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FLUCTUATIONS AND POWER LAW DISTRIBUTION FUNCTION IN NONEQUILIBRIUM SYSTEMS

The Fokker–Planck equation is formulated for the distribution functions of macroscopic open systems in the space of slowly changing physical variables (energy, adiabatic invariants, etc.). The stationary solution of such equations determines a quasi-equilibrium distribution function in the relevant space. The proposed approach involves the evolution of systems under the action of dissipation and diffusion in the space of the appropriate variables. It is shown that the well-known power law distribution can be obtained by considering internal and external fluctuations in statistical systems.

Keywords: Fokker–Planck equation, power law distribution function, fluctuations in statistical systems, Langevin equations.

1. Introduction

According to the basic principles of thermodynamics, a macroscopic system in contact with the environment reaches the state of equilibrium during the relaxation time. The relaxation time is determined by the physical nature of the system under consideration and the properties of the external environment (see, for example, [1–3]). Establishing the equilibrium in a system interacting with the thermostat leads both to the establishment of the thermodynamic parameters, equivalent to the parameters of the heat bath, and to the absence of flows in the equilibrium system. In the case of nonequilibrium open systems, flows are present, but steady states may exist. Such states can be interpreted as "quasi-equilibrium" in the sense that they do not change with time, but the thermodynamic parameters of the system and the environment are different. Nonequilibrium steady states are observed in a system of hot electrons in semiconductors [4], a system of photons with inhomogeneous scattering, where the diffraction coefficient depends on the frequency of photons [5, 6], a system of high-energy particles generated in the collisions of heavy ions in accelerators, and systems of dust particles in plasmas [7–9]. The distribution functions of such systems usually are different from the known equilibrium distributions.

An example of such a stationary distribution may be, in particular, a distribution with "heavy tails" for large values of the variable (power law distribution) that is extremely important from both theoretical and practical points of view [10]. Many power distributions have been found in physics, biology, and social sciences. The main feature is their qualitative and quantitative difference from the normal (Gaussian) distribution.

In physics, all distribution functions depend on the energy that determines a hypersurface in the phase

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space and completely determines all probable states of the system. The distribution function in the equilibrium case depends only on the energy of the system, and we can assume that the nonequilibrium distribution function also can be described in terms of the energy of the macroscopic system. For physical systems, energy is the slowest variable or control parameter. In economic systems, such a parameter may be the amount of money involved in financial transactions. For linguistics, this is a set of words that are used in this or that language. Therefore, in the further consideration we will analyze the changes of the slowest control parameter of the system.

Any system is usually in a nonequilibrium state *a priori*. Nevertheless, we may assume that, in some cases, the state of a subsystem of the large system is described by a local Gaussian distribution that evolves to provide the smallest energy and the largest entropy of that subsystem. On the other hand, in the general case, the evolution of a nonequilibrium system can be described on the basis of the Fokker–Planck equation, where the diffusion and dissipation coefficients are related to the corresponding Langevin equations [11–18]. The advantage of the Fokker–Planck equation is that it accounts for both the relaxation of the distribution function due to dissipation and the random effect of fluctuations of various nature (these can be both internal fluctuations and fluctuations of the parameters of the external environment). Thus, using the Fokker–Planck equation, it is possible to find probable stationary distributions generated, in particular, by multiplicative noise, which arisee as a result of the nonlinear interaction of internal and external fluctuations. In addition, stationary solutions of the Fokker–Planck equations under certain conditions indicate the probable existence of a phase transition to a state that can be associated with the stationary state of the system. We suggest that multiplicative fluctuations can lead to a power-law stationary distribution, if we properly account for them.

The purpose of this work is to substantiate the occurrence of a power law distribution in the space of various control parameters and to find the conditions for the formation of such a distribution. The solution of the Fokker–Planck equation for the distribution function of the macroscopic system, that describes the probable quasi-stationary states of the system is obtained. The proposed approach describes the evolution of the system between different states with regard for the dissipation and the external influence of the environment. For economic systems and systems of other nature, where the slow variable may be another quantity (such as money) [10], it has been shown that this approach can yield a known power law distribution.

