https://doi.org/10.15407/ujpe68.4.235

A. BOUMALI,[1] R. ALLOUANI,[2] A. BOUZENADA,[3] F. SERDOUK[4]

Laboratory of Theoretical and Applied Physics,
Echahid Cheikh Larbi Tebessi University
*(Algeria; e-mails: [1]boumali.abdelmalek@gmail.com, [2]rihemeallouani72@gmail.com,
[3]abdelmalek.bouzenada@univ-tebessa.dz, abdelmalekbouzenada@gmail.com,
[4]fadila.serdouk@univ-tebessa.dz)*

EFFECT OF THE APPLIED ELECTRIC FIELD ON THE THERMAL PROPERTIES OF THE RELATIVISTIC HARMONIC OSCILLATOR IN ONE DIMENSION

We study the relativistic harmonic oscillators (Dirac and Klein–Gordon ones) in a constant external electric field. The solutions obtained are exact. These solutions allowed us to focus on the effect of the external electric field on the thermal properties of such oscillators. These properties are calculated by means of the Zeta-based method. Some figures have been built to show the mentioned effect.

Keywords: relativistic harmonic oscillator, thermal properties, external applied field, partition function, zeta function.

1. Introduction

The harmonic oscillator has been a basic tool in physics for many centuries. The importance of the harmonic oscillator as a basis for the development of theoretical physics probably appeared with the birth of quantum mechanics. This was the first example to which the quantification rules were applied, and since then its spectra, wave functions, symmetries, *etc.* have had countless applications not only in direct calculations, but also as a model to increase our understanding of more complex problems.

The Dirac oscillator (DO) was, for the first time, studied by Ito *et al.* [1]. They estimated a Dirac equation in which the moment $\boldsymbol{p}$ is replaced by $\boldsymbol{p} - - im\omega\gamma^0\boldsymbol{r}$ of which r is the position vector, m the mass of the particle, and ω the frequency of the oscillator. Interest in the problem was revived by Moshinsky and Szczepaniak [2, 3], who gave it the name of the Dirac oscillator, because, within the non-relativist limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term.

Bruce and Manning [4] were the first who considered a new type of linear interaction in the Klein–Gordon equation known as the Klein–Gordon oscillator (KGO). They introduced this interaction in the same way as in the Dirac oscillator case as follows:

$$\boldsymbol{P} \to \boldsymbol{P} - im\hat{\gamma}\hat{\Omega}\boldsymbol{Q}, \quad \boldsymbol{P} = -i\frac{\partial}{\partial \boldsymbol{q}}, \tag{1}$$

where

$$\boldsymbol{P} = \hat{\eta}\boldsymbol{p}, \quad \boldsymbol{Q} = \hat{\eta}\boldsymbol{q}, \tag{2}$$

with $\hat{\eta}^2 = 1$. Dvoeglazo [5], in the commentary on Ref. [4], mentioned that the physical meaning for implementing $\hat{\gamma}$ and $\hat{\eta}$ matrices in [4] is obscure. He

Citation: Boumali A., Allouani R., Bouzenada A., Serdouk F. Effect of the applied electric field on the thermal properties of the relativistic harmonic oscillator in one dimension. *Ukr. J. Phys.* **68**, No. 4, 235 (2023). https://doi.org/10.15407/ujpe68.4.235.

Цитування: Бумалі А., Аллуані Р., Бузенада А., Сердук Ф. Вплив зовнішнього електричного поля на теплові властивості одновимірного релятивістичного гармонічного осцилятора. *Укр. фіз. журн.* **68**, №4, 234 (2023).

found, when reformulating the Klein–Gordon equation into the Hamiltonian form [6, 7], that the introduction of this interaction was just to replace p^2 by $(\boldsymbol{p}-im\omega\boldsymbol{r})(\boldsymbol{p}+im\omega\boldsymbol{r})$. Recently, following Ref [8] and instead of the definition of Dvoeglazo [5], the authors chose to replace it by $(\boldsymbol{p}+im\omega\boldsymbol{r})(\boldsymbol{p}-im\omega\boldsymbol{r})$. Under this minimal substitution, the wave equation becomes quadratic in both the momentum and the coordinates, vis. a harmonic oscillator for the squared energies. Both oscillators have attracted a lot of interest, both because it is one of the few examples that have accurate solutions, as well as because of its many applications in physics [3, 9–33].

Finally, it should also be noted that these oscillators have been demonstrated experimentally by Franco–Villafane *et al.* [34]. The authors present, for the first time, the experimental realization of the unidimensional Dirac oscillator that can be considered as a paradigm for resolvable relativistic systems. The experiment is are based on a relation between the Dirac oscillator and an associated strong coupling system. This tight-linkage system is implemented as a microwave system through a chain of coupled dielectric discs, where the coupling is evanescent and can be adjusted appropriately. The resonances of the finite microwave system give the spectre of a unidimensional Dirac oscillator with and without the mass term. The flexibility of the experimental assembly enables the implementation of other unidimensional equations like the Dirac one.

In this direction, we also mention the work by Fujiwara *et al.* [35] on the experimental realization of a relativistic harmonic oscillator. They experimented and quantitatively studied a harmonic oscillator in the relativistic regime using ultra-cold lithium atoms in the third band of an optical network. In addition, these oscillators were well designed and realized by the model of a system of trapped atomic ions. Using this model, the Dirac oscillator is well achievable experimentally as predicted by Bermudez [12, 31] and Blatt [32].

In addition, Yang *et al.* [36] proposed to introduce the Dirac oscillator on an entirely relativistic base that possesses all the desired attributes of the ordinary harmonic oscillator while incorporating naturally a strong spin-orbit coupling. This choice is due to the power and flexibility of the base of the Dirac oscillator in solving the problems of a nuclear structure within the framework of the theory of covariant density functional. There results reproduce with great precision those obtained using the Runge–Kutta method and suggest a clear way for a generalization to systems with axial symmetry. Therefore, they suggest expanding their study to the consideration of systems without spherical symmetry, as required in constrained calculations of nuclear excitations.

Now, after briefly introducing the Dirac and Klein–Gordon oscillators, we are ready to set out our main objectives of this paper. These goals can be expressed as follows:

• First, we determine the form of the energy spectrum of the unidimensional KGO in the presence of a constant electric field. Laba and Tkachuk [37] recently reviewed the case of one-dimensional DO. They obtained the exact expression for the energy spectrum using the supersymmetry method (SUSY).

• Through the form of the spectrum of energy, all the thermal properties of the KGO in the presence of a constant electric field can be determined by a method based on the Zeta function [16, 38–41].

