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MATHEMATICAL MODELS AND METHODS ON HIGHER DIMENSIONAL BULK VISCOUS STRING COSMOLOGY WITH THE FRAMEWORK OF LYRA GEOMETRY

We investigate a cosmological scenario generated by a cloud of strings containing particles in the framework of the Lyra geometry by considering five-dimensional Bianchi type-III line element. We assume two physically plausible conditions (i) shear scalar (σ) proportional to the expansion factor (θ), which leads to $P = Q^n$; $n \neq 0$ is a constant, P and Q being scale factors and (ii) $\xi = \xi_0 =$ constant, ξ being the coefficient of bulk viscosity, deterministic models of our Universe are obtained. We have solved the modified Einstein's field equations of a homogeneous Bianchi type-III metric. The bihaviors of cosmographic parameters for the different values of time (t) and redshift (z) are presented in detail to study the proposed model. It has been found that the displacement vector (β) behaves itself like the cosmological term, and the solution is consistent with the recent observations of SNeIa. The physical and geometrical properties of the model are premeditated, and it has been discussed in detail regarding the possibilities and prospects that can be happen throughout the evolution of the Universe. It is found that the bulk viscosity plays a crucial role in the evolution of the Universe, and the strings dominate in the early Universe and eventually disappear from the Universe during a sufficiently large time. So, our model can be treated as a realistic one.

Keywords: Lyra geometry, bulk viscosity, evolution, early Universe, string.

1. Introduction

The generalization of gravitation theory by A. Einstein in his general theory of relativity inspired several authors to geometrize other physical fields. It is familar to all of us that strings play a crucial role in the initial stage of the evolution of the Universe. The learning of the origin of the Universe and the early stage of its creation is still a fascinating area of investigations to find out its mysterious facts that have still

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to be experienced to study the ultimate fate of the Universe. The vast Universe contains various mysterious particles from different views of researchers and from observational findings. So, to study the fate of the Universe, a fresh exploration is required to clarify the unknown fundamentals of the mysterious Universe. In the context of the general relativity, the study of strings was made by Letelier [1,2] and Stachel [3]. Lately, several renowned researchers worked in the domain of cosmic strings in general relativity, as it is fundamental to understand the early stages of the evolution of the universe [4, 5] giving rise to density perturbations due to which galaxies are formed [6, 7].

The observations of the Universe with the aid of modern technological tools also prove that, in the early Universe, there exists a large scale network of strings. These strings possess the stress-energy and are coupled with the gravitational field. Anisotropy of the Universe is due to the presence of strings; however, strings are not visible nowadays. Strings do not pose a threat to the cosmological models, however, they show the way to incredibly exciting astrophysical consequences, as opposed to domain walls and monopoles. Strings are also useful to explain both the nature and the fundamental configuration of the early Universe. In string theory, all the matters and forces are combined to give us a single theoretical structure. The theory explains the initial state of the formation of a Universe in terms of (vibrating) strings. The string theory is very appropriate in describing the early evolution of the Universe. Many researchers nowadays are focusing on studying the cosmological models with strings of the universe to gain a comprehensive understanding of the development of the Universe. According to the GUT (grand unified theories) [4–6, 8–10], after the big-bang explosion, these strings arose, when the cosmic temperature goes down below some critical value due to the symmetry breaking throughout the phase transition in the early Universe. These cosmic strings can couple toward the gravitational field and possess the stress energy. Therefore, it may be fascinating to study the gravitational effect due to strings.

From the recent observations [11–13] it can be said that the space-time Universe is not perfectly isotropic. Rather, it is anisotropic. Therefore, the metric components in the line element should be different functions of the time. To study the homogeneous and anisotropic cosmological model with anisotropic property, Bianchi type space-times are used, and the isotropization process of these models may also be studied. From the theoretical perception, the anisotropic cosmological models have possessed a better generalization than the isotropic model of the Universes. Considering the spatial isotropy, [14, 15] showed that the Bianchi-type string cosmological model generalizes FRW models over and above those with asymptotics or less than SO (3) isotropy.

Several researchers [16–21] have investigated the dynamics of the Universe in different contexts by formulating the cosmological models containing cosmic strings in the fluid. Recently in [22–33, 35–40, 59, 60] some cosmological models with various modified theories of relativity in different contexts were studied in a Bianchi-type space-times. To obtain exact solutions of the Letelier string cosmological model, in [41, 42], Bianchi type II, VI0, VIII, and IX line elements in different contexts were analyzed. Considering the Bianchi type I space-time, in [43], a string cosmological model was studied, and exact solutions of it were obtained by taking the bulk viscosity coefficient in terms of a power function of the energy density. Assuming a hybrid expansion law for the average scale factor, Vinutha *et al.* [44] obtained Bianchi types I and III dark energy cosmological models with strings in the Saez–Ballester theory. Consideing the sum of the energy density and tension density as zero, Adhav *et al.* [45] determined a cosmological model with strings in the B–D theory. Not only the abovementioned authors [46–73] investigated cosmological models with cosmic strings in different contexts and in various geometries. Recently, Pradhan and Jaiswal [74] studied Bianchi type V geometries with massive strings and determined a class of anisotropic & homogeneous cosmological models.