The paper is organized as follows. A general approach to the derivation of the Fokker–Planck equation from the master equation is presented in Section 2. In Section 3, we formulate the specific Langevin equation that leads to the power law distribution function. Analysis of the application of the Fokker–Planck equation in the energy representation is given in Section 4. The results obtained are formulated in the Conclusions (Section 5). An illustration of the description of the known results (ordinary Brownian motion) is given in the Appendix.

2. Master Kinetic Equation

As is well-known, the macroscopic state of a statistical system can be described in terms of distribution functions that determine all macroscopic properties of the system under consideration [2, 11]. Contrary to the microscopic functions, such distribution functions depend only on a few macroscopic parameters. To describe the evolution in terms of distribution functions, these parameters, as control parameters of nonequilibrium systems, have to be "slow variables". In the absence of any other knowledge about the nonequilibrium systems, there is no reason to favor any definite state of the system. Actually, the current state of the system is determined by the control parameters. The nonequilibrium distribution function $\rho(\varepsilon, t)$ includes the dependence on the control parameter of the system ε and time. The distribution function, in general case, may be obtained from the master kinetic equation which describes the system evolution during a long period of time and takes the fast random processes in it into account. In terms of an arbitrary control parameter, the basic kinetic equation for the nonequilibrium distribution function may be written as [11]

$$\frac{\partial\rho(\varepsilon,t)}{\partial t} = \int \Big\{W(\varepsilon,\varepsilon')\rho(\varepsilon',t) - \\ - W(\varepsilon',\varepsilon)\rho(\varepsilon,t)\Big\}d\varepsilon', \qquad (1)$$

where $W(\varepsilon|\varepsilon')$ is the probability of a transition between different values of the control parameter of the system in the unit of time. This basic kinetic equation is the balance equation for the probabilities of states. All solutions of the basic kinetic equation for $t \to \infty$ have fundamental properties: they relax to stationary solutions that may be interpreted as "quasi-equilibrium" states for these systems. In general, such an evolution satisfies the law of entropy growth in the sense of the general concept [18]. However, the basic kinetic equation is nonlinear, and the transition probabilities themselves depend on the distribution function. The transition to a linear equation may be performed only for the Brownian motion, when there is no interaction between individual Brownian systems. The form of the Fokker–Planck equation for the Brownian system depends on the path of decomposition of the transition probabilities with the change $\Delta\varepsilon = \varepsilon - \varepsilon'$.

Regardless of the process leading to the changes of the distribution function on the control parameter, the Fokker-Planck equation may be written in the general form given by [11–18]

$$\frac{\partial \rho(\varepsilon,t)}{\partial t} = \frac{\partial}{\partial \varepsilon}\Big(A(\varepsilon)\rho(\varepsilon,t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2}\Big(D(\varepsilon)\rho(\varepsilon,t)\Big). \quad (2)$$

The coefficients $A(\varepsilon)$ and $D(\varepsilon)$ depend on the control parameters, and their physical sense is determined by the nature of the processes under consideration. The explicit form of the Fokker–Planck coefficients is of no particular importance, since, in different presentations, there is a direct relationship between them [11–18]. To choose one of them, it is necessary to involve physical arguments. In particular, the Fokker–Planck equation can be obtained from the chain of Bogolyubov equations in the approximation of the weak interaction (a small parameter at the interaction potential) or the smallness of the masses of a liquid or gas molecule to the mass of an impurity particle [19].

The sense of the coefficients may be clarified using the dynamic equation for the control parameter. In the general case [14, 18], one can suppose that the stochastic dissipation equation can be written as

$$\frac{d\varepsilon}{dt} = f(\varepsilon) + \sqrt{D(\varepsilon)}L(t). \quad (3)$$

The solution of this stochastic dissipation equation depends on the external influences and initial conditions. The external influence is manifested in the change of the control parameter of the system under the random influence of the external environment acting together with the dissipation. The dissipation is described by the first part of the above equation. This part may be obtained from the dynamic equation of the macroscopic system with the known interaction of the system with the environment.