• Finally, the thermal properties of DO in a constant electric field are calculated in the same manner as for the above element.

The paper is organized as follows. Section II is devoted to the presentation of solutions for a one-dimensional relativistic harmonic oscillator in the presence of an eternal electric field. In Section III, the thermal properties of these oscillators are established. Finally, Section IV will be a conclusion. In all of the paper, we put $\hbar = c = 1$ and $m\omega = 1$.

2. The (1 + 1)-Dimensional Harmonic Relativistic Oscillators in the Presence of a Constant Electric Field

2.1. A one-dimensional KGO

The general equation for a KGO is [8, 19, 20, 42, 43]

$$\left[(p_x + im\omega x)(p_x - im\omega x) - E^2 + m^2\right]\psi = 0. \tag{3}$$

In the presence of the electric constant field, this equation becomes

$$\left[(p_x + im\omega x)(p_x - im\omega x) - (E - \kappa x)^2 + m^2\right]\psi = 0. \tag{4}$$

The potential $V = q\zeta x = \kappa x$ is associated with the electric field ζ in one dimension with $\kappa = q\zeta$.

Equation (4) can be written in the detailed form as

$$[P_x^2 + \left(m^2\omega^2 - \kappa^2\right) x^2 + 2E\kappa x -$$
$$- E^2 - m\omega + m^2]\psi(x) = 0. \quad (5)$$

By reorganizing all the terms of the equation above, we obtain

$$\left[P_x^2 + \left(m^2\omega^2 - \kappa^2\right)\left(x^2 + \frac{2E\kappa x}{m^2\omega^2 - \kappa^2}\right) -\right.$$
$$\left. - E^2 - m\omega + m^2\right]\psi(x) = 0, \quad (6)$$

or

$$\left[P_x^2 + \left(m^2\omega^2 - \kappa^2\right)\left(x + \frac{E\kappa}{m^2\omega^2 - \kappa^2}\right)^2 -\right.$$
$$\left. - \frac{E^2\kappa^2}{(m^2\omega^2 - \kappa^2)} - E^2 - m\omega + m^2\right]\psi(x) = 0. \quad (7)$$

Putting

$$y = x - x_0, \quad (8)$$

with

$$x_0 = -\frac{E\kappa}{m^2\omega^2 - \kappa^2}, \quad (9)$$

(7) becomes

$$\left[\frac{P_y^2}{2m} + \frac{1}{2}m\left(\frac{m^2\omega^2 - \kappa^2}{m^2}\right)y^2\right]\psi(y) = \tilde{E}\psi(y), \quad (10)$$

with

$$\tilde{E} = \frac{1}{2m}\left[\frac{E^2m^2\omega^2}{(m^2\omega^2 - \kappa^2)} + m\omega - m^2\right]. \quad (11)$$

Here, x_0 can be interpreted as the equilibrium position of the oscillator in question.

Equation (10) refers to the differential equation for a standard harmonic oscillator in one dimension. So, the eigenvalues are given by

$$\tilde{E} = \frac{1}{2m}\left[\frac{E^2m^2\omega^2}{(m^2\omega^2 - \kappa^2)} + m\omega - m^2\right] =$$
$$= \sqrt{\frac{m^2\omega^2 - \kappa^2}{m^2}}\left(n + \frac{1}{2}\right). \quad (12)$$

The new frequency of the oscillator in the presence of the electric field, ω' reads as follows:

$$\omega' = \omega\sqrt{1 - \left(\frac{\kappa}{m\omega}\right)^2}. \quad (13)$$

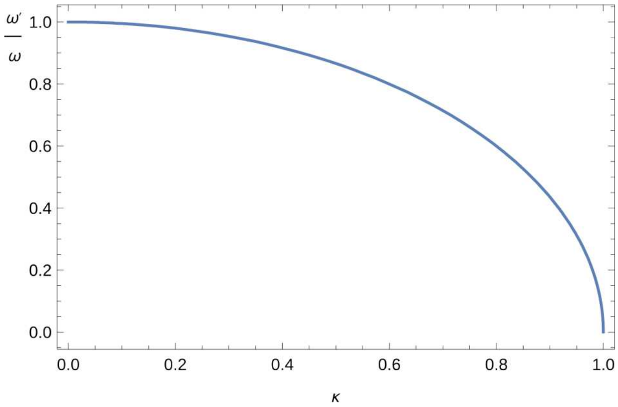

Fig. 1. The variation of the ratio $\frac{\omega'}{\omega}$ with the parameter κ

Figure 1 shows the variation of the ratio $\frac{\omega'}{\omega}$ with the parameter κ. In the absence of an electric field represented by the $\kappa = 0$ condition, the two frequencies are equal. Now, when the field increases, the ratio decreases until it is cancelled at $\kappa = 1$. This situation can be argued in the following way: in the very strong electric field, a very strong force emerges and acts very strongly on the oscillator. Therefore, there is no chance of getting the oscillator back.

Now, rearranging all the terms of Eq. (12), we reach the form of the energy spectrum as

$$E_n(r, a) =$$
$$= \pm m\sqrt{r\left(1 - a^2\right)^{3/2}(2n + 1) + (1 - r)\left(1 - a^2\right)}, \quad (14)$$

with $a = \frac{\kappa}{m\omega}$ and $r = \frac{\omega}{m}$ to be a parameter which controls the nonrelativistic limit. Taking $m\omega = 1$, Eq. (14) becomes

$$E_n = \pm m\sqrt{r\left(1 - \kappa^2\right)^{3/2}(2n + 1) + (1 - r)\left(1 - \kappa^2\right)}. \quad (15)$$

Equation (15) shows the energy spectrum of the one-dimensional KGO in the presence of an electric field. In the absence of the field ($\kappa = 0$), the usual one-dimensional KGO energy spectrum is recovered [16]. Now, when the electric field value is higher than the critical value $|\kappa| > 1$, eigenvalues become complex, which means that the bounded eigenstates are not present.

Before discussing the thermal properties of the KGO, let us briefly review the proper values of the one-dimensional (1D) DO in the presence of an electric field.