Here, we will study a Bianchi type III string cosmological model in the Lyra geometry. Some physical and dynamical properties of the model universes are studied, and it will be seen that throughout the evolution of the proposed model universe, the strings disappear, whereas the particles will remain. Interestingly, it is seen that the bulk viscosity plays a great role throughout the evolution of the model Universe, and our model can be thought of a realistic universe supporting the present-day observational findings. This paper is organized as follows. The field equations and line elements are presented in Section 2; in Section 3, we obtain the exact solutions of the field equations. In

Section 4, the dynamical and Physical properties of the model are discussed in detail, and Section 5, deals with conclusions.

2. The Metric and Field Equations

The Bianchi type III metric for a five-dimensional cosmological model Universe is taken as

$$ds^2 = P^2\left(dx^2 + e^{-2\alpha x}dy^2 + dz^2\right) + Q^2 dm^2 - dt^2, \quad (1)$$

where $P(t)$ and $Q(t)$ are the scale factors, $\alpha \neq 0$ is a constant, m is the space-like fifth coordinate [75] and the spatial curvature is set to be zero. [76]

Einstein's field equations for Lyra's manifold in the normal gauge [77, 78] read

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi^k\phi_k = -T_{ij}, \quad (2)$$

with $\frac{8\pi G}{c^2} = 1$ in geometrical units, where R_{ij}, R, T_{ij}, and ϕ_i are, respectively, the Ricci tensor, Ricci scalar, energy-momentum tensor, and displacement vector field. Let us define ϕ_i as

$$\phi_i = (0, 0, 0, 0, \beta(t)). \quad (3)$$

The energy-momentum tensor T_{ij} for a bulk viscous fluid is taken as

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j - \xi\theta(g_{ij} + u_i u_j), \quad (4)$$

where, $\rho = \lambda + \rho_p$ is the energy density for a cloud of strings (loaded with particles), λ denotes the string tension density, ρ_p is the particle energy density, $\theta = u^k_{;k}$ is the expansion factor, ξ is the bulk viscosity coefficient, x^i is the unit space-like vector representing the direction of strings, and u^i is the five-velocity vector of a fluid flow.

The velocity vector u^i and direction of a string x^i are given by

$$u^i = (0, 0, 0, 0, 1) \quad (5)$$

and

$$x^i = \left(0, 0, 0, \frac{1}{Q}, 0\right). \quad (6)$$

In co-moving coordinates, the velocity vector u^i and direction of a string x^i satisfy the conditions

$$u_i u^i = -x_i x^i = -1 \quad \text{and} \quad u^i x_i = 0. \quad (7)$$

Therefore, we get

$$T^1_1 = T^2_2 = T^3_3 = -\xi\theta; \quad T^4_4 = -\lambda - \xi\theta;$$

$$T^5_5 = -\rho \quad \text{and} \quad T^i_j = 0 \quad \text{for} \quad i \neq j.$$

For the line element (1), essential physical parameters like the spatial volume V, the average scale factor R, the expansion factor θ, the Hubble expansion factor H, the shear scalar σ^2, and the mean anisotropy parameter Δ are found as follows:

$$V = R^4 = P^3 Q, \quad (8)$$

$$\theta = u^i_{;i} = 3\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q}, \quad (9)$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[3\frac{\dot{P}^2}{P^2} + \frac{\dot{Q}^2}{Q^2} - \frac{1}{4}\theta^2\right], \quad (10)$$

$$4H = \theta = u^i_{;i} = 3\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q}, \quad (11)$$

$$\Delta = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{H_i - H}{H}\right)^2, \quad (12)$$

where H_j; $j = 1, 2, 3, 4$ are, respectively, the directional Hubble's parameter along the direction of x, y, z & m which are defined as

$$H_1 = H_2 = H_3 = \frac{\dot{P}}{P} \quad \text{and} \quad H_4 = \frac{\dot{Q}}{Q}.$$

Henceforth, overhead dots are used to denote derivatives with respect to the time t.