The random influence of the environment is taken into account by the second part of the equation. Usually, it is assumed that the fluctuation sources are not correlated, and the correlation between two fluctuation values at two different times $\langle L(t)L(t')\rangle = \phi(t-t')$ may not be zero only during the time interval that is equal to the interaction time. The symbol $\langle ...\rangle$ implies the statistical averaging of the corresponding value. The function $\phi(t-t')$ should have a sharp peak around zero and satisfy the condition $\int \phi(\tau)d\tau = \sigma^2$ for the white noise [18]. However, the properties of the environment may also be changed randomly. As a result, a system that cannot reach equilibrium after rapid environmental changes should relax to a new state. This process indicates degradation of the system upon the contact with the environment. Since the general stochastic equation has the same form as that for an ordinary Brownian particle, henceforth we will call such systems Brownian.

A true nonlinear Langevin equation should have an equivalent equation for the probability distribution function that can be written with regard for the specific properties of the statistical system. Today, we know two different approaches for the consideration of stochastic processes. If the diffusion coefficient $D(\varepsilon)$ depends on the control parameter at the initial point, then the equation for the nonequilibrium distribution function may be obtained in the Ito form. If this coefficient depends on the control parameter before and after the transition, the diffusion equation can be written in the Stratonovich form. If we use the breakdown of the symmetric and asymmetric parts of the transition probability, we obtain the kinetic form of the representation of the Fokker–Planck equation. They are all related to the representation of the "scattering" and diffusion forces, i.e.,

$$A(\varepsilon) = f(\varepsilon) + \nu\,\frac{dD(\varepsilon)}{d\varepsilon}, \quad (4)$$

where the coefficient $\nu = 1, 1/2, 0$ in the Ito, Stratonovich, and kinetic representations, respectively. The explicit form of the coefficients through the microscopic parameters is defined as [11–18]

$$f(\varepsilon) = \int \Delta\varepsilon W(\Delta\varepsilon) d\Delta\varepsilon,$$
$$D(\varepsilon) = \int (\Delta\varepsilon)^2 W(\Delta\varepsilon) d\Delta\varepsilon. \tag{5}$$

The equation for the nonequilibrium distribution function in this case may be rewritten in the form of the local conservation law for the probability, i.e.,

$$\frac{\partial\rho(E,t)}{\partial t} = \frac{\partial J(\rho(\varepsilon,t))}{\partial\varepsilon}, \tag{6}$$

where

$$J = A(\varepsilon)\rho(\varepsilon,t) + \frac{\partial}{\partial\varepsilon}D(\varepsilon)\rho(\varepsilon,t). \tag{7}$$

The stationary solution of the Fokker–Planck equation for $J(\rho(\varepsilon,t)) = 0$ is given by

$$\rho_s(\varepsilon) = \frac{C}{D^\nu(\varepsilon)} \exp\left(-\int\limits_0^\varepsilon \frac{f(\varepsilon')d\varepsilon'}{D(\varepsilon')}\right). \tag{8}$$

This distribution function has the extreme value for the control parameter satisfying the equation

$$\nu D'(\tilde{\varepsilon}) = f(\tilde{\varepsilon}), \tag{9}$$

where $'$ stands for the control parameter derivative. Equation (8) establishes the relation between the system dissipation and diffusion in the stationary case and completely determines a new "quasi-equilibrium" state of the system [15, 16].

3. Power Law Distribution Function

The power law stationary distribution of the control parameter has been observed for many systems of different types. Many articles and reviews [10] are related to this important problem. We would like to draw attention to one of the general probable reasons for the appearance of such a stationary distribution both in the general approach and for specific physical examples. First of all, we will focus on the stationary distribution function (8). It is obvious that, in the presence of a relation between the drift of the system under the influence of external factors and the diffusion coefficient $f(\varepsilon) \sim D'(\varepsilon)$, it is always possible to obtain a power law dependence on the control parameter. Practically, this means that there is a relationship between the average value of the drift of the system, when the control parameter changes, and the diffusion in the space of this control parameter. This leads to the fact that the relation between the dispersion of the system and the fluctuations is present, and it provides a steady state with the stationary distribution function in the power form $\rho_s = D^\mu(\varepsilon)$, where the degree μ will be determined by the ratio between the degradation coefficient and the fluctuation correlations. This makes it possible to conclude that the Pareto-type distribution function arises under the influence of a multiplicative noise in the system, namely, due to fluctuations in the parameters of the dissipation process.

