gas. Pressure versus inverse isospin density $v = 1/n_I$ ($a$) and versus isospin density $n_I$ ($b$). The $T = 160$ MeV isotherm is the approximate beginning of the QGP phase. The red dashed curve $p_{\rm lim}$ indicates the pressure of an ideal gas at $\mu = m$ and separates thermal phase from the BEC phase




*a* *b*

***Fig. 4.*** Phase diagrams of the $\pi^-$-$\pi^+$ system with repulsion only between particles, i.e., $\kappa = 0$ (the same notations as in the former Fig. 3)

from zero and go along a specific isotherm $T$, we come to the point $n_{\rm cd}$ on the critical curve[5]. At this point we reach the maximum density of the thermal particles. The further increase of the particle density is due only to increase of the density of condensate particles. At the same time, in the condensate phase, with an increase in $n_I$, the total density of particles in the system increases only due to an increase in the density of condensed particles with $\boldsymbol{k} = 0$, which does not contribute to the pressure.

[5] The values $T_{\rm cd}$ and $n_{\rm cd}$ corresponds to the same point on the critical curve $n_{\rm lim}(T)$. The notation $T_{\rm c}$ is reserved for the critical isotherm in the description of the liquid-gas phase transition.

On the next step, the system with only repulsion between particles, i.e., $\kappa = 0$, was examined. The phase structure of the system in this case is depicted on two lower panels in Fig. 4. In each panel, we see three different phases: 1) "Thermal phase" – particles and antiparticles are both in thermal states; 2) "BEC" – the subsystem of particles have the Bose–Einstein condensate contribution and the subsystem of antiparticles, $\pi^+$-mesons, is in thermal phase; 3) "QGP" – the phase, where the quark-gluon plasma occurs. This phase is separated by the isotherm $T = T_{\rm qgp} = 160$ MeV (we assume a melting of all pion states at temperatures $T > T_{\rm qgp}$). The line $p_{\rm lim}$ is the pressure of $\pi^-$ mesons on the critical curve $n_{\rm lim}$. As it is evidently seen in the "condensate" area

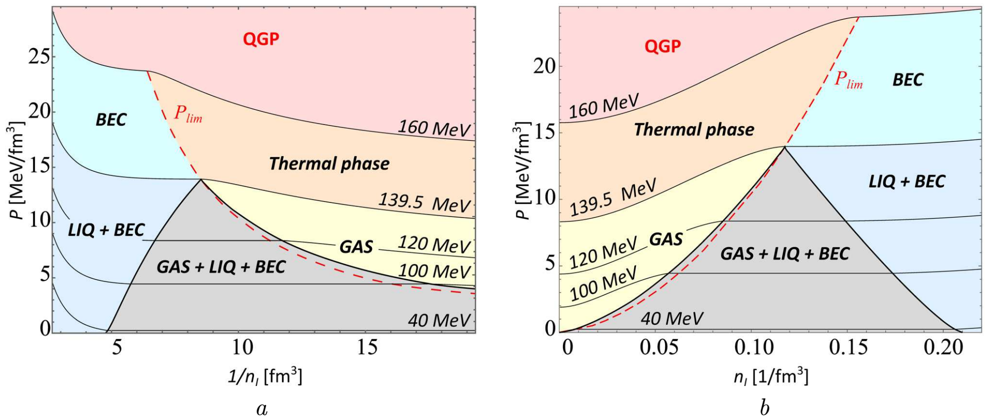



***Fig. 5.*** Phase diagrams: pressure versus inverse isospin density $v = 1/n_I$ ($a$) and versus isospin density $n_I$ ($b$) in $\pi^-$-$\pi^+$ interacting system at $\kappa = 0.2$. The isotherm $T_{\rm qgp} = 160$ MeV separates the QGP phase. The red dashed curve $p_{\rm lim}$ indicates the pressure of an ideal gas at $\mu = m$ and separates thermal phase from the BEC phase. The grey dashed "triangle" represents the mixed phase of the gas and liquid, which is almost in the $\pi^-$-meson condensate

the behavior of isotherms is different in comparison to isotherms in ideal gas: the pressure increases with the isospin density. This effect is due to the presence of positive excess pressure $P_{\rm ex}(n)$ as an additional contribution along with the kinetic pressure in the system.

If there is an attraction between particles, then the isotherms for temperatures from the interval $T < T_{\rm c}$ show a "sinusoidal" behavior in the finite interval of $n_I$. According to the standard thermodynamic approach, this specific isotherm behavior can be considered as a liquid-gas phase transition. To solve the problem, we apply Maxwell's generalized rules (see the Appendix A), which, unlike the textbook presentation of Maxwell's design, deal with the isospin (charge) density rather than the total particle density. As it follows from the generalized Maxwell's rules, the pressure associated with the isotherms, which cross the mixed liquid-gas phase, has a constant value, as well as the chemical potential. As a result, we obtain the binodal, which determines the region of the liquid-gas phase transition in a similar way as was done in Ref. [30]. The resulting phase diagram is shown in Fig. 5. The mixed liquid-gas phase (shaded gray area) appears to be almost entirely in the condensate phase (labeled GAS + LIQ + BEC). Remind that the density of condensate in the two-component pion system is created by $\pi^-$ mesons only, i.e., $n_{\rm cond} = n^{(-)}_{\rm cond}$. This means that a certain part of $\pi^-$ mesons consists of particles with $\boldsymbol{k} = 0$. At the same time, the thermal $\pi^-$ mesons together with $\pi^+$ mesons create the mixture of a gas and a liquid.