Using Eqs. (1)–(7), in the system of co-moving coordinate, we have the survival field equations as

$$2\frac{\ddot{P}}{P} + \frac{\ddot{Q}}{Q} + \frac{\dot{P}^2}{P^2} + 2\frac{\dot{P}\dot{Q}}{PQ} + \frac{3}{4}\beta^2 = \xi\theta, \quad (13)$$

$$2\frac{\ddot{P}}{P} + \frac{\ddot{Q}}{Q} + \frac{\dot{P}^2}{P^2} + 2\frac{\dot{P}\dot{Q}}{PQ} - \frac{\alpha^2}{P^2} + \frac{3}{4}\beta^2 = \xi\theta, \quad (14)$$

$$3\frac{\ddot{P}}{P} + 3\frac{\dot{P}^2}{P^2} - \frac{\alpha^2}{P^2} + \frac{3}{4}\beta^2 = \lambda + \xi\theta, \quad (15)$$

$$3\frac{\dot{P}^2}{P^2} + 3\frac{\dot{P}\dot{Q}}{PQ} - \frac{\alpha^2}{P^2} - \frac{3}{4}\beta^2 = \rho. \quad (16)$$

3. Solution of the Field Equations

Considering $\xi = \xi_0$(constant), the determinate solutions of the above field equations (13)–(16) are obtained in the following section.

Note that there are 4 highly nonlinear independent equations (13)–(14) involving 6 unknown variables (P, Q, β, λ, ρ, and θ). To get determinate solutions of the above system of equations, we need two extra equations. We assume the following two physically plausible conditions.

(i) Considering shear scalar (σ) proportional to the expansion factor (θ), we can obtain

$$P = Q^n, \tag{17}$$

where $n \neq 0$ is a constant. The above assumption is based on the observations of the velocity and redshift relation for an extragalactic source and predicts that the Hubble expansion is 30%-isotropic, which is supported in the works by Thorne [79]; Kristian and Sanchs [80]. In particular, it can be said that $\frac{\sigma}{H} \leq \leq 0.30$, where σ and H are, respectively, shear scalar & Hubble constant. Collins *et al.* [81] showed that if the normal to the spatially homogeneous line element is congruent to the homogeneous hypersurface, then $\frac{\sigma}{\theta} =$ constant, θ being the expansion factor.

(ii) Berman's [82] suggestion regarding a variation of Hubble's parameter H provides us a model universe that expands with a constant deceleration defined by

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \text{const.} \tag{18}$$

It is well known that, whenever q (deceleration parameter) is negative, then the model universe is expanded with acceleration, whereas a positive q explains a decelerating universe. Although the present observations like CMBR and SNeIa suggested the negative value of q (accelerating model universe), but it can be remarkably state that they are not able to deny about the decelerating expansion of the Universe. Vishwakarma [83] showed that the decelerating Universe is also not inconsistent by means of these observations. Solving Eq. (18) for $q =$ constant, it can be found that

$$R = (at + b)^{\frac{1}{1+q}}, \tag{19}$$

where $a \neq 0$ & b are integrating constants, and $q > > -1$ for the accelerating model Universe.

Now, from Eqs. (8), (17), and (19), we have

$$P = (at + b)^{\frac{4n}{(3n+1)(1+q)}}, \tag{20}$$

$$Q = (at + b)^{\frac{4}{(3n+1)(1+q)}}. \tag{21}$$

Taking $a = 1$ and $b = 0$ in Eqs. (20) and (21), which do not affect the generality, by the suitable choice of the coordinate, we reduce Eqs. (20) and (21) to

$$P = t^{\frac{4n}{(3n+1)(1+q)}}, \tag{22}$$

$$Q = t^{\frac{4}{(3n+1)(1+q)}}. \tag{23}$$

Using Eqs. (22) and (23), the metric (1) can be written as

$$ds^2 = t^{\frac{8n}{(3n+1)(1+q)}} \left(dx^2 + e^{-2\alpha x} dy^2 + dz^2\right) + + t^{\frac{8}{(3n+1)(1+q)}} dm^2 - dt^2. \tag{24}$$

The model given by Eq. (24) represents the bulk viscosity in the Bianchi type III model in the Lyra geometry, where $q > 1$ is a constant.

4. Some Physical and Geometrical Properties

The energy density ρ is obtained from Eqs. (13) and (16) as follows:

$$\rho = -\frac{4\xi_0}{(1+q)t} - \frac{4(2n+1)(q-3)}{(3n+1)(1+q)^2 t^2} - \alpha^2 t^{\frac{-8n}{(3n+1)(1+q)}}. \tag{25}$$

From Eqs. (13) and (15), the value of λ (tension density) is obtained as

$$\lambda = -\frac{4(n-1)(q-3)}{(3n+1)(1+q)^2 t^2} - \alpha^2 t^{\frac{-8n}{(3n+1)(1+q)}}. \tag{26}$$

Therefore, by using Eqs. (25) and (26) in the relation $\rho = \lambda + \rho_p$, the energy density of particles is found as follows:

$$\rho_p = -\frac{4\xi_0}{(1+q)t} - \frac{8(n+2)(q-3)}{(3n+1)(1+q)^2 t^2}. \tag{27}$$