Let us consider a simple case concerning the evolution of the system in the inhomogeneous dissipation environment. In this case, the characteristics of the system may be considered in terms of a different value of the dissipation coefficient that depends on the control parameter as a random quantity. In this case, we can use the general results [14] where the present approach is used to describe the noise induced phase transition. In the case under consideration the Langevin equation for slow control parameter can be written in the form

$$\frac{d\varepsilon}{dt} = -\gamma_t\varepsilon,$$

where $\gamma_t = \gamma + \xi_t$ consists of constant part γ, which determine the average dissipation coefficient and chaotic part ξ_t which describes the influence of a random dissipation change. In the case of the white noise, we can write the following Fokker–Planck equation for the nonequilibrium distribution function in the standard form of the Stratonovich interpretation [14]

$$\frac{\partial\rho(\varepsilon,t)}{\partial t} = \frac{\partial}{\partial\varepsilon}\left(\gamma\varepsilon\rho(\varepsilon,t)\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\varepsilon^2}\varepsilon^2\rho(\varepsilon,t), \tag{10}$$

where σ^2 determines dispersion of the dissipation coefficient fluctuations γ. The stationary solution of this equation can be presented as [14]

$$\rho_s(\varepsilon,t) = C(\varepsilon)^{-[\nu+\gamma/\sigma^2]}. \tag{11}$$

This stationary solution has the form of a Pareto distribution. This result has been tested experimentally and found to be valid in many cases, such as the distribution of money in society or the distribution of words in literary works, *etc.* It may be argued that the distribution of the Pareto function is generated by fluctuations in the environment in which the dynamics of an unbalanced system takes place. In addition, it should be considered that σ^2 is the variance of only the dissipation coefficient γ and may be determined by external conditions. The dependence of dissipation properties on the control parameter may be nonlinear. In such a case, we can introduce additional nonlinear term in the function

$$f(\varepsilon) = -\gamma\varepsilon + \delta\varepsilon^n.$$

As is seen, if $n = 1$, nothing will change compared to the previous case, except the power $-[\nu+(\gamma-\delta)/\sigma^2]$. In the case of a nonlinear power, we get the following result for the distribution function

$$\rho(\varepsilon) = A\varepsilon^{-[\nu+\gamma/\sigma^2]}\exp\left(\frac{\delta}{\sigma}^{2}\varepsilon^{n-1}\right). \tag{12}$$

The resulting distribution does not work well for both low and high control parameter values and can only occur in a certain range of it. The multiplier before the exponent can be interpreted as a dependence on the density of states. It turns out that the physical properties of the system, such as heat capacity, depend on the density of states. Based on the behavior of the latter, conclusions are made about the type of the phase transition in the equilibrium case [20].

4. Energy Space

For a better understanding of the results thus obtained, let us apply this approach to the description of physical systems, where the energy of the system may be a controlling parameter. In the nonequilibrium case, the energy of the macroscopic system changes depending on the external influences and initial conditions, i.e., the system makes a transition from some energy state to some other one. In order to describe such evolution, it is necessary to consider the dissipation and influence of the environment [7–16].

A generalization of the statistical description of such systems using the Gibbs distribution in the energy representation [21] is not always possible. In order to understand the reasons for this, let us recall the key points of this approach. The canonical Gibbs distribution in the phase space is given by

$$\rho(q,p)d\Gamma = \exp\left\{\frac{F-H(q,p)}{\Theta}\right\}d\Gamma, \tag{13}$$

where $H(q,p) = E$ is the Hamiltonian on the hypersurface of the constant energy E, $d\Gamma = \prod_i dq_i dp_i$ is an element of the phase space, $\Theta = kT$, T is the temperature, and F is the free energy that may be found from the normalization condition

$$\int \exp\left[\frac{F-H(q,p)}{\Theta}\right]d\Gamma = 1.$$

The phase space, as is known, is determined by the energy of the system and by external parameters [21]. We introduce the quantity $\Sigma = \ln(d\Gamma)/(dE)$. Then we can consider the distribution in the energy space

$$\rho(E)dE = C\exp\left\{\frac{F-E}{\Theta} + \Sigma(E)\right\}dE. \tag{14}$$

The normalization condition yields

$$\int c\exp\left[\frac{F-E}{\Theta} + \Sigma(E)\right]dE = 1.$$

In order to select the states with dominant contributions into the partition function, we employ the condition for the temperature given by $(d\Sigma)/(dE) = 1/\Theta$.