2.2. The one-dimensional DO

The general equation for the $(1+1)$-dimensional DO reads [16]

$$[\sigma_x (p_x - im\omega x\sigma_z) + \sigma_z]\,\psi_D = \varepsilon\psi_D, \tag{16}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17}$$

are Pauli matrices, and $\psi_D = (\psi_1, \psi_2)^T$. In the presence of the exterior electric field, the eigenvalue equation for the DO is given by

$$[\sigma (p_x - im\omega x\sigma_z) + \sigma_z]\,\psi_D = (\varepsilon - \kappa x)\,\psi_D. \tag{18}$$

Using the new transformation [37]

$$\psi_D = (\sigma_x p_x - \sigma_y m\omega x + m\sigma_z + (\varepsilon - \kappa x))\,\bar{\psi}_D, \tag{19}$$

Eq. (19) becomes

$$[p_x^2 + m^2\omega^2 x^2 + m^2 - (\varepsilon - \kappa x)^2 + \\ + (-m\omega\sigma_z + i\kappa\sigma_x)]\bar{\psi}_D = 0. \tag{20}$$

Putting now

$$\bar{\psi}_D = \chi\phi\,(x)_D, \tag{21}$$

where χ is the spin part of the wave function satisfying the equation [37]:

$$(-\sigma_z + i\kappa\sigma_x)\,\chi = \lambda\chi. \tag{22}$$

Using (17), the eigenvalues of (22) are

$$\lambda = \pm\sqrt{1-\kappa^2}, \tag{23}$$

and the Eq. (20) is transformed into

$$\left[p_x^2 + m^2\omega^2(1-\kappa^2)x^2 + 2m\omega\varepsilon\kappa x - m\omega\sqrt{1-\kappa^2}\right] \times \\ \times \phi = \left(\varepsilon^2 - m^2\right)\phi. \tag{24}$$

This equation can be cast into

$$\left(p_x^2 + m^2\omega^2\left(1-\kappa^2\right)(x-x_0)^2\right)\phi = \tilde{\epsilon}\phi, \tag{25}$$

where

$$\tilde{\epsilon} = \frac{\varepsilon^2}{1-\kappa^2} - m^2 + m\omega\sqrt{1-\kappa^2}, \tag{26}$$

and

$$x_0 = -\frac{\kappa\varepsilon}{m\omega\,(1-\kappa^2)},\quad \omega' = \omega\sqrt{1-\kappa^2}. \tag{27}$$

Equation (10) describes the differential equation for the standard harmonic oscillator in one dimension. The eigenvalues are given by

$$\varepsilon_n = \pm m\sqrt{(1-\kappa^2)\left(2r\sqrt{1-\kappa^2}n + 1\right)}, \tag{28}$$

with $r = \frac{\omega}{m}$.

Equation (14) shows the energy spectrum of the 1D DO in the presence of an electric field. In the absence of a field $\kappa = 0$, we recover the well-known energy spectrum of a one-dimensional DO [16]. It is of interest to note that when the value of the electric field is larger than the critical one $|\kappa| > 1$, the bounded eigenstates are absent.

2.3. Results and Discussion

Figures 2 and 3 show the energy spectrum E_n as a function of both the quantum number n and the parameter κ for the two oscillators. In the light of these figures, the following observations may be made:

- There is no degeneration of the energy spectrum.
- For the curves, where the variation is with the quantum number n, they are all below the curve of the habitual relativistic harmonic oscillator, when $\kappa = 0$.
- Now, for the curves, where the variation is a function of the field via the parameter κ, all curves converge at $\kappa = 1$, and bounded states with $\kappa > 1$ are not permitted. When the electric field value is greater than $\kappa = 1$ (critical value), the energy spectrum becomes imaginary. When $\kappa = 1$, the energy spectrum $E_n = 0$. This state corresponds to the vacuum state and signifies that the bound states in this region are not allowed, and any states can not exist, when $\kappa \geq 1$.
- To argue the above item, we have $\kappa = e\zeta = 1$ which implies an electric field value around 10^{18} V/m. As we know, the large electric fields produce electron-positron pairs by means of the so-called Schwinger effect [44]. The Schwinger effect is a physics-predicted phenomenon by which the matter is created by a strong electric field. When the electric field is high enough, near the Schwinger limit 10^{18} V/m, the production by pair drains the energy of the field, which reduces it. Electric fields larger than the Schwinger limit are unstable, since they would “decay” into charge pairs. Based on this argument, we can understand why the energy spectrum of the two Klein–Gordon and Dirac oscillators is not allowed in the $\kappa > 1$ region. In this region, the sufficiently strong electric

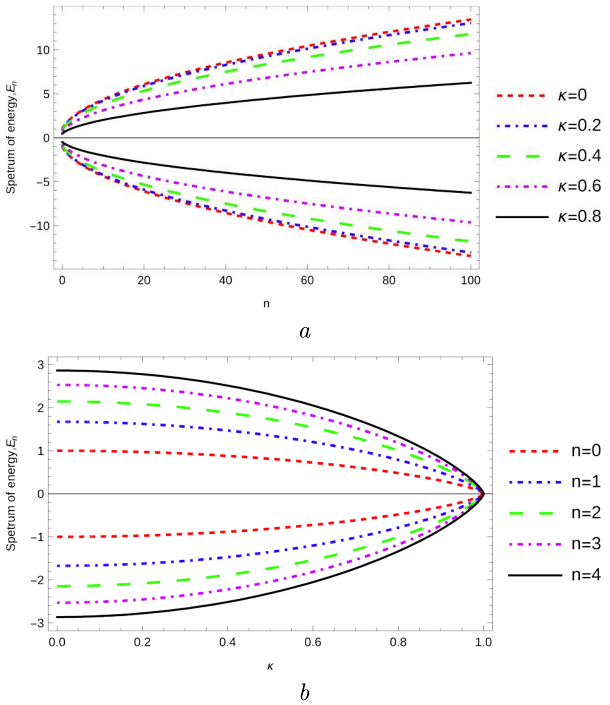

Fig. 2. Spectrum of energy of the 1D KGO in the presence of an external electric field: here, $r = 0.9$. Spectrum of energy E_n versus: the quantum number n (a) and the parameter κ (b)

field destroys the bounded states. So, there are no bounded eigenstates.

Moreover, our results indicate that we have obtained the equilibrium positions of the two oscillators x_0. Figures 4 and 5 present the equilibrium position x_0 as a function of n and κ for the two oscillators. Based on these figures, we can summarize our results as follows:

• The equilibrium positions of both oscillators are quantified, contrary to the usual case. The reason for this quantification is that an external electric field is present.

• Each energy level oscillates according to its own equilibrium position.

• In both $\kappa = 0$ and $\kappa = 1$ boundary cases, we have: (i) the first boundary in which $\kappa = 0$ gives us a relativistic oscillator in the absence of an electric field: in that case, all the levels have the same equilibrium point at $x = 0$. (ii) In contrast, at the other boundary, where $\kappa = 1$, the energy spectrum of the two oscillators is zero (unstable region as described above).