The displacement vector β can be obtained from Eq. (13) as

$$\beta^2 = \frac{16\xi_0}{3(1+q)t} - - \frac{64(3n^2 + 2n + 1) - 16(2n+1)(3n+1)(1+q)}{3(3n+1)^2(1+q)^2 t^2}. \tag{28}$$

From Eqs. (8)–(12), (19), and (24), the physical and geometrical quantities like V, R, θ, H, σ^2, and Δ are obtained as follows:

$$R = t^{\frac{1}{1+q}}, \tag{29}$$

$$V = t^{\frac{4}{1+q}}, \tag{30}$$

$$\theta = \frac{4}{(1+q)t}, \tag{31}$$

$$H = \frac{1}{(1+q)t}, \tag{32}$$

$$\sigma^2 = \frac{6(n-1)^2}{(3n+1)^2(1+q)^2t^2}, \tag{33}$$

and

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} = (\text{const}). \tag{34}$$

The redshift z defined by $z = -1 + \frac{R_0}{R(t)}$, where $R(t)$ is the average scale factor, and R_0 is the present value of $R(t)$, is found as

$$z = -1 + t^{-\frac{1}{1+q}}, \tag{35}$$

and the EoS parameter $\overline{\omega}$ defined by $\overline{\omega} = \frac{\rho}{\lambda}$ is given by

$$\overline{\omega} = \frac{\frac{4\xi_0}{(1+q)t} + \frac{4(2n+1)(q-3)}{(3n+1)(1+q)^2t^2} + \alpha^2 t^{-\frac{8n}{(3n+1)(1+q)}}}{\frac{4(n-1)(q-3)}{(3n+1)(1+q)^2t^2} + \alpha^2 t^{-\frac{8n}{(3n+1)(1+q)}}} \Rightarrow \overline{\omega} =$$

$$= \frac{4\xi_0(3n+1)(1+q)t + 4(2n+1)(q-3)}{4(n-1)(q-3)+\alpha^2(3n+1)(1+q)^2t^{2-\frac{8n}{(3n+1)(1+q)}}} +$$

$$+ \frac{\alpha^2(3n+1)(1+q)^2t^{2-\frac{8n}{(3n+1)(1+q)}}}{4(n-1)(q-3)+\alpha^2(3n+1)(1+q)^2t^{2-\frac{8n}{(3n+1)(1+q)}}}. \tag{36}$$

Equations (32) and (35) lead to

$$H = H_0(1+z)^{1+q}, \tag{37}$$

where H_0 is the present value of the Hubble parameter.

It is worth to note that we constrain the model parameters of the Universe in the derived model by using 46 $H(z)$ data sets. The complete list of $H(z)$ data points are compiled in Refs. [84, 85]. The quantity χ^2 for $H(z)$ data is read as

$$\chi^2_{H(z)} = \sum_{i=1} \left[\frac{H_{\rm th}(z_i) - H_{\rm obs}(z_i)}{\sigma_i}\right]^2, \tag{38}$$

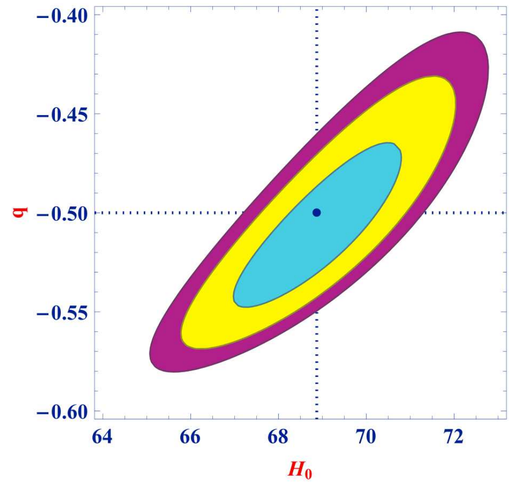


Fig. 1. Two-dimensional contours in the $H_0 - q$ plane at 1σ, 2σ and 3σ confidence regions by bounding our model with $H(z)$ data

where $H_{\rm th}(z_i)$ and $H_{\rm obs}(z_i)$ denote the theoretical and observed values, respectively, and σ_i^2 denotes the standard deviation of each $H_{\rm obs}(z_i)$.

Figure 1 depicts two-dimensional contours at 1σ, 2σ, and 3σ confidence regions by bounding our model with 46 $H(z)$ data sets. We constrained the model parameters H_0 and q as 68.87 km/s/Mpc and −0.50. That is why, we choose $q = -0.50$ for the graphical analysis of various parameters of the derived model.