When doing so, we assume that the relation between the changes of the value of the phase space on the energy E is known. In terms of this definition and within the context of fundamental principles of statistical mechanics [13], we come to the conclusion that

$$\Sigma = \ln\frac{d\Gamma}{dE} = S$$

reproduces the entropy of the system bearing in mind that the temperature describes the dependence of entropy only on the energy, but not on other thermodynamic functions. It also follows that integration over energy in the continual sense yields an expression for the partition function. It is obvious that the extremum contribution into the partition function is associated with the states for which $F = E - \theta S$ and that for any deviations from the latter condition the contribution into the partition function is negligibly small similarly to the contribution of quantum corrections to the classical trajectories [8, 16].

In the case of open systems, the additional dissipation can occur (for example, the dissipative characteristics of the external environment can have significant random changes during the evolution), which can lead to a slow change in the energy of the system under external influence. The random walking of the system is also a result of the interaction with the environment, the influence of which is manifested in the random sequence of changes in the energy of the system. If the system does not have enough time to return to its initial distribution after a random change in environmental parameters, it has to relax to a new state. Therefore, the contribution of states with energy changed by fluctuations of external parameters and which do not satisfy the condition of the dominant contribution to the partition function in the equilibrium case, can be significant, and the Gibbs distribution is no longer valid.

The general equation that describes the variable energy (3) as a slow control parameter $\varepsilon = E$ and accounts for the dissipation energy and the random walk in the energy space may be represented in the form

$$\frac{\partial E}{\partial t} = f(E) + L(t), \tag{15}$$

where $f(E)$ determines changes in the energy due to the dissipation processes, and $L(t)$ is a fluctuation that cannot be associated with the changing dissipation and accounts for the random disturbances of the environment. The dependence of the dissipation function on the energy includes the influence of all external factors leading to the energy dissipation. Below, we consider some examples of such dependence. In the case of the white noise, we use the simple Fokker–Plank equation (2) for the nonequilibrium distribution function given by [14, 18]

$$\frac{\partial \rho(E,t)}{\partial t} = -\frac{\partial}{\partial E}\left(f(E)\rho(E,t)\right) + \frac{\sigma^2}{2}\frac{\partial^2 \rho(E,t)}{\partial E^2}. \tag{16}$$

The stationary solution of this equation has the following simple form:

$$\rho(E) = A\exp\left(\int\limits_0^E \frac{f(E')}{\sigma^2}dE'\right). \tag{17}$$

In the case $f(E) = -\gamma E$ that involves the efficient dissipation of the energy in the external medium. The stationary solution may be written in the form

$$\rho(E) = A\exp\left(-\frac{\gamma E^2}{\sigma^2}\right). \tag{18}$$

To determine the physical meaning of the coefficient of energy dissipation, we should return to the dynamic equation for the energy. The solution of this equation may be written in the form

$$\left\langle E^2\right\rangle = E_0^2\exp(-2\gamma t) + \frac{\sigma^2}{2\gamma}\left(1-\exp(-2\gamma t)\right), \tag{19}$$

with the necessary condition $\lim_{t\to\infty}\left\langle E^2\right\rangle = \frac{\sigma^2}{2\gamma}$. In such case, the nonequilibrium distribution function is given by

$$\rho(E) = A\exp\left(-\frac{E^2}{\langle E^2\rangle}\right) \tag{20}$$

which is the equilibrium "Maxwell distribution function" for the energy close to the average value of the fluctuating energy. It can be treated as a generalization of the energy distribution in the conservative system that does not interact with the thermostat. In the latter case $f(E) = 0$, and the stationary solution transforms to a constant. The equation for the nonequilibrium distribution function in the case of an open system has the form of the diffusive equation that has the solution

