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QUANTUM ROTATING BLACK HOLES (RECOVERING GEOMETRY IN A QUANTUM WORLD) [1]

Classical geometries for spherically symmetric systems can be effectively obtained from quantum coherent states for the relevant degrees of freedom. This description replaces the classical singularity of black holes with integrable structures in which tidal forces remain finite, and there is no inner Cauchy horizon. It is then shown how the extension to rotating systems can avoid the classical inner horizon provided the rotation is not ultra-rigid.

Keywords: classical geometry, quantum rotating black holes, quantum gravity, Planck scale, gravitational collapse, Schwarzschild geometry.

1. Quantum Gravity and the Planck Scale

For a particle of proper mass m, the Compton–de Broglie length $\lambda_m = \hbar/m\,c$ is comparable with the gravitational (Schwarzschild) radius $R_{\rm H} = 2\,G_{\rm N}\,m$, if the mass is near the Planck mass

$$m_{\rm p} = \sqrt{c\,\hbar/G_{\rm N}} \simeq 10^{-8}\,{\rm kg}, \tag{1.1}$$

corresponding to the Planck length

$$\ell_{\rm p} = \sqrt{\hbar\,G_{\rm N}/c^3} \simeq 10^{-35}\,{\rm m}. \tag{1.2}$$

Quantum gravity is, therefore, usually associated with processes occurring at the Planck mass $m_{\rm p}$ and length $\ell_{\rm p}$. However, the Compton length is relevant only in the scatterings of (asymptotically free standard model) particle excitations [1], whereas processes involving bound states are usually characterised by (significantly) larger length scales. For example, the hydrogen atom is a few orders of magnitude bigger than the Compton length of the electron, superconductors and Bose–Einstein condensates can be macroscopic in size, and neutron stars are quantum objects of several kilometres in radius.

The situation is more complicated in General Relativity, because the nonlinearity of the Einstein equations makes it difficult to separate (mass or length) scales like we can do in electrodynamics or in the weak-field approximation of gravity. For compact objects of ADM [2] mass M and radius $R_{\rm s}$, we cannot discard nonlinear effects, when the compactness $X = G_{\rm N}\,M/R_{\rm s} \sim 1$. In fact, $X \simeq 1$ for a particle of mass $m = m_{\rm p}$ and Compton length $\lambda_m = \ell_{\rm p}$, which is, therefore, expected to be a quantum black hole [3, 4]. One might, therefore, wonder, if quantum physics becomes generically relevant for self-gravitating systems of compactness $X \sim 1$, leading to a departures from classical expectations similar to the case of the hydrogen atom.

2. Quantum Gravity and the Gravitational Collapse

Objects with compactness $X \sim 1$ should appear in the Nature during the gravitational collapse that would classically end with a black hole geometry characterised by the existence of a (outer) event horizon. Such geometries, however, suffer of serious physical issues: once the horizon forms, the interior becomes geodesically incomplete, with regions, where tidal forces diverge [5]; moreover, the evolution of quan-

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tum fields on such classical backgrounds leads to possible violations of unitarity [6]. We expect the quantum theory will fix this inconsistent (semi)classical picture, much in the same way that quantum mechanics explains the stability of atoms by not admitting quantum states corresponding to the classical ultraviolet catastrophe.

The problem at hand is again much more involved, because reaching $X \sim 1$ requires (several) solar mass amounts of matter. In terms of the Standard Model of particles, this means $M \simeq M_\odot \simeq 10^{57}$ neutrons and, according to Bekenstein's counting of gravitational excitations involved in the corresponding geometry, $M_\odot^2/m_\mathrm{p}^2 \simeq 10^{76}$ gravitons [7]. These numbers clearly put the gravitational collapse beyond any present and future possibility of detailed modelling, and we must do what we always do: find a simplified description which allows for mathematical treatment, like the Oppenheimer–Snyder model [8]. In Refs. [9, 10], the quantum theory of a ball of dust was studied to show that a ground state core of size $R_\mathrm{s} \simeq G_\mathrm{N}\, M$ might indeed replace the classical singularity (see Fig. 1). This conclusion is also required by a consistent quantum description of the geometry.