• Finally, the equilibrium position of the two oscillators shows two signs: the positive sign corresponds

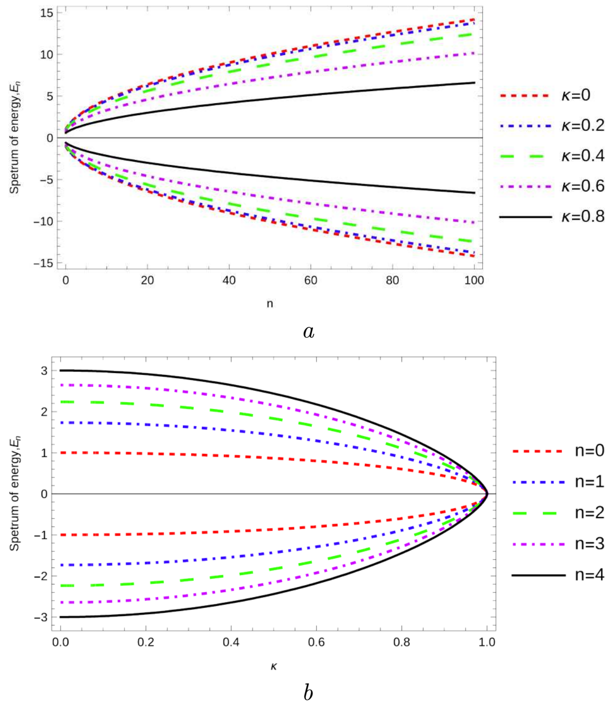

Fig. 3. Spectrum of energy of 1D DO in the presence of an external electric field: here, $r = 0.9$. Spectrum of energy E_n versus the quantum number n (a) and the parameter κ (b)

to the antiparticles, but the negative sign corresponds to the particles.

After discussing the results obtained from the eigensolutions, we are ready to discuss the effect of an applied electric field on the thermal properties of the Klein–Gordon and Dirac oscillators in one dimension.

3. The Thermodynamic Properties of the Relativistic Harmonic Oscillator in One Dimension

3.1. Methods

First, we focus on calculating the partition function Z in order to obtain all thermodynamic quantities of the relativistic oscillators in question. The partition function is defined by

$$Z = \sum_n e^{-\beta E_n} = \sum_n e^{-\beta a\sqrt{n+b}}. \tag{29}$$

In order to evaluate the above partition function, we use the formula of the Mellin transformation [16, 38, 45]. The Mellin transformation $\mathcal{M}$ is the operation mapping the function f into the function F defined

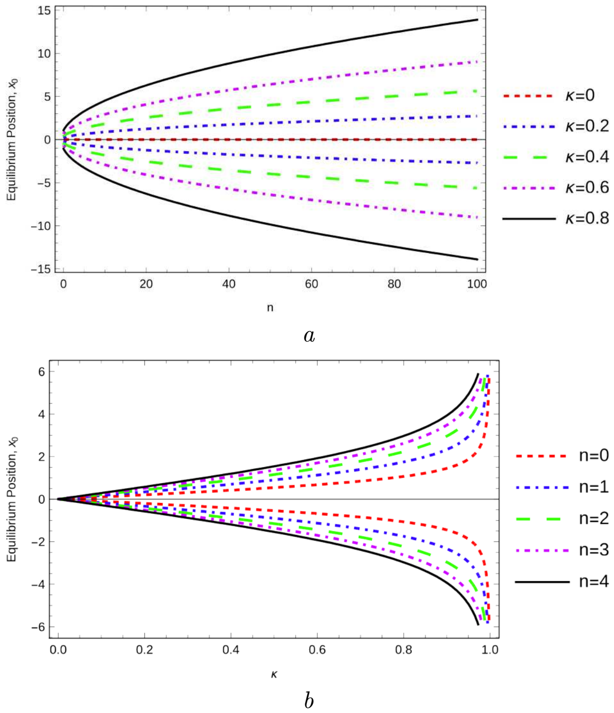

Fig. 4. Equilibrium position of 1D KGO in the presence of an external electric field: here, $r = 0.9$. Equilibrium position versus the quantum number n (a) and the parameter κ (b)

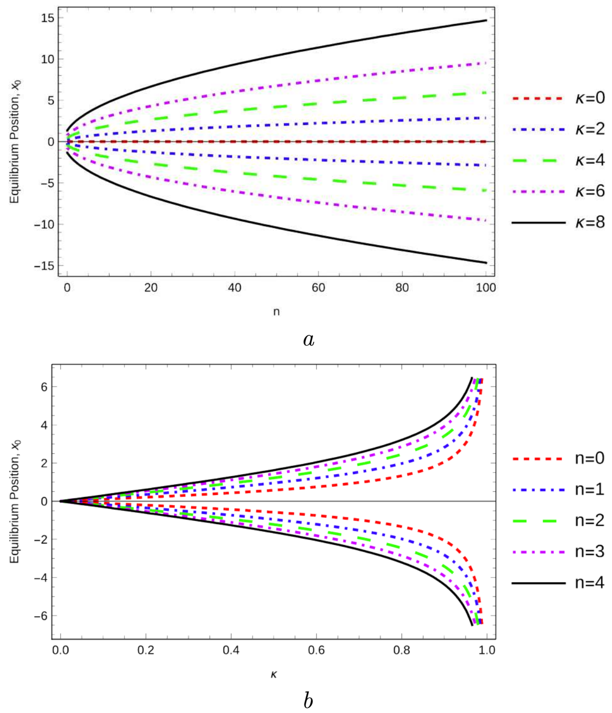

Fig. 5. Equilibrium position of 1D DO in the presence of an external electric field: here, $r = 0.9$. Equilibrium position versus the quantum number n (a) and the parameter κ (b)

on the complex plane by the relation [46]:

$$\mathcal{M}\left[f,s\right] \equiv F\left(s\right) = \int\limits_{0}^{\infty} x^{s-1} f\left(x\right) dx. \tag{30}$$

The inverse transform is

$$\left\{\mathcal{M}F\right\}^{-1}\left(x\right) \equiv f\left(x\right) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} x^{-s} F\left(s\right) ds. \tag{31}$$

In the case of the exponential function, the Mellin transform of e^{-x} is

$$\Gamma\left(s\right) = \int\limits_{0}^{\infty} x^{s-1} e^{-x} dx, \tag{32}$$

where $\Gamma\left(s\right)$ is the gamma function. The inverse transform gives

$$e^{-x} = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} ds x^{-s}\Gamma(s), \tag{33}$$

where $c \geq 0$ is real. This integral is known as the Cahen–Mellin integral [38].