$$\rho(E) = A\frac{1}{\sqrt{8\pi\sigma}}\exp\left(-\frac{E^2}{4\sigma^2 t}\right) \tag{21}$$

that describes free migration of a fuzzy (blurred) system in the energy space. The measure of fuzziness increases in time according to the law $\left\langle E^2\right\rangle = 4\sigma^2 t$. This solution describes the evolution of the system, that in the initial state, was described by the distribution function of the closed system $\rho(E) = \delta(E-E_0)$. All states of the system at the initial time are located at the points of the energy conservation surface. Fluctuations in external conditions lead to the blurring of the microcanonical distribution. In this situation, a stationary probability density does not exist. In this sense, it is necessary to consider that $E = 0$ is not only an internal limit, but also a stationary point at which all probable states of the system have similar probability densities, and, therefore, transfer and diffusion do not occur. This point is also an attraction point, and all the "mass" of the probability density is concentrated at zero and must satisfy the normalization condition for the distribution function [14]. According to the probability density, it should correspond to the microcanonical distribution function. The stationary solution in the general nonequilibrium case can be presented as given by

$$\rho_s(E) = \frac{C}{D^{\nu}(E)} \exp\left(-\int\limits_0^E \frac{f(E')d\varepsilon'}{D(E')}\right). \tag{22}$$

This distribution function has the extreme value by the control parameter that can be found as a solution of the equation

$$\nu D'(\widetilde{E}) = f(\widetilde{E}). \tag{23}$$

If we assume that the dissipation $f(E)$ is a nonlinear function of the system state, many interesting situations may arise, including noise-induced transitions to new nonequilibrium states that are more stable than the previous state. In this sense, it is possible to realize a new nonequilibrium state, with longer lifetime of the system. In this special state, the system manifest new properties that were not manifested under the initial conditions. Fluctuations of the environment determine the temperature of the system and all possible states of this system. The temperature of the system is determined by the diffusion in the energy space, and this diffusion is a universal characteristic of the environment. The temperature is determined by the process under consideration. As an example, we calculate the entropy at the extreme value of the distribution function, i.e.,

$$S = -\int \rho_s(E) \ln \rho_s(E) dE = \langle U(E) \rangle \tag{24}$$

and determine by definition, the temperature of the system at the saddle point:

$$\frac{1}{\Theta} = \frac{dS}{d\langle E \rangle} = \frac{d\langle U(E) \rangle}{d\langle E \rangle}. \tag{25}$$

This means that the temperature does not correspond to the average value of the energy in the system and, in the general case, represents a certain function of the average energy. Below, we will show this in some cases. In the simplest case of the Brownian particle, we can take $f(E) = -\gamma E$, where γ is the coefficient of dissipation and $g(E) = 1$ for the white noise, then $U(E) = \frac{2\gamma}{\sigma^2} E$ and $\frac{1}{\Theta} = \frac{2\gamma}{\sigma^2}$. The temperature for such systems is determined only through the noise intensity, that is natural.

5. Conclusion

The cause of the Pareto distribution in many nonequilibrium systems is shown to be associated with the fluctuations of the environmental properties.

The justification of the previously proposed approach to the statistical description of nonequilibrium systems in the energy space [15] is presented, and a general description of the evolution of the nonequilibrium distribution function in the energy space is proposed. On the basis of the Fokker–Planck equation for the distribution function of the macroscopic system, a new equation was obtained that accounts for fluctuations in the state of the environment. The stationary solution of such equation describes steady states of nonequilibrium systems. The proposed approach involves probable transitions between different states due to the dissipation and diffusion of the system in the energy space.

To describe the evolution of a nonequilibrium system, we should find solutions to the Fokker–Planck equation with nonlinear kinetic coefficients that may be the subject of further research. It is natural to describe the evolution in the energy space using kinetic coefficients, which should be found from the relevant Langevin equations for the evolution of the control parameter of the system interacting with the fluctuations of the environment and its macroscopic characteristics. A decrease in the power of the initial ground state does not contradict the H-theorem, since it is known that distributions whose evolution is governed by the Fokker–Planck equation lead to an increase in the entropy. Therefore, the picture proposed in this article seems to be quite consistent.