3. Coherent States for Classical Geometry

We will first review how to describe a spherically symmetric metric

$$\mathrm{d}s^2 = -\,(1+2\,V)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+2\,V} + r^2\,\mathrm{d}\Omega^2, \tag{3.1}$$

where $V = V(r)$ is recovered as the mean field of the coherent state of a scalar field (for more details, see, *e.g.,* Ref. [11–15]). We define the canonically normalised real scalar field $\Phi = \sqrt{m_\mathrm{p}/\ell_\mathrm{p}}\, V$ and then quantize Φ as a massless field satisfying the free wave equation [2]

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right)\right]\Phi(t,r) = 0. \tag{3.2}$$

Normal modes of Eq. (3.2) can be written as

$$u_k(t,r) = e^{-i\,k\,t}\, j_0(k\,r), \tag{3.3}$$

where $j_0 = \sin(k\,r)/k\,r$ are spherical Bessel functions with $k > 0$. The quantum field operator and its conjugate momentum read

$$\hat{\Phi}(t,r) = \int\limits_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} \times$$
$$\times \sqrt{\frac{\hbar}{2\,k}}\left[\hat{a}_k\, u_k(t,r) + \hat{a}_k^\dagger\, u_k^*(t,r)\right], \tag{3.4}$$

$$\hat{\Pi}(t,r) = i\int\limits_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2} \times$$
$$\times \sqrt{\frac{\hbar\,k}{2}}\left[\hat{a}_k\, u_k(t,r) - \hat{a}_k^\dagger\, u_k^*(t,r)\right] \tag{3.5}$$

and satisfy the equal time commutation relations

$$\left[\hat{\Phi}(t,r), \hat{\Pi}(t,s)\right] = \frac{i\,\hbar}{4\,\pi\,r^2}\,\delta(r-s), \tag{3.6}$$

provided the creation and annihilation operators obey the commutation rules

$$\left[\hat{a}_k, \hat{a}_p^\dagger\right] = \frac{2\,\pi^2}{k^2}\,\delta(k-p). \tag{3.7}$$

The Fock space of quantum states can now be built starting from the vacuum $\hat{a}_k\,|0\rangle = 0$ for all $k > 0$. It is very important to remark that the flat Minkowski metric in Eq. (3.2) corresponds to this vacuum state representing a completely empty spacetime, devoid of any matter source and without excitations of the gravitational field. In this respect, the number of space and time dimensions in Eq. (3.2) is formal and arbitrary, what matters is that only those modes and corresponding dimensions of relevance to the geometry (3.1) will be assumed to get excited above the vacuum. On the other hand, the choice (3.2) is also consistent with the weak field expansion in which the Newtonian potential in three spatial dimension is obtained from longitudinally polarized gravitons [17].

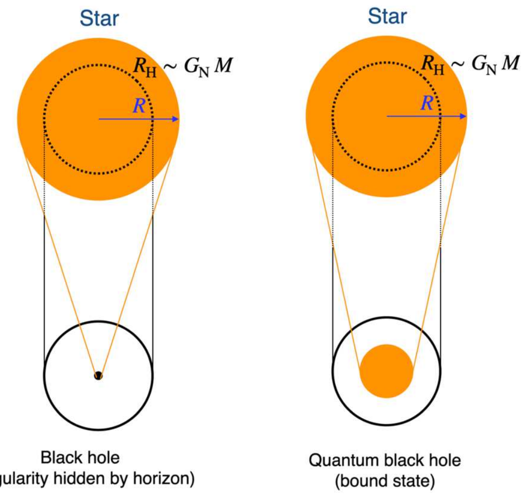

Fig. 1. Black hole formation in classical General Relativity (left panel) versus quantum gravity (right panel)

[2] Units with $c = 1$ will be used from now, so that $G_\mathrm{N} = \ell_\mathrm{p}/m_\mathrm{p}$ and $\hbar = \ell_\mathrm{p}\, m_\mathrm{p}$ [16].