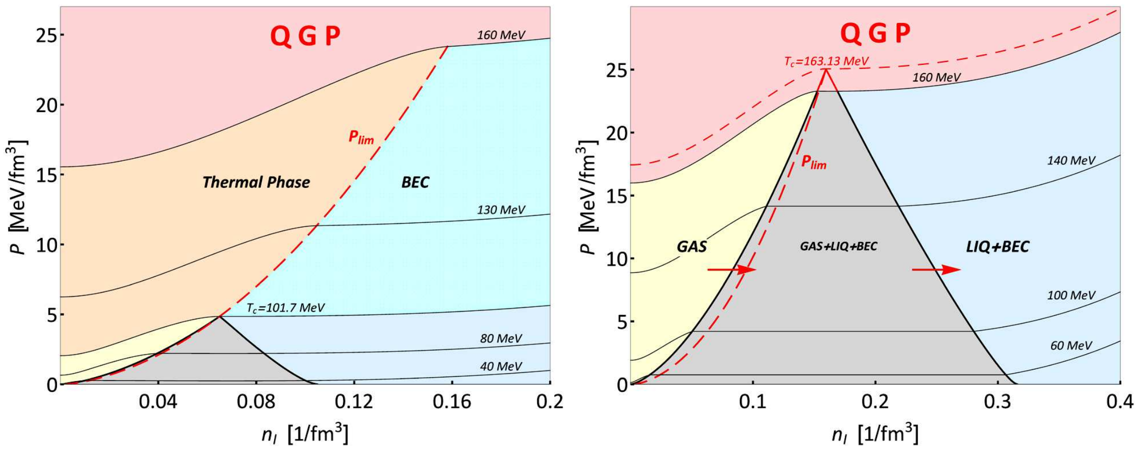
### *4.1. The liquid-gas phase transition in the quantum particle-antiparticle system of bosons*

We return to discussion of the interacting $\pi^+$-$\pi^-$ system. If we consider the quantum statistics, when dealing with the liquid-gas phase transition, then, due to the appearance of the condensate, the situation becomes different from that which was in the case of the Boltzmann statistics. It turns out that the presence of the condensate strongly affects the position of the local maximum of pressure. Indeed, this maximum is localized now on the curve $p_{\rm lim}(n_I)$, which represents the pressure in the system that is determined by the states belonged to the critical curve $n_{\rm lim}(T)$ of the $\pi^-$-meson subsystem. Remind, only this subsystem of mesons develops the Bose-condensate in the case of the weak attraction with $n^{(-)}\big|_{\rm crit.\ curve} = n_{\rm lim}(T)$. The states $(T, n_{\rm lim}(T))$ determine the total density of particles $n = 2n_{\rm lim}(T) - n_I$. Thus, the total pressure at these states reads

$$p_{\rm lim}(n_I) = p_{\rm kin}(T, n_I) + P_{\rm ex}(n), \tag{27}$$

where the kinetic pressure at the states $(T, n_{\rm lim}(T))$ looks like

$$p_{\rm kin} = \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\,\frac{\mathbf{k}^2}{\omega_k}\big[f_{\rm BE}(\omega_k, m) + {} + f_{\rm BE}\big(E(k,n), -\mu_I\big)\big] \tag{28}$$




*a* *b*

***Fig. 6.*** Phase diagrams: the pressure versus isospin density in the interacting $\pi^-$-$\pi^+$ system at $\kappa = 0.1$ ($a$) and $\kappa = 0.3$ ($b$). Below the critical isotherm $T_c = 101.7$ MeV the liquid-gas phase transition takes place. The isotherm $T = 160$ MeV is the limit of the QGP phase. The red dashed curve $p_{\lim}$ separates thermal phase from the BEC phase ($a$). The virtual critical isotherm $T_c = = 163$ MeV, calculated at $\kappa = 0.3$, lies in the QGP phase, which is bounded by the isotherm $T_c = 160$ MeV ($b$)

with $E(k,n) = \omega_k + U(n)$ and $\mu_I = U(n) + m$. The curve $p_{\lim}(n_I)$ separates the pressure that correspond to the condensate states (shaded area marked as BEC) from the pressure that correspond to the the thermal states of the boson system, see Figs. 4–6.