APPENDIX A.
Ordinary Brownian Motion

To more fully illustrate the advantages of the proposed approach, let us consider the known results of the theory of Brownian motion. Now, we will show that, for an ordinary Brownian particle, the proposed approach to the description of the stochastic dynamics in the energy space is completely equivalent to the description in the velocity space. The dynamics of Brownian particles can be described in terms of the velocity v by the Langevin equation

$$\frac{dv}{dt} = -\gamma v + F(t), \tag{A1}$$

where γ is the coefficient of friction, and $F(t)$ is the random force that describes the action of the environment on the particle with the average over the equilibrium ensemble equal to zero $\langle F(t) \rangle = 0$ and $\langle F(t)F(t') \rangle = \phi^2 \delta(t - t')$, which satisfy the white noise condition and describe the uncorrelated process of particle motion. For a Brownian particle, the energy $E = Mv^2/2$ and a change in the energy may be defined as

$$\frac{dE}{dt} = Mv\frac{dv}{dt} = -2\gamma E + \sqrt{2ME}F(t) \qquad \text{(A2)}$$

that is the Eq. (15) with $f(E) = -2\gamma E$, $g(E) = \sqrt{E}$ and $L(t) = \sqrt{2M}F(t)$. Using the solution of the Langevin equation for the velocity, we obtain [18] $\langle v^2(\infty)\rangle = \phi^2/(2\gamma) = (kT)/M$ and, so, $\langle E\rangle = (kT)/2$, where T is the temperature of the thermal bath. Using the solution of Eq. (A2) and disregarding the correlation of the energy fluctuations, we obtain also

$$\sqrt{\langle E\rangle^2} = \frac{\sigma^2}{4\gamma} \equiv \frac{\phi^2}{4\gamma}2M = kT$$

that, as in the previous result, fully satisfies the equilibrium condition. Different descriptions of the process occurring in a nonequilibrium system are equivalent, but the energy representation is preferred, because it provides a possibility to find the conditions for the "steady" states of the nonequilibrium system. This approach is valid for various systems for which the direct influence of interaction with the environment and probable random nonequilibrium fluctuations can be determined. This is preferable, because the energy is the slowest variable on which relaxation of the system depends.

The energy presentation may be more transparent for the understanding, if we compare it with the usual way of describing the equilibrium states. For example, for an ordinary Brownian particle, the stationary solution can be written as

$$\rho_s(E) = A\exp\left\{-\frac{4\gamma}{\sigma^2}E - \ln\sqrt{E}\right\} \equiv A\frac{1}{\sqrt{E}}\exp(-\beta E), \qquad \text{(A3)}$$

where the well-known relation $(2\gamma)/\sigma^2 = \beta$ was used. In view of the normalization condition

$$\int \rho_s(E)dE \equiv \int \rho_s(p)dp,$$

we obtain the equilibrium distribution function in the momentum space in the form given by

$$\rho_s(p) = A\exp\left(-\beta\frac{p^2}{2M}\right) = A\exp\left(-\frac{Mv^2}{2kT}\right). \qquad \text{(A4)}$$

The stationary solution completely represents the well-known equilibrium distribution function for ordinary Brownian particles.

Let us analyze another case where the energy is constantly introduced into the system and dissipated, i.e., $f(E) = \alpha - \gamma E$. For such system, we obtain the following stationary distribution function for fluctuations of the dissipation coefficient $\gamma = \gamma + \xi_t$

$$\rho(\varepsilon) = AE^{-[1+(2\gamma)/\sigma^2]}\ \exp\left(-\frac{2\alpha}{\sigma^2 E}\right). \qquad \text{(A5)}$$

Here, σ^2 is the variance of only the dissipation coefficient γ. For large values of the energy, we observe only power law distribution function.