The sum in (29) is transformed into an integral as follows:

$$\sum_{n} e^{-\beta a\sqrt{n+b}} = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} ds\left(\beta a\right)^{-s}\sum_{n}\left(b+n\right)^{\frac{-s}{2}}\times$$

$$\times\Gamma\left(s\right) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} ds\left(\beta a\right)^{-s}\zeta_H\left(\frac{s}{2},b\right)\Gamma\left(s\right), \tag{34}$$

with $x = \beta a\sqrt{n+b}$, $\Gamma(s)$ and $\zeta_H\left(\frac{s}{2},b\right)$ are, respectively, the Euler and Hurwitz zeta function [38]. As is known, the gamma function [47] has simple poles at the non-positive integers for all $s = -n$, $n \in \mathbb{N}$, and their residues at negative poles are

$$\mathrm{Re}\left\{\Gamma,-n\right\} = \frac{\left(-1\right)^{n}}{n!}. \tag{35}$$

On the other hand, the Hurwitz zeta function $\zeta_H\left(s',b\right) = \sum_{n=0}^{\infty}\frac{1}{(n+b)^{s'}}$ [38]. It is a series that converges only when $\mathcal{R}\left(s'\right) > 1$ and $\mathcal{R}\left(\alpha\right) > 0$. It can be extended by analytic continuation to a meromorphic function defined for all complex numbers s' with $s' \neq 1$. At $s' = 1$, it has a simple pole with residue 1. In addition, the condition on the convergence of the series is $s' = s/2 \geq 1$ that implies $c \geq 2$. The integration in (34) can be closed only in the negative parts of the complex plane, where the poles $s = \{0, 2, \mathbb{Z}^-\}$. So, applying the theorem of residues to Eq. (34) gives [48] (see Fig. 6)

$$\sum_n e^{-\beta a\sqrt{n+b}} = (\beta ma)^{-2} + \zeta_H(0, 1+b) + \\ + \sum_{n=1}^{\infty} \zeta_H\left(-\frac{n}{2}, 1+b\right) \frac{(-\beta ma)^n}{n!}. \quad (36)$$

The desired partition function Z is written down in terms of the Hurwitz zeta function using the theorem of residues for the poles $s = \{0, 2, \mathbb{Z}^-\}$ by [48]

$$Z(\beta) = (\beta a)^{-2} + \zeta_H(0, 1+b) + \\ + \sum_{n=1}^{\infty} \zeta_H\left(-\frac{n}{2}, b\right) \frac{(-1\beta a)^n}{n!}. \quad (37)$$

Equation (37) can be written for both relativistic oscillators as follows:

• For the case of 1D KGO with the substitutions

$$a = \sqrt{2r}\left(1-\kappa^2\right)^{3/4}, \quad b = 1 + \frac{(1-r)}{2r\sqrt{1-\kappa^2}}, \quad (38)$$

and Eq. (37) becomes

$$Z_{\text{KGO}}(\beta, \kappa) = \frac{1}{2r\left(1-\kappa^2\right)^{3/2}} \frac{1}{\beta^2} + \\ + \zeta_H\left(0.2 + \frac{(1-r)}{2r\sqrt{1-\kappa^2}}\right) + \frac{(1-r)}{r\sqrt{1-\kappa^2}} + \\ + \sum_{n=1}^{\infty} \zeta_H\left(-\frac{n}{2}, 2 + \frac{(1-r)}{2r\sqrt{1-\kappa^2}}\right) \times \\ \times \frac{\left(-\beta\sqrt{2r}\left(1-\kappa^2\right)^{3/4}\right)^n}{n!}. \quad (39)$$

• Now, in the the case of 1D DO, we have

$$a = \sqrt{2r}\left(1-\kappa^2\right)^{3/4}, \quad b = \frac{1}{2r\left(1-\kappa^2\right)^{1/2}}, \quad (40)$$

that gives

$$Z_{\text{DO}}(\beta, \kappa) = \frac{1}{2r\left(1-\kappa^2\right)^{3/2}} \frac{1}{\beta^2} + \\ + \zeta_H\left(0, \frac{1}{2r\left(1-\kappa^2\right)^{1/2}}\right) + \\ + \sum_{n=1}^{\infty} \zeta_H\left(-\frac{n}{2}, 1 + \frac{1}{2r(1-\kappa^2)^{1/2}}\right) \times \\ \times \frac{\left(-\beta\sqrt{2r}(1-\kappa^2)^{3/4}\right)^n}{n!}. \quad (41)$$

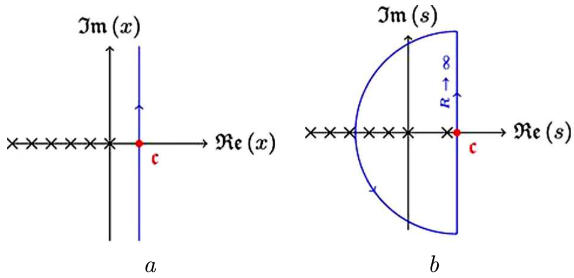

Fig. 6. Contour of both Γ and $\xi_H(n, s)$ functions (a). Contour of the Mellin transformation of our problems (b). (Figure from Ref. [48])

Now, having determined the partition function for the two relativistic oscillators, all the thermal properties of these oscillators can be determined. The most important thermal quantities, namely, free energy F, mean energy U, entropy S, and specific heat C_v may be obtained from the following expressions:

$$\frac{F}{m} = -\tau \ln(z), \quad \frac{U}{m} = \tau^2 \frac{\partial \ln(z)}{\partial \tau}, \quad (42)$$

$$\frac{S}{k_b} = \ln(z) + \tau \frac{\partial \ln(z)}{\partial \tau}, \\ \frac{C_v}{k_b} = 2\tau \frac{\partial \ln(z)}{\partial \tau} + \tau^2 \frac{\partial^2 \ln(z)}{\partial \tau^2}, \quad (43)$$

where $\tau = \frac{1}{\beta m}$ corresponds to the reduced temperature, and k_b is the Boltzmann constant.