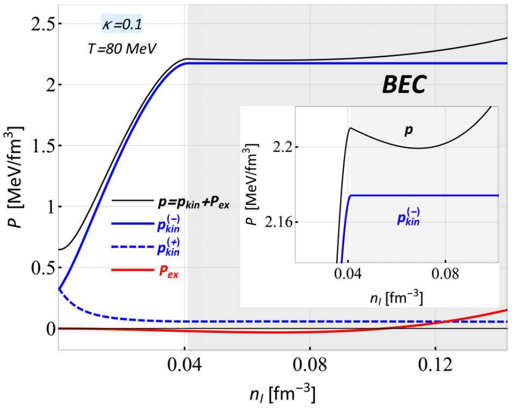
Next, we are going to prove two features that are inherent to the behavior of the pressure in the condensate phase. The first feature: the kinetic pressure along each isotherm in the condensate phase is approximately constant

$$p_{\text{kin}}(T, n_I)\big|_{T=\text{const}} \approx \text{const}. \tag{29}$$

a) This will be an exact equality in the absence of interaction between particles, when $p = p_{\text{kin}} + P_{\text{ex}}$ reduces to $p = p_{\text{kin}}(T, n_I)$. The pressure of the two-component ideal gas is depicted in Fig. 3. We see the constant pressure in the condensate phase (shaded blue area). The effect, which is indicated in Eq. (29), arises, because the increase of the variable $n_I$ in the condensate phase occurs only due to the increase of condensed particles, whereas the density of thermal particles remains constant along isotherm. But, the increase of the number of particles in the system with zero momentum $\boldsymbol{k} = 0$ do not contribute to the kinetic pressure. Here, we discuss the pressure of $\pi^-$ mesons, which develop the condensate states. The partial pressure of $\pi^+$ mesons on the same isotherm $T$ is calculated for $\mu = m$, and it is created only by thermal particles, whose density is also constant in the condensate phase. By this, we prove the rigorous validity of Eq. (29) in the ideal $\pi^-$ - $\pi^+$ meson system.

b) In the system with interaction, equality (29) is approximately valid. As it is seen from Eq. (28), the first contribution of pressure, i.e., the kinetic partial pressure of $\pi^-$ mesons, which contributes 98% to $p_{\text{kin}}$, is still constant, when we increase $n_I$, because $U(n) - \mu_I = -m$ in the condensate phase. The kinetic partial pressure of $\pi^+$ mesons, which are in the thermal phase, is suppressed, because the distribution function looks like $f_{\text{BE}} = 1/\{\exp[(\omega_k + 2U(n) + m)/T] - 1\}$, and the contribution of $\pi^+$ mesons to the kinetic pressure is not more than 2 %. Hence, we can adopt that, in the condensate phase, the kinetic pressure in $\pi^-$-$\pi^+$ meson system is constant with a good accuracy, and Eq. (29) is approximately valid.

The second feature: The pressure has the following structure, $p = p_{\text{kin}} + P_{\text{ex}}$. It is obvious that the kinetic pressure is always positive, $p_{\text{kin}} > 0$, and the excess pressure is not, its sign depends primarily on the density $n_I$. As can be seen in Fig. 2 on the left panel, the excess pressure $P_{\text{ex}}$ is negative ($U(n)$, and $P_{\text{ex}}$ have the same sign). Due to this, with increasing $n_I$, the pressure on each isotherm $T < T_c$ begins to go down in the condensate phase after crossing the line $p_{\lim}(n_I)$. This decreasing of the pressure is going on up to the point of the local minimum, as it is shown in Fig. 7, *a*. In this figure, we present the re-

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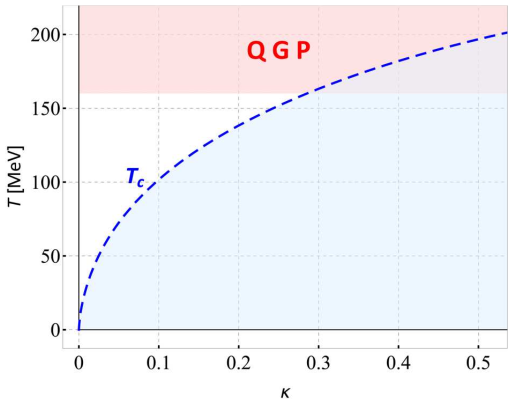


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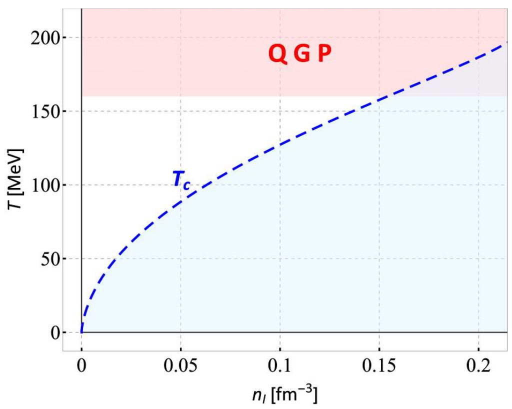


*c*

***Fig. 7.*** Pressure contributions vs isospin density in the $\pi^-$-$\pi^+$ interacting system at $T = 80$ MeV and $\kappa = 0.1$. A local maximum at the edge of the condensate region and a local minimum that is due to metastable and unphysical pressure segments are shown in the small window. Shaded area indicates the Bose–Einstein condensate (BEC) states ($a$). Critical temperature $T_{\rm c}$ of the liquid-gas phase transition vs attraction parameter $\kappa$ ($b$). Critical temperature $T_{\rm c}$ of the liquid-gas phase transition vs isospin critical density $n_{\rm Ic}$ ($c$)

sults of calculation of the kinetic pressure $p^{(-)}_{\rm kin}$ of the $\pi^{(-)}$ mesons, $p^{(+)}_{\rm kin}$ of the $\pi^{(+)}$ mesons, and the excess pressure $P_{\rm ex}$.