1. N.N. Bogoliubov. *Problems of Dynamic Theory* (Geophysics Research Directorate, AF Cambridge Research Laboratories, Air Force Research Division, United States Air Force, 1960).
2. R. Balescu. *Equilibrium and Nonequilibrium Statistical Mechanics* (J. Wiley and Sons, 1975) [ISBN: 978-0471046004].
3. Yu.L. Klimontovich. *Kinetic Theory of Nonequilibrium Processes* (Springer, 1982).
4. D. Zubaryev. *Nonequilibrium Statistical Thermodynamics* (Nauka, 1971).
5. C.W. Gardiner, P. Zoller. *Quantum Noise* (Springer, 2000) [ISBN: 3-540-22301-0].
6. D.F. Wells, G.J. Milburn. *Quantum Optics* (Springer, 1994) [ISBN-13: 978-3-540-58831-3].
7. B.I. Lev, A.G. Zagorodny. Structure formation in system of Brownian particle in dusty plasma. *Phys. Lett. A* **376**, 1101 (2009).
8. B.I. Lev, A.G. Zagorodny. Statistical description of Coulomb-like systems. *Phys. Rev. E* **84**, 061115 (2011).
9. B.I. Lev, V.B. Tymchyshyn, A.G. Zagorodny. Brownian particle in non-equilibrium plasma. *Cond. Matt. Phys.* **12** (4), 593 (2009).
10. V. Guerriero. Power law distribution: Method of multi-scale inferential statistics. *J. Mod. Math. Frontier* **1** (1), 21 (2012).
11. Yu.L. Klimontovich. Nonlinear Brownian motion. *Uspekhi Fizicheskikh Nauk* **164** (8), 811 (1994).
12. D. Huang. *Statistical Mechanics* (W.A. Benjamin, Inc., 1969).
13. L.D. Landau, E.M. Lifshitz. *Statistical Physics* (Elsevier, 2013).
14. W. Horsthemke, R. Lefever. *Noise-Induced Transition: Theory, Applications in Physics, Chemistry and Biology* (Springer-Verlag, 1984) [ISBN: 978-3540113591].
15. B.I. Lev, A.D. Kiselev. Energy representation for nonequilibrium Brownian-like systems: Steady states and fluctuation relations. *Phys. Rev. E* **82**, 031101 (2010).
16. B.I. Lev. Brownian system in energy space: Non-equilibrium distribution function in energy representation. *Eur. Phys. J. (Special Topics)* **216**, 37 (2013).
17. A.J. Lichtenberg, M.A. Liberman. *Regular and Stochastic Motion* (Springer-Verlag, 1984).
18. N.G. van Kampen. *Stohastic Process in Physics and Chemistry* (North-Holland, 1990).
19. N.N. Bogoliubov. On stochastic processes in dynamic system. In: *Physics of Elementary Particles and Nuclei (PEPAN)* (JINR (Dubna), 1978), **9**, No. 4.
20. Fugao Wang, D.P. Landau. Determining the density of states for classical statistical models: A random walk algorithm to produce a flat histogram. *Phys. Rev. E* **64** (5), 056101 (2001).
21. J. Willard Gibbs. *Elementary Principles in Statistical Mechanics, Developed with Especial Reference to the Rational Foundation of Thermodynamics* (Cambridge University Press, 1902) [ISBN: 9780511686948].

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ФЛУКТУАЦІЇ ТА СТЕПЕНЕВІ ФУНКЦІЇ РОЗПОДІЛУ В НЕРІВНОВАЖНИХ СИСТЕМАХ

Сформульовано рівняння Фоккера–Планка для функцій розподілу макроскопічних відкритих систем у просторі повільно змінних фізичних величин (енергії, адіабатичних інваріантів тощо). Стаціонарний розв'язок отриманих рівнянь визначає квазірівноважну функцію розподілу у такому просторі. Запропонований підхід враховує еволюцію систем під дією дисипації та дифузії у просторі відповідних змінних. Показано, що відомий степеневий закон розподілу можна отримати, якщо врахувати внутрішні та зовнішні флуктуації в статистичних системах. Наведено приклад рівнянь Ланжевена, які генерують степеневі функції розподілу.

Ключові слова: рівняння Фоккера–Планка, степенева функція розподілу, флуктуації в статистичних системах, рівняння Ланжевена.