3.2. Results and discussion

It is worth to mote that we have limited ourselves to stable states with positive energy, because the two oscillators have an exact Foldy–Wouthuysen (FWT) transformation [49] which is one of the corner stones of relativistic quantum mechanics. It offers a simple and practical way to get an adequate physical interpretation of relativistic wave equations. The very existence of the exact FWT for the considered relativistic problem justifies its quantum mechanical treatment, inasmuch as, in this case, the transitions between positive and negative energy states are forbidden [50]. Therefore, positive and negative energy solutions never mix. Consequently, we assume that

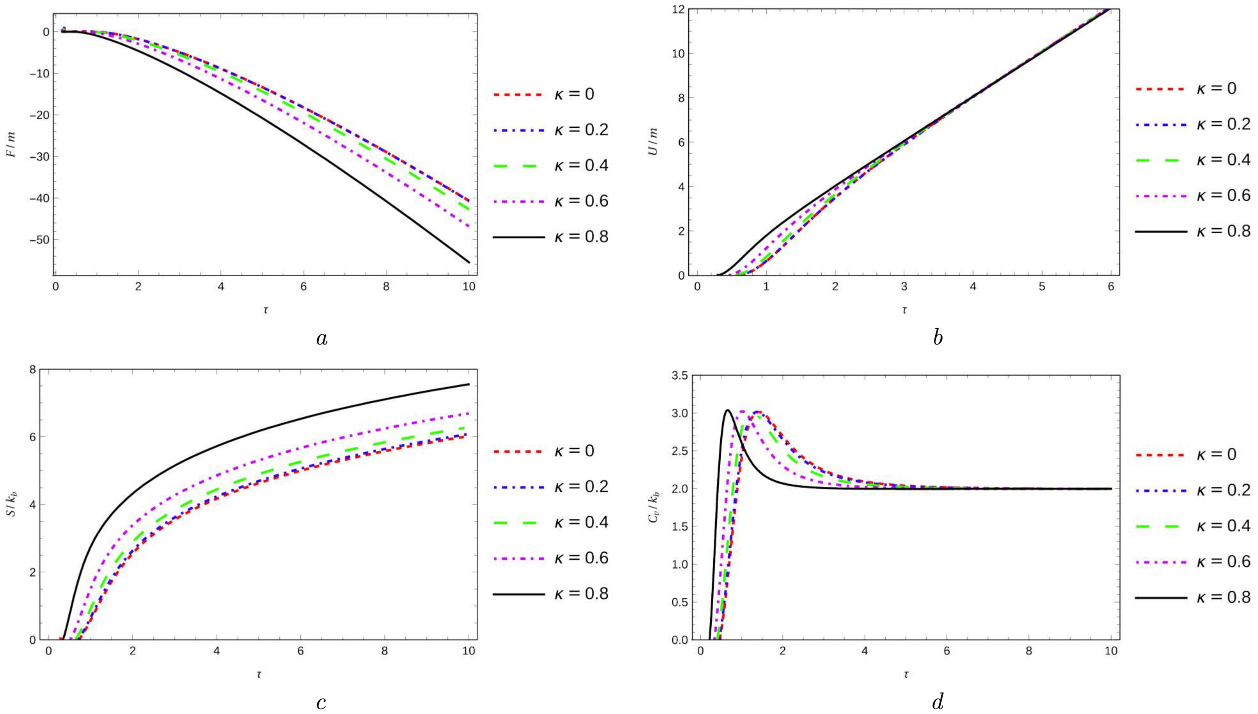

Fig. 7. Thermal properties of 1D KGO as a function of τ in the presence of an electric field: here, $r = 0.9$. Free energy F/m versus τ (a), total energy U/m versus τ (b), entropy S/k_b versus τ (c), specific heat C_v/k_b versus τ (d)

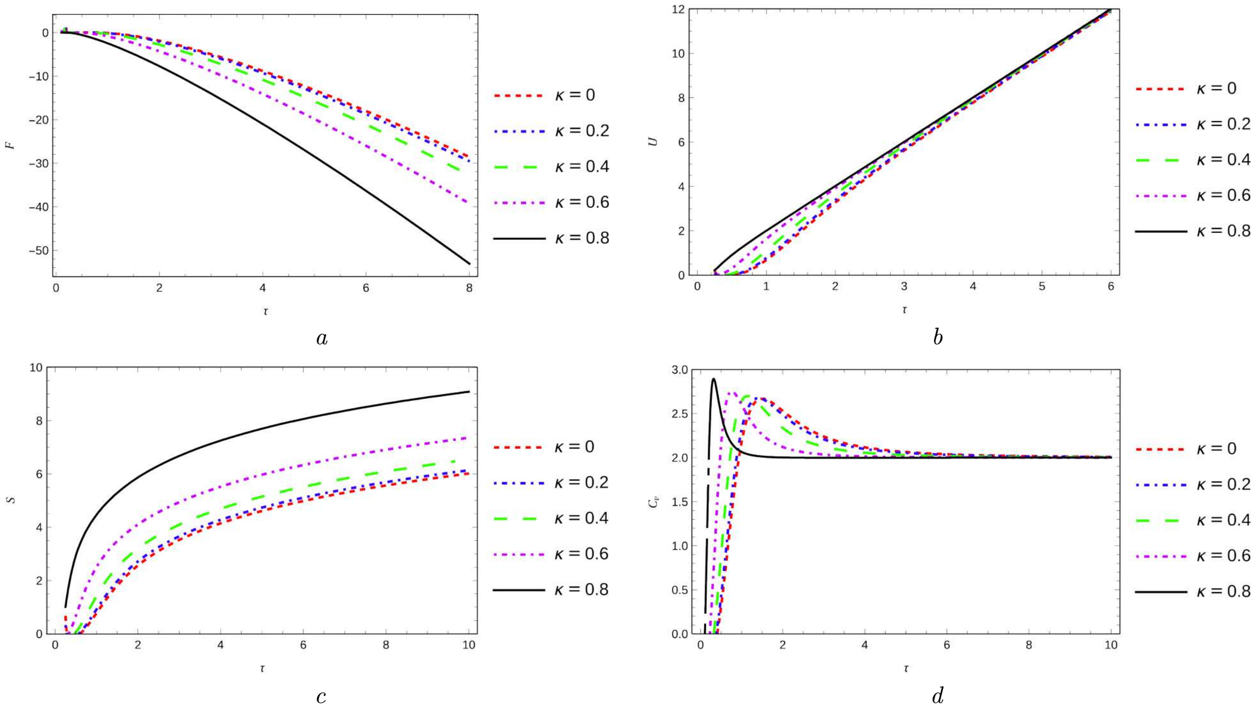

Fig. 8. Thermal properties of 1D DO as a function of τ in the presence of an electric field: here, $r = 0.9$. Free energy F/m versus τ (a), total energy U/m versus τ (b), entropy S/k_b versus τ (c), specific heat C_v/k_b versus τ (d)

only positive-energy particles are available to determine the thermodynamic properties of the oscillators in question.