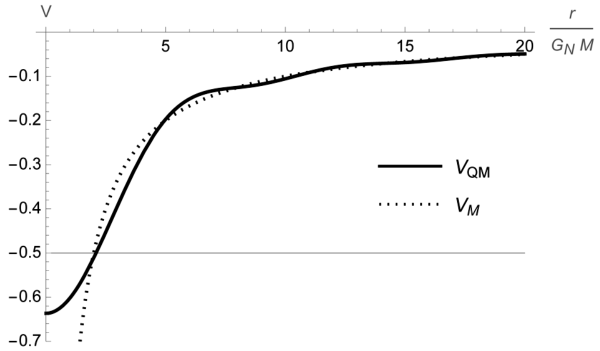

Fig. 2. Quantum metric function $V_{\mathrm{Q}M}$ in Eq. (4.6) (solid line) compared to V_M (dashed line) for $R_{\mathrm{s}} = G_{\mathrm{N}}\, M$. The horizontal thin line marks the location of the horizon, where $V = -1/2$

We next assume that classical configurations (3.1) of the metric can be realized in the quantum theory by coherent states $\hat{a}_k\,|g\rangle = g_k\, e^{i\,\gamma_k(t)}\,|g\rangle$ such that

$$\sqrt{\frac{\ell_{\mathrm{p}}}{m_{\mathrm{p}}}}\,\langle g|\,\hat{\Phi}(t,r)\,|g\rangle = V(r). \tag{3.8}$$

From the expansion (3.4), we obtain

$$\langle g|\,\hat{\Phi}(t,r)\,|g\rangle = \int\limits_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\times$$
$$\times\sqrt{\frac{2\,\ell_{\mathrm{p}}\,m_{\mathrm{p}}}{k}}\,g_k\,\cos[\gamma_k(t)-k\,t]\,j_0(k\,r). \tag{3.9}$$

We can eliminate the time dependence from the normal modes (3.3) by setting $\gamma_k = k\,t$, and the coherent state finally reads [15]

$$|g\rangle = e^{-N_{\mathrm{G}}/2}\,\exp\left\{\int\limits_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,g_k\,\hat{a}_k^\dagger\right\}|0\rangle\,, \tag{3.10}$$

where the coefficients

$$g_k = \sqrt{\frac{k}{2}}\,\frac{\tilde{V}(k)}{\ell_{\mathrm{p}}} \tag{3.11}$$

are determined from

$$V = \int\limits_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,\tilde{V}(k)\,j_0(k\,r). \tag{3.12}$$

The normalization factor

$$N_{\mathrm{G}} = \int\limits_0^\infty \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,g_k^2 \tag{3.13}$$

can be interpreted as the total occupation number of the state $|g\rangle$, and it is a measure of its distance from the vacuum of the Fock space $|0\rangle$ [15].

4. Coherent States for Schwarzschild Geometry

The Schwarzschild geometry is given by Eq. (3.1) with

$$V = V_M = -\frac{G_{\mathrm{N}}\,M}{r}, \tag{4.1}$$

from which we find

$$\tilde{V}_M = -4\,\pi\,G_{\mathrm{N}}\,\frac{M}{k^2} \tag{4.2}$$

and the coefficients

$$g_k = -\frac{4\,\pi\,M}{\sqrt{2\,k^3}\,m_{\mathrm{p}}}. \tag{4.3}$$

The total occupation number now reads

$$N_{\mathrm{G}} = 4\,\frac{M^2}{m_{\mathrm{p}}^2}\int\limits_0^\infty \frac{\mathrm{d}k}{k}, \tag{4.4}$$

which diverges logarithmically in the infrared (IR) and the ultraviolet (UV) [13, 15]. The latter divergence is due to the vanishing size of the source in the classical Schwarzschild geometry, and the very existence of a proper quantum state $|g\rangle$ requires the coefficients g_k to depart from their purely classical expression (4.3) for $k \to 0$ and $k \to \infty$. We, therefore, remove the UV divergence by introducing a cut-off $k_{\mathrm{UV}} \sim 1/R_{\mathrm{s}}$, where R_{s} could be the finite radius of the matter core at the end of the collapse displayed in Fig. 1. Likewise, we introduce a IR cut-off $k_{\mathrm{IR}} = 1/R_\infty$ to account for the finite size R_∞ of the Universe, so that

$$N_{\mathrm{G}} = 4\,\frac{M^2}{m_{\mathrm{p}}^2}\int\limits_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \frac{\mathrm{d}k}{k} = 4\,\frac{M^2}{m_{\mathrm{p}}^2}\,\ln\left(\frac{R_\infty}{R_{\mathrm{s}}}\right). \tag{4.5}$$