The liquid-gas phase transition occurs, when the pressure in the system possesses firstly a local maximum and then a local minimum, when the isospin density $n_I$ increases. Let us consider the structure of the pressure in the condensate region. For the derivative of the total pressure after some algebra (see Appendix B), we get

$$\frac{\partial p(T,n_I)}{\partial n_I} = \left(1 - \frac{2n^{(+)}}{n}\right) \frac{\partial P_{\rm ex}(n)}{\partial n} \frac{\partial n}{\partial n_I} = 0. \qquad (30)$$

Here, we use $\partial p^{(-)}_{\rm kin}(T,n_I)/\partial n_I = 0$, For a positive value of the bracket $(1 - 2n^{(+)}/n) > 0$, where $n = 2n^{(+)} + n_I$, Eq. (30) leads to

$$\frac{\partial P_{\rm ex}(n)}{\partial n} = 0 \to n(-A + 2B\, n) = 0, \qquad (31)$$

where, for the second equation, we use the explicit form of $P_{\rm ex}$ given in Eq. (15).

Therefore, the minimum of the pressure is in the point

$$n^{(\rm min)} = \frac{A}{2B} = \kappa \sqrt{\frac{m}{B}}, \qquad (32)$$

where $A = 2\kappa\sqrt{mB}$. Indeed, it is a minimum, because the sign of the second derivative at this point is positive, $\frac{\partial^2 P_{\rm ex}(n)}{\partial n^2} = A > 0$. For the temperature $T = T_{\rm c}$, i.e., on the critical isotherm, this total particle density obtained in Eq. (32), or the point $\left(T_{\rm c}, n^{(\rm min)}\right)$, determines the critical point. Actually, one can make transform: $\left(T_{\rm c}, n^{(\rm min)}\right) \to \left(T_{\rm c}, n_I^{(\rm min)}\right)$, thus, it will be determined a critical point in $(T, n_I)$-plane.

The critical temperature can be found as a solution of the equation, when $p_{\rm max}$ coincides with $p_{\rm min}$

$$p_{\rm max}(T_{\rm c}) = p_{\rm min}(T_{\rm c}). \qquad (33)$$

However, in the case of a presence of the condensate, it is not possible to determine the maximum of the pressure using the equation $\partial p(T,n_I)/\partial n_I = 0$, because the isotherm is not a smooth function on the edge of the condensate. In accordance with the textbook procedure: to find the maximum and minimum of the smooth function in the region with the edges, one has to compare the values of the pressure given by solutions of Eq. (30) with the values of the pressure on the edges of the region. As we argued above, Eq. (30) determines only a local minimum of the pressure, whereas a local maximum is on the edge of the condensate region that is to the left of a local minimum, see Fig. 7, *a*. In the plots of the phase diagrams, see Fig. 6, the edge between the thermal phase and the condensate phase is indicated as the pressure $p_{\rm lim}(n_I)$, which corresponds to the states $(T, n_{\rm lim}(T))$ on the critical curve $n_{\rm lim}(T)$, see the red dashed line in Fig. 1. Hence, the local maximum of every isotherm $T \le T_{\rm c}$ belongs to the curve $p_{\rm lim}(n_I)$.

We are going now to discuss the algorithm for calculation of the critical temperature $T_{\rm c}$. The total density of particles in the particle-antiparticle system reads

$$n(T) = n^{(-)}(T) + n^{(+)}(T). \tag{34}$$

For the states on the critical curve, where the condensate disappears, for any temperature, one has $n^{(-)}(T) = n_{\rm lim}(T)$. Because the density of $\pi^{(-)}$ mesons at the critical temperature $T_{\rm c}$ belongs also to the critical curve, we get $n^{(-)}(T_{\rm c}) = n_{\rm lim}(T_{\rm c})$. Hence, at the critical temperature, one can rewrite Eq. (34) as

$$n(T_{\rm c}) = n_{\rm lim}(T_{\rm c}) + n^{(+)}_{\rm th}(T_{\rm c}), \tag{35}$$

where $n^{(+)}_{\rm th}$ is the density of the thermal $\pi^+$ mesons. As we argued, the density $n(T_{\rm c})$, determined in Eq. (35), corresponds to the local maximum of the isotherm pressure. In the liquid-gas phase transition the critical isotherm crosses the point, where the local maximum coincides with the local minimum of the pressure, that is, the isotherm temperature equals to the critical temperature $T_{\rm c}$. Thus, if, in the place of $n(T_{\rm c})$ in Eq. (35), we put the density $n^{(\rm min)}$ from Eq. (32), we come to equation that determines $T_{\rm c}$:

$$n_{\rm lim}(T_{\rm c}) + n^{(+)}_{\rm th}(T_{\rm c}) = n^{(\rm min)}. \tag{36}$$

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\exp\left[(\omega_k - m)/T_{\rm c}\right] - 1} + \\ + \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\exp\left\{\left[\omega_k + 2U(n^{(\rm min)}) + m\right]/T_{\rm c}\right\} - 1} = \\ = n^{(\rm min)}, \tag{37}$$

where $n^{(\rm min)} = A/2B = \kappa\sqrt{m/B}$, and we use the definition of the $n_{\rm lim}$ given in Eq. (20). Solution of this equation determines the curve $T_{\rm c}(\kappa)$, which is depicted in Fig. 7, *b*. We account for that the second integral of Eq. (37) that represents $n^{(+)}_{\rm th}$ is small in comparison to the first term. Therefore, we neglects it.