5. Physical Interpretation of the Solutions

In this section, we have presented the analysis of cosmological parameters to study the nature of the proposed model. For this model, the expressions for the average scale factor R, volume V, and expansion scalar θ, as obtained in Eqs. (29)–(31), show that the model Universe begins with the initial singularity at $t = 0$ from $V = 0$, i.e., our model Universe starts from the zero volume at $t = 0$ with the big-bang. As the time progresses, it is expanding, i.e., when $t \to \infty$, $V \to \infty$. From Eq. (31), it is seen that $\frac{dH}{dt}$ is negative, indicating that the expansion rate in the model is accelerating. But when $t \to \infty$, $\theta \to 0$ showing that the expansion stops at the infinite time. These behaviors of the model parameters are presented in Figs. 2 and 3.

From the value of the redshift given by Eq. (35), it has been observed that $z \to -1$ as $t \to \infty$. This variation can be seen from Fig. 4 depicting the nature of

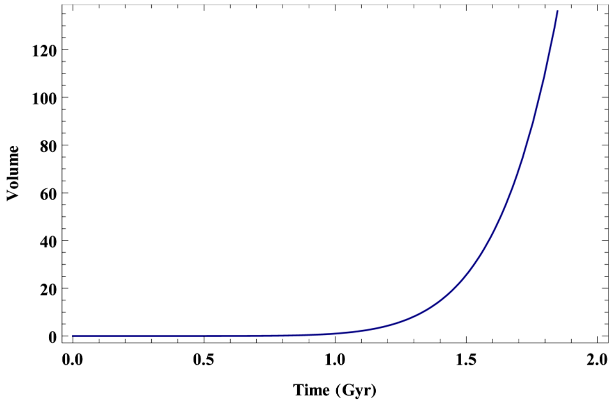


Fig. 2. The graph of volume V vs. cosmic time t for $n = 2$, $q = -0.5, \alpha = -1$ and $\xi_0 = 1$

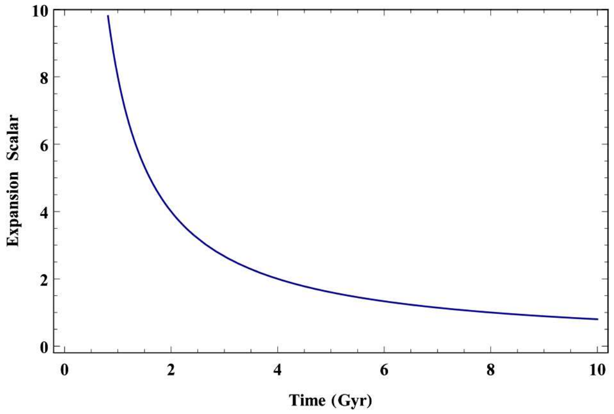


Fig. 3. The graph of expansion factor θ vs. cosmic time t for $n = 2$, $q = -0.5$, $\alpha = -1$ and $\xi_0 = 1$

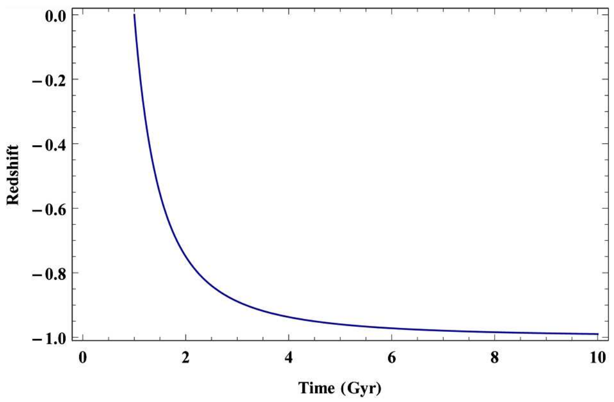
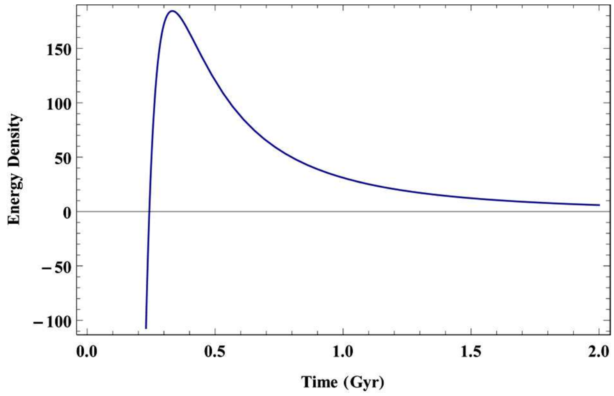
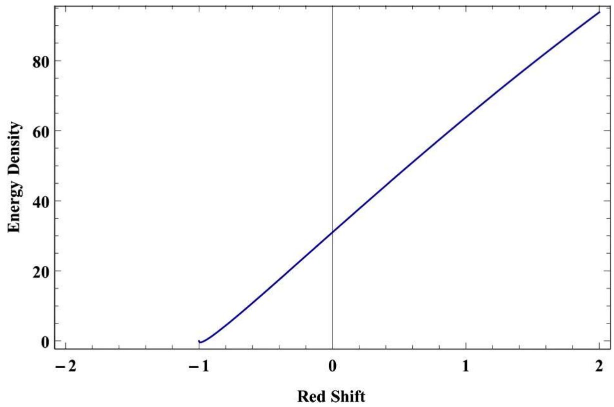