Figures 7 and 8 represent the one-dimensional thermal properties of the Dirac and Klein–Gordon oscillators, when the electric field is present. The values of this field are limited to the interval $0 < \kappa < 1$ in order to keep the energy spectrum real. All the thermal quantities are plotted versus the reduced temperature τ. As can be seen, both oscillators exhibit the same behavior. In order to understand the influence of the field on these quantities, the case of relativistic oscillators was added in the absence of an applied electric field. The effect occurs in the region of the strong electric field about 10^{18} V/m (Schwinger effect). In the region, where $\kappa \to 0$ (lower electric field), the energy spectrum of the two oscillators recovers the well-known relation as in Ref. [16]. Therefore, the effect of the field in that area is not present.

For the following, we consider only the area, where the electric field is strong. According to these figures (Figs. 7 and 8), we observe the following:

• The free energy curves $\frac{F}{m}$ vs the reduced temperature τ are computed for various values of the parameter κ. They show that if the electric field increases, the curves are lower than in the usual case, when $\kappa = 0$.

• The curves of the total energy $\frac{U}{m}$ vs the reduced temperature τ are computed for various values of the parameter κ. In these figures, there are two regions: in the region, where $\tau < 3$, the influence of the electric field is observed. However, in another region, all curves coincide, which testifies to the beginning of the saturation regardless of the field value. Furthermore, when $\kappa \to 1$, the behavior of the total energy becomes linear.

• For the curves of the numerical entropy function $\frac{S}{k_b}$, no sudden changes near a certain temperature observed in the specific heat curves were identified. This means that the curvature, observed in the specific heat curve $\frac{C_v}{k_b}$, does not exhibit or indicate the existence of a phase transition at this temperature. Furthermore, the saturation phenomena may be observed in the specific heat curves for both oscillators.

• All specific heat curves tend toward the same limit 2 at high temperatures: it is the Dulong–Petit law for an ideal relativistic gas [9, 51].

All these results are explained in the following way: in the zone, where the electric field is very strong, the number of allowed states has decreased. When κ changes from zero to one, the energy gap in the energy spectrum curves decreases to $E_n = 0$, which corresponds to the vacuum state. This explains why the saturation on specific heat curves is very fast for κ values mear 1.

To conclude, our problem is a very interesting one both theoretically and experimentally: theoretically, it provides a realistic model that, for example, has exact solutions. But, on the experimental level, our oscillators are well conceived and experimentally produced by Franco–Villafane *et al.* [34].

4. Conclusion

We have attempted to deduce the thermodynamic properties of both Klein–Gordon and Dirac oscillators in the presence of a constant electric field. Both oscillators have demonstrated their usefulness in different models of modern physics. The influence of the electric field on them is well established. This effect may be summarized as follows.

• In the region, where $0 < \kappa < 1$, the spectrum of energy decreases, when the electric field increases. In a vicinity of $\kappa = 1$, (very strong electric field), the spectrum tends to zero (vacuum state). This case is analogous to the famous Schwinger effect.

• In the region, where $\kappa > 1$, these energy spectra become complex: the bounded eigenstates are missing (unstable state). Thus, the sufficiently strong electric field destroys the bounded states of the two oscillators.

Based on the form of the energy spectrum, the thermal function curves are constructed using the zeta-based method. The effect of the field on these features in the region $0 < \kappa < 1$ is well observed for both oscillators.

1. D. Ito, K. Mori, E. Carriere. An example of dynamical systems with linear trajectory. *Nuovo Cimento A* **51**, 111, (1967).
2. M. Moshinsky, A. Szczepaniak. The Dirac oscillator. *J. Phys. A: Math. Gen.* **22**, L817 (1989).
3. C. Quesne, M. Moshinsky. Symmetry Lie algebra of the Dirac oscillator. *J. Phys. A: Math. Gen.* **23**, 2263 (1990).
4. S. Bruce, P. Minning. The Klein–Gordon oscillator. *II Nuovo Cimento.* **106**, 711 (1993).
5. V.V. Dvoeglazo. Comment on “the Klein–Gordon oscillator” by S. Bruce and P. Minning. *II Nuovo Cimento.* **107**, 1411 (1994).