### *4.2. Discussion*

We can compare the characteristics of the liquid-gas phase transition in the classical and quantum interacting boson systems. The interaction in both systems was studied in the framework of the mean-field model with the mean field $U(n) = -An + Bn^2$, at the same values of attractive coefficient $A$ and repulsive coefficient $B$. As a result, we obtain that the presence of the condensate, which is due to the quantum Bose statistics, sufficiently increases the value of the critical temperature of the liquid-gas phase transition. Indeed, at the attraction parameter $\kappa = 0.2$ for the classical gas, we get the critical temperature $T_{\rm c} = 2.8$ MeV ($T_{\rm c} = A^2/8B$) and, for the quantum system, we get $T_{\rm c} = 139$ MeV. Let us discuss this phenomena.

It is necessary to point out that the quantum bosonic system possesses the increasing of $T_{\rm c}$ with an increase of the attractive parameter $A$. In fact, for $\kappa = 0.1$ it was obtained $T_{\rm c} = 101$ MeV and for $\kappa = 0.3$ it was obtained $T_{\rm c} = 163$ MeV, see (compare) left and right panels in Fig. 6. The increase in $T_{\rm c}$ is caused primarily by the peculiarities of the pressure behavior in the condensate phase. An additional contribution is made by "turning on" quantum statistics in the boson system.

Indeed, in a quantum system, the Bose-statistics "generates" an effective attraction between particles, whereas the Fermi statistics "generates" an effective repulsion. Hence, a "switching on" of the Bose-statistics in the system results in the effective increasing of the attraction between particles or leads to an effective increasing of the attraction coefficient $A$ in the mean field $U(n)$. Indeed, the statistically induced interaction potential $V(r)$ for the first order quantum correction reads

$$V(r) = -T \ln\left[1 \mp e^{-2\pi r/\Lambda^2}\right], \tag{38}$$

where $r$ is the distance between particles, $\Lambda = \sqrt{2\pi/mT}$ is the thermal wave length, and the upper sign corresponds to the Fermi statistics, whereas, the lower sign to the Bose statistics. Evidently, this two-particle effective potential possesses the attractive behavior in the case of the Bose statistics and the repulsive behavior in the case of the Fermi statistics (the correction is valid, when $r \gg \Lambda$). Hence, the transit from the classical gas to the quantum one is twofold. First, the effective attraction between particles goes up, and, second, the condensate that is presented now in the system determines the increase of the critical temperature. We emphasize that a presence of the condensate that is due to the quantum statistics and the interplay of attraction and repulsion between particles make this "jump up" of the critical temperature. The main reason for this is that, the kinetic pressure in the condensate is almost constant.

At the same time, we can name the opposite effect which is due to the Fermi statistics. As was shown in Ref. [31], the quantum van der Waals equation that was applied to the description of the nuclear matter gives, for the liquid-gas phase transition, the critical temperature $T_c = 19.7$ MeV. This value of the critical temperature is close to the experimental estimates given in Refs. [32, 33]. On the other hand, the classical van der Waals equation gives, for the liquid-gas phase transition, the value $T_c = 29$ MeV. As we have discussed in this section and in view of Eq. (38), the Fermi statistics effectively amplifies the repulsion between particles what means an effective decreasing of the attraction between particles. This quantum-statistical effect of the decreasing of the attraction in the system can be explanation of the decreasing of the critical temperature from the value $T_c = 29$ MeV to the value $T_c = 19.7$ MeV when moving from the classical description of the liquid-gas phase transition in the nuclear matter to the quantum-statistical one [31].

## 5. Discussion and Concluding Remarks

The article presents a thermodynamically consistent approach for description of the Bose–Einstein and liquid-gas phase transitions in a dense self-interacting bosonic system at a conserved isospin (charge) density $n_I$. As an example, we have considered the system of meson particles with $m = m_\pi$ and zero spin; we name these bosonic particles as "pions" just conventionally. This choice was made, because the charged $\pi$-mesons are the lightest nuclear particle and the lightest hadrons that couple to the isospin number. For the same reason, a "temperature creation" of the particle-antiparticle pairs in the temperature interval $T \leq 200$ MeV becomes a common problem for the quantum-statistical methods. Description of thermodynamic properties of the system was carried out using the Canonical Ensemble formulation, where the chemical potential $\mu_I$ is a thermodynamic quantity which depends on the canonical variables $(T, n_I)$. To obtain phase diagram, which reflects the liquid-gas phase transition, we calculated the dependence of the pressure with respect to the isospin density for different isotherms. Then we modified the pressure dependence in accordance with the generalized Maxwell rules (see Appendix).