Fig. 4. The graph of redshift z vs. cosmic time t for $n = 2$, $q = -0.5$, $\alpha = -1$ and $\xi_0 = 1$

changes in the redshift z with the changes in the cosmic time. From this result, we can say that our model Universe corresponds to the expanding Universe.



a



b

Fig. 5. The graph of energy density ρ vs. time t (a) and redshift z (b) for $n = 2$, $q = -0.5$, $\alpha = -1$ and $\xi_0 = 1$

Equation (33) gives us an idea about the shear scalar σ which approaches the zero, as $t \to \infty$, explaining a shear-free model Universe at the infinite time. For all values of t, the anisotropy parameter $\Delta \neq 0$ for $n \neq 1$. But if $n = 1$, $\Delta = 0$. Thus, the model Universe is anisotropic for $n \neq 1$ and isotropic, whenever $n = 1$ throughout its evolution.

From the expressions for the energy density ρ and tension density λ given by Eqs. (25) and (26), we have observed that both of them are negative at the initial epoch of the time. But, as the time progresses, they will change the sign from negative to positive and finally becomes zero, when $t \to \infty$. Figures 5, a and b present the variations of the energy density with respect to the cosmic time t and the redshift z, respectively. Figures 5, a and b clearly indicate that, at the infinite time (or when $z \to -1$), $\rho \to 0$. The nature of the variations of the tension density λ versus the time t and the redshift z are, respectively,

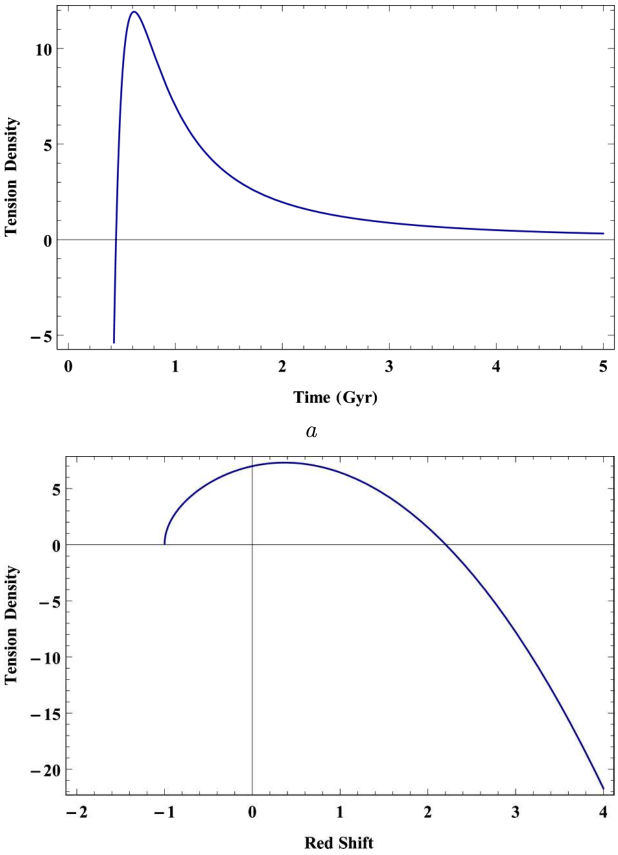


Fig. 6. The graph of tension density λ vs. time t (a) and redshift z (b) for $n = 2$, $q = -0.5$, $\alpha = -1$ and $\xi_0 = 1$