6. M. Taketani, S. Sakata. On the wave equation of meson. *Proc. Phys. Math. Soc.* **22**, 757 (1940).
7. H. Feshbach, F. Villars. Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles. *Rev. Mod. Phys.* **30**, 24 (1958).
8. A. Boumali, A. Hafdallah, A. Toumi. Comment on "Energy profile of the one-dimensional Klein–Gordon oscillator". *Phys. Scr.* **84**, 1 (2011).
9. A. Boumali, H. Hassanabadi. The thermal properties of a two-dimensional Dirac oscillator under an external magnetic field. *Eur. Phys. J. Plus* **128**, 124 (2013).
10. B.P. Mandal, S. Verma. Dirac oscillator in an external magnetic field. *Phys. Lett. A* **374**, 1021 (2010).
11. B.P. Mandal, S.K. Ray. Noncommutative Dirac oscillator in an external magnetic field. *Phys. Lett. A* **376**, 2467 (2012).
12. A. Bermudez, M.A. Martin-Delgado, A. Luis. Nonrelativistic limit in the $2+1$ Dirac oscillator: A Ramsey-interferometry effect. *Phys. Rev. A* **77**, 063815 (2008).
13. O. Bertolami, J.G. Rosa, C.M.L. deArago, P. Castorina, D. Zappala. Noncommutative gravitational quantum well. *Phys. Rev. D* **72**, 025010 (2005).
14. A. Boumali, L. Chetouani. Exact solutions of the Kemmer equation for a Dirac oscillator. *Phys. Lett. A* **346**, 261 (2005).
15. A. Boumali. One-dimensional thermal properties of the Kemmer oscillator. *Phys. Scr.* **76**, 669 (2007).
16. A. Boumali. The one-dimensional thermal properties for the relativistic harmonic oscillators. *EJTP* **12**, 1 (2015).
17. A. Boumali, F. Serdouk, S. Dilmi. Superstatistical properties of the one-dimensional Dirac oscillator. *Physica. A* **553**, 124207 (2020).
18. H. Hassanabadi, S.S. Hosseini, A. Boumali, S. Zarrinkamar. The statistical properties of Klein-Gordon oscillator in noncommutative space. *J. Math. Phys.* **55**, 033502 (2014).
19. B. Mirza, M. Zarei. Non-commutative quantum mechanics and the Aharonov-Casher effect. *Eur. Phys. J. C* **32**, 583 (2004).
20. B. Mirza, R. Narimani, M. Zarei. Relativistic oscillators in a noncommutative space and in a magnetic field. *Eur. Phys. J. C* **48**, 641 (2006).
21. S.K. Moayedi, F. Darabi. Exact solutions of Dirac equation on a 2D gravitational background. *Phy. Lett. A* **322**, 173 (2004).
22. H. Motavalli, A.R. Akbarieh. Klein–Gordon equation for the coulomb potential in noncommutative space. *Mod. Phys. Lett. A* **25**, 2523 (2010).
23. Y. Nedjadi. R.C. Barrett. A generalized Duffin–Kemmer–Petiau oscillator. *J. Phys. A: Math.Gen.* **31**, 6717 (1998).
24. E. Sadurni, J.M. Torres, T.H. Seligman. Dynamics of a Dirac oscillator coupled to an external field: a new class of solvable problems. *J. Phys. A: Math. Theor.* **43**, 285204 (2010).
25. E.S. Santos, G.R. deMelo. The schrπ̈Sdinger and Pauli–Dirac oscillators in noncommutative phase space. *Int. J. Theor. Phys.* **50**, 332 (2011).
26. S. Sargolzaeipor, H. Hassanabadi, A. Boumali. Morse potential of the q-deformed in the Duffin–Kemmer–Petiau equation. *Int. J. Geom. Method. Mod. Phys.* **14**, 1750112 (2017).
27. J. Wang, K. Li. Klein–Gordon oscillators in noncommutative phase space. *Chinese. Phys. C* **32**, 803 (2008).
28. B.C. Lutfuoglu, J. Kriz, P. Sedaghatnia, H. Hassanabadi. The generalized Klein–Gordon oscillator in a cosmic space-time with a space-like dislocation and the Aharonov–Bohm effect. *Eur. Phys. J. Plus.* **135**, 691 (2020).
29. B. Hamil, B.C. Lutfuoglu. Dunkl–Klein–Gordon equation in three-dimensions: The Klein–Gordon oscillator and Coulomb potential. *Few-Body. Syst.* **63**, 74 (2022).
30. B. Hamil, B.C. Lutfuoglu. Thermal properties of relativistic Dunkl oscillators. *Eur. Phys. J. Plus* **137**, 812 (2022).
31. A. Bermudez, M.A. Martin-Delgado, E. Solano. Exact mapping of the 2+1 Dirac oscillator onto the Jaynes–Cummings model: Ion-trap experimental proposal. *Phys. Rev. A* **76**, 041801 (2007).
32. R. Blatt, C.F. Roos. Quantum simulations with trapped ions. *Nature. Phys.* **8**, 277 (2012).
33. M.H. Pacheco, R.R. Landim, C.A.S. Almeida. One-dimensional Dirac oscillator in a thermal bath. *Phys. Lett. A* **311**, 93 (2003).
34. J.A. Franco-Villafane, E. Sadurni, S. Barkhofen, U. Kuhl, F. Mortessagne, T.H. Selig-man. First experimental realization of the Dirac oscillator. *Phys. Rev. Lett.* **111**, 170405 (2013).
35. K.M. Fujiwara, Z.A. Geiger, K. Singh, R. Senaratne, S.V. Rajagopal, M. Lipatov, T. Shimasaki, D.M. Weld. Experimental realization of a relativistic harmonic oscillator. *New. J. Phys.* **20**, 063027 (2018).
36. J. Yang, J. Piekarewicz. Dirac oscillator: An alternative basis for nuclear structure calculations. *Phys. Rev. C* **102**, 054308 (2020).
37. H.P. Laba, V.M. Tkachuk. Exact energy spectrum of the generalized Dirac oscillator in an electric field. *Eur. Phys. J. Plus* **133**, 279 (2018).
38. E. Elizalde. *Ten Physical Applications of Spectral Zeta Functions* (Springer-Verlag, 1995).
39. A. Boumali. Thermodynamic properties of the graphene in a magnetic field via the two-dimensional Dirac oscillator. *Phys. Scr.* **90**, 045702 (2015).
40. M.-A. Dariescu, C. Dariescu. Persistent currents and critical magnetic field in planar dynamics of charged bosons. *J. Phys: Condens. Matter.* **19**, 256203 (2007).
41. M.-A. Dariescu, C. Dariescu. Chiral electrons in static fields at finite temperature. *Rom. Journn. Phys.* **56**, 1043 (2011).

42. B. Mirza, R. Narimani, S. Zare. Relativistic Oscillators in a Noncommutative Space and in a Magnetic Field. *Comm. Theor. Phys.* **55**, 405 (2011).
43. N.A. Rao, B.A. Kagali. Energy profile of the one-dimensional Klein–Gordon oscillator. *Phys. Scr.* **77**, 015003 (2008).
44. J. Schwinger. On Gauge invariance and vacuum polarization. *Phys. Rev.* **82**, 664 (1951).
45. M. Abramowitz, I.A. Stegun. *Handbook of Mathematical Functions* (Dover, 1965).
46. R.B. Paris, D. Kaminski. *Asymptotics and Mellin-Barnes Integrals* (Cambridge University Press, 2001), Vol. 85.
47. G. Andrews, R. Askey, R. Roy. *Special Functions* (Cambridge University Press, 1999).
48. A.M. Frassino, D. Marinelli, O. Panella, P. Roy. Thermodynamics of quantum phase transitions of a Dirac oscillator in a homogenous magnetic field. *J. Phys. A: Math. Theor.* **53**, 185204 (2020).
49. L.L. Foldy. S. Wouthuysen. On the Dirac theory of spin 1/2 particles and its non-relativistic limit . *Phys. Rev.* **78**, 29 (1950).
50. A.G. Nikitin. On exact Foldy–Wouthuysen transformation. *J. Phys. A: Math. Gen.* **31**, 3297 (1998).
51. M.H. Pacheko, R.V. Maluf, C.A.S. Almeida, R.R. Landim. Three-dimensional Dirac oscillator in a thermal bath. *Eur. Phys. Lett.* **108**, 10005 (2014).

Received 27.09.22

А. Бумалі, Р. Аллуані, А. Бузенада, Ф. Сердук

ВПЛИВ ЗОВНІШНЬОГО ЕЛЕКТРИЧНОГО ПОЛЯ НА ТЕПЛОВІ ВЛАСТИВОСТІ ОДНОВИМІРНОГО РЕЛЯТИВІСТИЧНОГО ГАРМОНІЧНОГО ОСЦИЛЯТОРА

Досліджено релятивістичні гармонічні осцилятори Дірака та Клейна–Гордона в постійному зовнішньому електричному полі. Отримано точні розв'язки, які дозволяють розглянути вплив зовнішнього електричного поля на теплові властивості цих осциляторів. Такі властивості розраховано з використанням дзета-функції. Побудовано графіки, які демонструють згаданий вплив.

Ключові слова: релятивістичний гармонічний осцилятор, теплові властивості, зовнішнє поле, статистична сума, дзета-функція.