It should be noted that the electrical charge of the condensate is negative in the case where the total charge of the system is negative. Vice versa, the electric charge of the condensate would be positive, if the total charge of the bosonic system is positive. It is shown that, at a fixed temperature, the dependencies of the particle densities $n^{(-)}(T, n_I)$, and $n_{\rm tot}(T, n_I)$ with respect to $n_I$ are almost linear and close to one another for $n_I > n_{Ic}$. This happens, since, for every fixed $T$, the value of particle number density $n_{\rm th}^{(-)}$ of thermal (kinetic) $\pi^-$-mesons does not change [19], and the value of $n^{(+)}$ is small and approximately constant (see Fig. 2, *b*). Because only $\pi^-$ mesons undergo the phase transition to the Bose–Einstein condensate, the increase of the densities $n^{(-)}$ and $n = n_{\rm tot}$ for $n_I > n_{Ic}$ is almost due to an increase of the density of condensate.

Phase diagrams were introduced in Fig. 4. The scale parameter of the model $\kappa = A/(2\sqrt{mB})$, which is itself a combination of the mean-field parameters $A$ and $B$ ($U(n) = -An + Bn^2$) and the particle mass, determines the different possible phase scenarios which occur in the particle-antiparticle boson system. However, when the attraction coefficient $A = \kappa A_c$ is zero (i.e., $\kappa = 0$), the system can be in the thermal or condensate phase, but cannot develop a liquid-gas phase transition.

In the case of $\kappa > 0$, the liquid-gas phase transition occurs in the system, and a transition from the thermal phase to the condensate one is possible both with the liquid-gas phase transition, if $T < T_c$, and without it when $T \geq T_c$. In other words, there is a region in the phase diagram, where the BEC and the mixed liquid-gas phase exist simultaneously (grey area on the left panel in Fig. 6). A similar situation is described in [30], where the Bose–Einstein condensation and the liquid-gas phase transition in $\alpha$-matter were investigated. The area above the isotherm $T = T_{\rm qgp} = 160$ MeV (QGP) is the phase, where the quark-gluon plasma occurs. We assume this to be a limitation of our model, since a melting of all pion states at temperatures higher than $T_{\rm qgp}$.

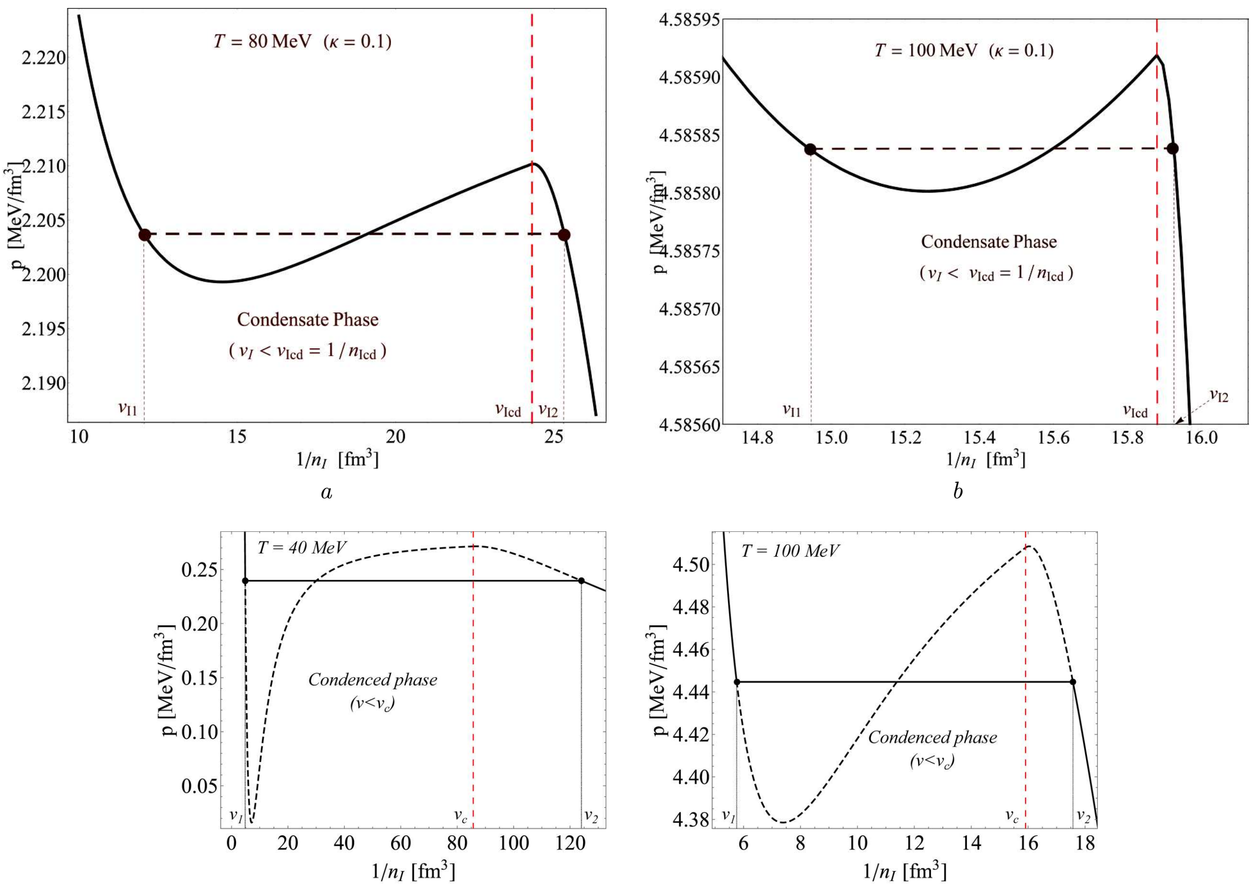


***Fig. 8.*** Pressure versus inverse isospin density $1/n_I$ in the interacting $\pi^+$-$\pi^-$ pion system in the framework of the mean-field model: in two top panels at $\kappa = 0.1$ ($a$ and $b$) and in two bottom panels at $\kappa = 0.2$ ($c$ and $d$). Onset of the liquid-gas phase transition is determined under the Maxwell rule

The role of neutral pions is left beyond the scope of the present paper. The present analysis can be improved by addressing these issues in more detail and also by generalizing the calculation to nonzero contribution to the mean field which depends on $n_I$. Authors plan to consider these problems elsewhere.