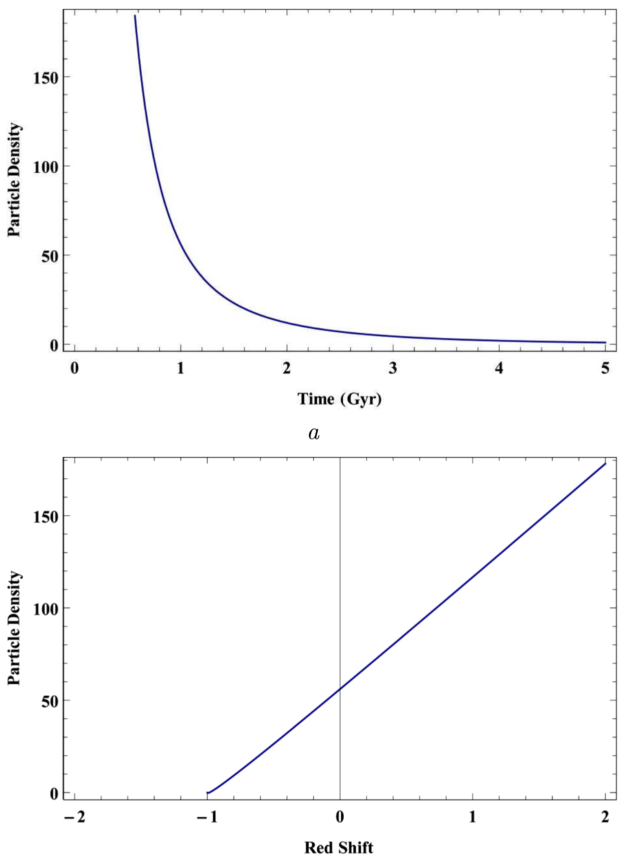


Fig. 7. The graph of particle density ρ_p vs. time t (a) and redshift z (b) for $n = 2$, $q = -0.5$, $\alpha = -1$ and $\xi_0 = 1$

depicted in Figs. 6, a and b. From Figs. 6, a and b, we can conclude that, when $t \to 0$ (or $z \to \infty$), λ is negative. But, with the passage of the cosmic time, it changes the sign from negative to positive. Finally, at the infinite time (i.e., when $z \to -1$), it becomes zero, which is supported by Letelier [2].

For the model Universe, the expression for the particle density ρ_p, as found in Eq. (27), and its variations versus the cosmic time and redshift, respectively, in Figs. 7, *a, b* show that ρ_p is always positive and decreases from $\rho_p = \infty$ as $t = 0$ to $\rho_p = 0$, as $t \to \infty$.

Figure 7 depicts that the tension density diminishes more rapidly than the particle density. Thus, our model begins from the infinite density with a big bang at $t = 0$, and then the strings will disappear leaving the particles only. Hence, our model is a realistic one.

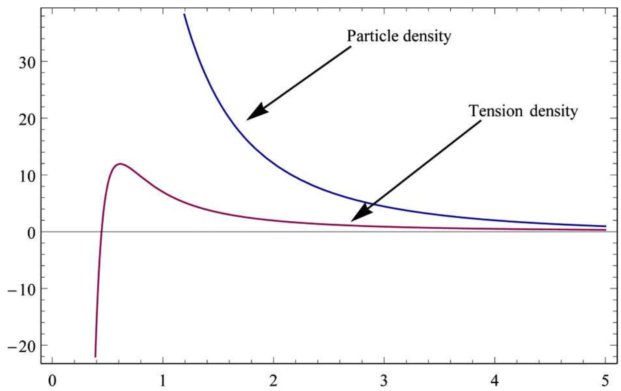


Fig. 8. Comparative variations of λ and ρ_p vs. time t for $n = 2$, $q = -0.5$, $\alpha = -1$ and $\xi_0 = 1$

In this model Universe at the initial epoch of the time, the gauge function β^2 given by Eq. (28) is found to be negative, increases gradually with the time, and

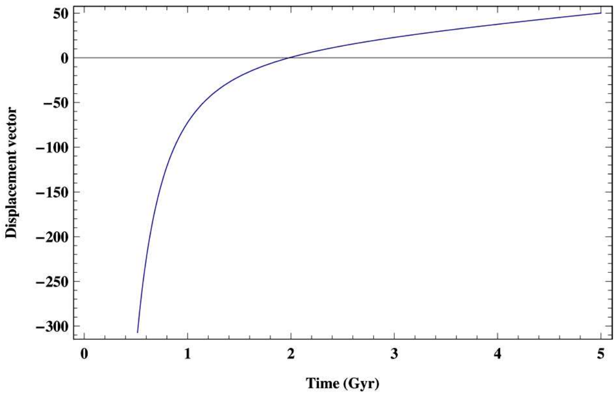
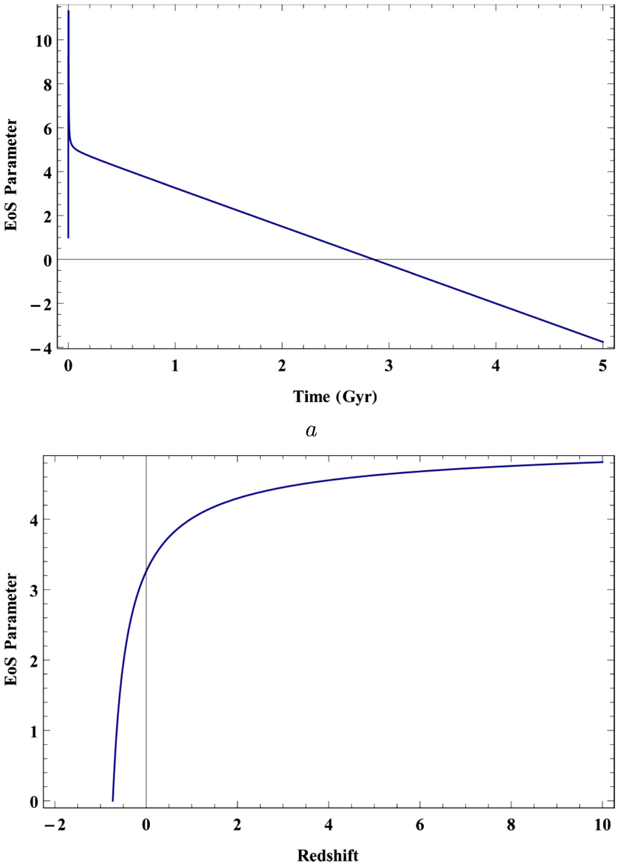