This yields an effective (quantum) metric (3.1) with

$$V = V_{\mathrm{Q}M} \simeq \int\limits_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \frac{k^2\,\mathrm{d}k}{2\,\pi^2}\,\tilde{V}_M(k)\,j_0(k\,r) \simeq$$
$$\simeq V_M\left\{1-\left[1-\frac{2}{\pi}\,\mathrm{Si}\left(\frac{r}{R_{\mathrm{s}}}\right)\right]\right\}, \tag{4.6}$$

where Si denotes the sine integral function (see Fig. 2 for an example).

5. Quantum Integrable Black Holes

In the classical Schwarzschild spacetime, the Kretschmann scalar $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \sim R^2 \sim r^{-6}$ for $r \to 0$, whereas, for the above quantum corrected metric, we have

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \simeq R^2 \simeq \frac{64 G_N^2 M^2}{\pi R_s^2 r^4}. \quad (5.1)$$

This ensures that tidal forces remain finite all the way to $r = 0$, and the spatial integration of the Einstein–Hilbert Lagrangian density is also finite, since the Ricci scalar $R \sim r^{-2}$. The point at $r = 0$ can, therefore, be seen as an integrable singularity [18], where some geometric invariants diverge with no harmful effects on matter.

From the Einstein tensor, one can find the effective energy density and pressure

$$\varepsilon = \frac{M}{2\pi^2 r^3} \sin\left(\frac{r}{R_s}\right) = -p_r \quad (5.2)$$

and the effective tension

$$p_t = \frac{M}{4\pi^2 r^3}\left[\sin\left(\frac{r}{R_s}\right) - \frac{r}{R_s}\cos\left(\frac{r}{R_s}\right)\right]. \quad (5.3)$$

Note, in particular, that the effective energy density can be interpreted in terms of a normalizable wave-function $\psi = \psi(r)$ for the matter in the core,

$$\varepsilon \sim |\psi(r)|^2 \sim r^{-2}, \quad \text{for } r \to 0, \quad (5.4)$$

and the Misner–Sharp–Hernandez mass function [19, 20] is given by

$$m(r) = 4\pi \int_0^r x^2 \varepsilon(x)\, dx \simeq \frac{2 M r}{\pi R_s}, \quad \text{for } r \to 0, \quad (5.5)$$

with $m(r \to \infty) = M$.

Another good feature of this quantum corrected geometry that follows from Eq. (5.5) is that there is no (inner) Cauchy horizon (whenever there exists the outer event horizon). The locations of horizons are, in fact, given by solutions of

$$\Delta \equiv r^2 (1 + 2V) = r^2 - 2 r G_N m(r) = 0. \quad (5.6)$$

If there is an outer (event) horizon $\Delta(r_+) = 0$ at $r = r_+ > 0$, the function Δ becomes negative for $r < r_+$ and remains negative all the way to $r = 0$ provided $R_s \lesssim 4 G_N M/\pi$ (see solid line in Fig. 3). Note that, for a regular black hole solution with $m(r) \sim r^3$,

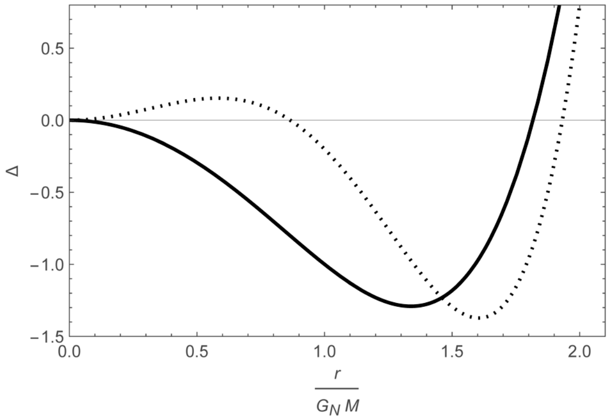

Fig. 3. Behaviour of Δ in Eq. (5.6) for integrable black holes (solid line) and regular black holes (dotted line)