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## APPENDIX A. Liquid-Gas Phase Transition: the Maxwell Rules in the System with Conservation of the Charge

To determine the points $v = 1/n_I$ which correspond to the Maxwell rule one needs to solve the system of two equations with respect to the points $v_{x1} = 1/n_{I2}$ and $v_{x2} = 1/n_{I1}$ (note, we obtain the pressure as function of $n_I$, i.e. $p(n_I)$). In case of the homogeneous system for isothermal process $T = \text{const}$ we have

$$dp = sdT + n_I d\mu_I \to dp = n_I d\mu_I. \tag{A1}$$

Then (1):

$$d\mu_I = 0 \to dp = 0 \to \int_1^2 dp = 0 \to p_1 = p_2. \tag{A2}$$

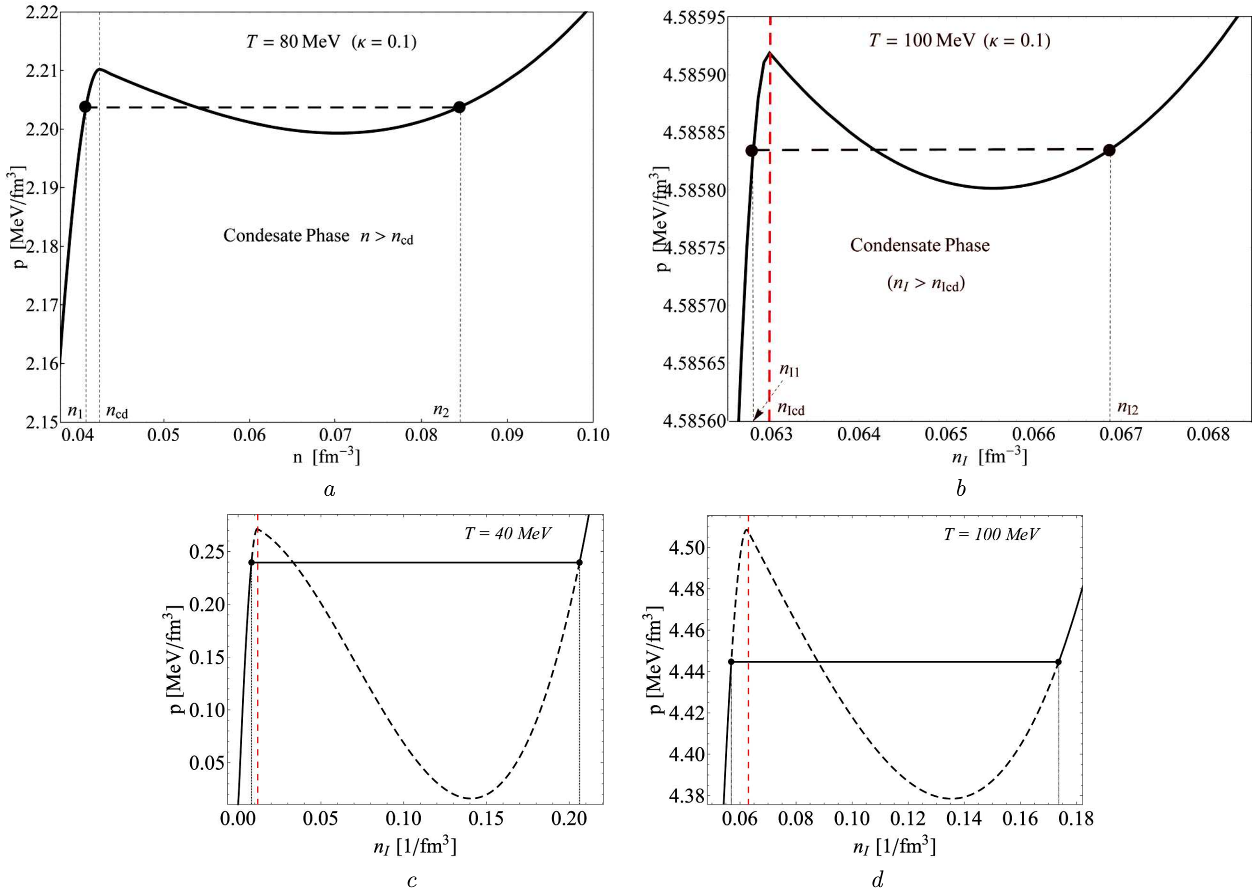


***Fig. 9.*** Pressure versus isospin density $n_I$ in the interacting $\pi^+$-$\pi^-$ pion system in the framework of the mean-field model: in two top panels at $\kappa = 0.1$ ($a$ and $b$) and in two bottom panels at $\kappa = 0.2$ ($c$ and $d$)