Fig. 9. The graph of displacement vector β vs. time t for $n = 2, q = -0.5, \alpha = -1$ and $\xi_0 = 1$



a

b

Fig. 10. The graph of EoS parameter $\overline{\omega}$ vs. time t (a) and redshift z (b) for $n = 2$, $q = -0.2$, $\alpha = 0.1$ and $\xi_0 = 1$

changes the sign from the negative one to a finite positive value. Then it again decreases, as the time increases. Finally, the gauge function $\beta^2 \to 0$, when $t \to \infty$. This is in agreement with the recent observations of SNeIa ([86–93]).

The variations of the EoS parameter ($\overline{\omega}$) given by Eq. (36) are presented in Figs. 10, a and b with respect to the cosmic time (t) and redshift (z), respectively. From both figures, we observe that the model Universe starts with the matter-dominated era. Then, with the progress of the time, it enters the dark energy era (quintessence type). Finally, it moves toward the phantom era.

6. Conclusion

We have investigated a homogeneous and anisotropic space-time described by the 5-dimensional Bianchi type III metric in the Lyra geometry with a constant deceleration parameter. Exact solutions of Einstein's field equations have been obtained. In this model, the Universe starts evolving with the null volume at the time $t = 0$ with big-bang and is expanding with acceleration. During the evolution of the Universe, it has been also observed that the strings disappear leaving only the particles. The shear scalar σ becomes zero as $t \to \infty$. So, our model represents a shear-free inflationary cosmological model Universe for large values of the cosmic time. It is worth to note that we obtained $n = 1$, which shows that the anisotropy, as well as the shear scalar, vanish, and the evolution of the Universe isotropizes. From Figs. 9 and 10, we also observe that the dynamical behavior of the EoS parameter corresponds to the accelerating phase of the have also observed that the model Universe starts with the matter-dominated era. Then, with the progress of the time, it enters the dark energy era (quintessence type). Finally it moves to the phantom era. In our model, the gauge function β^2 is found to be negative, increases gradually with the time, and changes the sign from negative to a finite positive value. Then it again decreases, as the time increasesS. Finally, it tends to zero in the late time, which is in agreement with the recent observations of SNeIa. It is worth to not that we have obtained $q = -0.50$ to describe the behavior of the physical parameters of the derived model by constraining our model with 46 observational Hubble's data. This shows the consistency of the model with observations. So, our investigation will be of interest to the young researchers to study the evolution of the present-day Universe.

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МАТЕМАТИЧНІ МОДЕЛІ І МЕТОДИ БАГАТОВИМІРНОЇ КОСМОЛОГІЇ СТРУН З ОБ'ЄМНОЮ В'ЯЗКІСТЮ В РАМКАХ ГЕОМЕТРІЇ ЛИРИ

Досліджується космологічний сценарій, за яким еволюціонує сукупність струн та частинок, в рамках геометрії Лири, беручи до уваги п'ятивимірний лінійний елемент Б'янкі типу III. Зроблено два фізично правдоподібні припущення: 1) скаляр зсуву (σ) є пропорційним фактору розширення (θ), що дає $P = Q^n$; $n \neq 0$ є константою, P та Q є масштабними коефіцієнтами; 2) $\xi = \xi_0 = \text{const}$, де ξ є коефіцієнтом об'ємної в'язкості. Знайдено розв'язки для модифікованих польових рівнянь Ейнштейна з однорідною метрикою Б'янкі типу III. Для вивчення запропонованої моделі детально розглянуто поведінку космографічних параметрів для різних значень часу (t) і червоного зсуву (z). Виявлено, що вектор зміщення (β) поводить себе як космологічний член, і розв'язок узгоджується із нещодавніми спостереженнями SNeIa. Знайдено, що об'ємна в'язкість відіграє принципову роль в еволюції Всесвіту, а струни домінують у ранньому Всесвіті і зникають протягом доволі значного часу. Таким чином, наша модель може розглядатися як реалістична.

Ключові слова: геометрія Лири, об'ємна в'язкість, еволюція, ранній Всесвіт, струна.