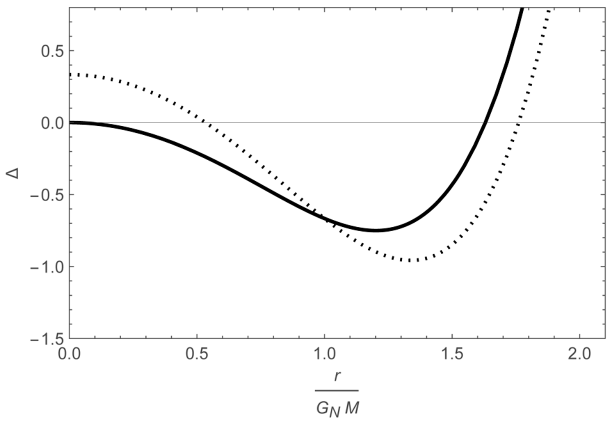

Fig. 4. Behaviour of Δ in Eq. (6.2) for integrable black holes with differential rotation (solid line) and rigid rotation (dotted line)

the function Δ will necessarily becomes positive in a neighborhood of the center (see dotted line in Fig. 3). The absence of inner horizons can be generalized to electrically charged black holes [21] and excludes potentially serious casual issues which are often present in regular black hole candidates (see, *e.g.,* Ref. [22] and references therein).

6. Rotating Integrable Black Holes

The ultra-rigid rotating black-hole-geometry for constant specific angular momentum $a = J/M$ can be written as

$$ds^2 = \left[1 - \frac{2 r m}{\rho^2}\right] dt^2 + \frac{4 a r m \sin^2\theta}{\rho^2}\, dt\, d\phi - \frac{\rho^2}{\Delta}\, dr^2 -$$

$$- \rho^2\, d\theta^2 - \frac{\Sigma\, \sin^2\theta}{\rho^2}\, d\phi^2, \qquad (6.1)$$

where $\rho^2 = r^2 + a^2 \sin^2\theta$ and $\Sigma = (r^2 + a^2)^2 - a^2\,\Delta\,\sin^2\theta$. Horizons are located so that

$$\Delta = a^2 + r^2 - 2\, r\, G_{\rm N}\, m(r) = 0, \qquad (6.2)$$

in which we assumed a mass function that depends only on the radial coordinate like in the spherically symmetric case [23].

A mass function which behaves itself like the one in Eq. (5.5) near $r = 0$ will again removes the (ring) singularity of the Kerr geometry that arises for $m = M$ [24]. However, $\Delta(0) = a^2 > 0$, and one necessarily encounters an inner horizon at $0 < r = r_- < r_+$ (see dotted line in Fig. 4). In order to eliminate the zero of Δ at $r = r_- > 0$, we must consider a differential rotation $a = a(r)$ such that $a \sim r^\alpha$ with $\alpha \geq 1$ for $r \to 0$ [25] (see the solid line in Fig. 4).

Building coherent states for differentially rotating geometries (6.1) with $m = m(r)$ and $a = a(r)$ represents a huge complication with respect to the spherically symmetric cases. As of now, only slowly rotating geometries with $a \ll M$ were investigated in Ref. [26].

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Р. Касадіо

КВАНТОВІ ОБЕРТОВІ ЧОРНІ ДІРИ
(ВІДНОВЛЕННЯ ГЕОМЕТРІЇ В КВАНТОВОМУ СВІТІ

Класичні геометрії для сферично-симетричних систем можуть бути ефективно отримані з квантових когерентних станів для відповідних ступенів вільності. Даний опис замінює класичну сингулярність чорних дір інтегровними структурами, в яких припливні сили залишаються скінченними і немає внутрішнього горизонту Коші. Потім показано, як узагальнення на обертові системи може уникнути класичного внутрішнього горизонту за умови, що обертання не є наджорстким.

Ключові слова: класична геометрія, квантові обертові чорні діри, квантова ґравітація, шкала Планка, ґравітаційний колапс, геометрія Шварцшильда.