Then (2):

$$\frac{1}{n_I}\,dp = d\mu_I, \quad d\mu_I = 0 \to \int\limits_1^2 \frac{1}{n_I}\,dp = 0. \tag{A3}$$

We write these two equations (A2), (A3) as a set with respect to $v_1$ and $v_2$,

$$p\,(1/v_1) = p\,(1/v_2), \tag{A4}$$

$$\int\limits_{v_1}^{v_2} dv\,p\,(1/v) = (v_2 - v_1)\,p\,(1/v_1), \tag{A5}$$

where last equation can be read as

$$\int\limits_{v_1}^{v_2} dv\,\left[p\,(1/v) - p\,(1/v_1)\right] = 0. \tag{A6}$$

Then, after we get solution one can determine the values $n_{I1} = 1/v_2$ and $n_{I2} = 1/v_1$.

The graphical examples of the application of this algorithm are depicted in Figs. 8–9 for the set of temperatures $T = 40, 80, 100$ MeV and two variations of the parameter $\kappa = 0.1$ and $\kappa = 0.2$.

## APPENDIX B. Derivative of the Pressure in the Condensate Phase

Here we present the details of calculation of the derivative of the pressure in the condensate phase. As was mentioned before, for the pressure in the interacting system one has $p = p_{\rm kin} + P_{\rm ex}$. For the derivative of the kinetic contribution we get

$$\frac{\partial p_{\rm kin}(T,n_I)}{\partial n_I} = \frac{\partial p^{(-)}_{\rm kin}(T,n_I)}{\partial n_I} + \frac{\partial p^{(+)}_{\rm kin}(T,n_I)}{\partial n_I}, \quad \frac{\partial p^{(-)}_{\rm kin}(T,n_I)}{\partial n_I} = 0. \tag{B1}$$

From Eq. (10) it follows

$$p^{(+)}_{\rm kin}(T,n_I) = -T\int \frac{d^3k}{(2\pi)^3}\times \times \ln\left[1 - \exp\left(-\frac{\sqrt{m^2+\mathbf{k}^2} + 2U(n) + m}{T}\right)\right], \tag{B2}$$

We calculate derivative of this partial pressure:

$$\frac{\partial p^{(+)}_{\rm kin}(T,n_I)}{\partial n_I} =$$
$$= -\int \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left(\frac{\sqrt{m^2+\mathbf{k}^2+2U(n)}+m}{T}\right)-1} \times$$
$$\times\, 2\frac{\partial U(n)}{\partial n}\frac{\partial n}{\partial n_I}. \tag{B3}$$

Hence, we get

$$\frac{\partial p_{\rm kin}(T,n_I)}{\partial n_I} = -2n^{(+)}(T,n_I)\frac{\partial U(n)}{\partial n}\frac{\partial n}{\partial n_I}. \tag{B4}$$

Now using Eq. (7) one can write

$$\frac{\partial p(T,n_I)}{\partial n_I} = -2n^{(+)}(T,n_I)\frac{\partial U(n)}{\partial n}\frac{\partial n}{\partial n_I} + \frac{\partial P_{\rm ex}(n)}{\partial n}\frac{\partial n}{\partial n_I} =$$
$$= \left(1 - \frac{2n^{(+)}(T,n_I)}{n(T,n_I)}\right)\frac{\partial P_{\rm ex}(n)}{\partial n}\frac{\partial n}{\partial n_I} = 0. \tag{B5}$$

With account for $n = 2n^{(+)}(T,n_I) + n_I$ one obtains that $2n^{(+)}/n < 1$ for finite isospin densities, $n_I > 0$. Thus, for a positive value of the bracket $(1 - 2n^{(+)}/n) > 0$, equation (B5) leads to

$$\frac{\partial P_{\rm ex}(n)}{\partial n} = 0. \tag{B6}$$


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ФАЗОВІ ДІАГРАМИ РЕЛЯТИВІСТСЬКОЇ САМОВЗАЄМОДІЮЧОЇ БОЗОННОЇ СИСТЕМИ

У рамках формалізму канонічного ансамблю та моделі середнього поля досліджується система взаємодіючих релятивістських бозонів при скінченних температурах і скінченних густинах ізоспіну. Середнє поле містить як притягальну, так і відштовхувальну складові. Отримано залежності термодинамічних величин від температури та густини ізоспіну. Показано, що у разі наявності притягання між частинками в такій бозонній системі, на фоні бозе-айнштайнівської конденсації додатково виникає фазовий перехід рідина–газ. Наведено відповідні фазові діаграми. Пояснено причини, чому наявність бозе-конденсату значно підвищує критичну температуру фазового переходу рідина–газ у порівнянні з температурою, отриманою для тієї ж системи в рамках статистики Больцмана. Отримані результати можуть застосовуватися при інтерпретації експериментальних даних, зокрема у питанні, наскільки критична точка змішаної фази чутлива до присутності конденсату Бозе–Айнштайна.

*Ключові слова*: релятивістська бозонна система, конденсація Бозе–Айнштайна, фазовий перехід.