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V.F. KHARCHENKO

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143, Ukraine; e-mail: vkharchenko@bitp.kiev.ua)

ON THE ANALYTICAL SOLVING OF THE TER-MARTIROSIAN–SKORNYAKOV EQUATION FOR THREE PARTICLES AT NEGATIVE ENERGIES

Simple analytical expression for the solution of the Ter-Martirosian–Skornyakov equation for three particles at a negative energy has been obtained.

Keywords: Ter-Martirosian–Skornyakov equation, Mellin transformation.

1. Analytical Solution of the Equation

The analytical expression for the solution of the Ter-Martirosian–Skornyakov equation [1] has the form

$$\varphi(p) = \varphi_0(p) + \mu \int_0^\infty \frac{1}{\alpha} - \sqrt{\lambda^2 + \frac{3}{4}q^2} \times$$
$$\times \ln \frac{p^2 + p\rho + q^2 + \lambda^2}{p^2 - p\rho + q^2 + \lambda^2}\, \varphi(\rho) d\rho. \tag{1}$$

Our main interest is in the case where $\lambda^2 \equiv \alpha^2 - \frac{3}{4}p_0^2$ contains large p_0^2 so that $\lambda^2 < 0$. Let us transform this equation into an equation with $\lambda^2 > 0$ and then perform the analytical continuation with negative λ^2. Now, we substitute

$$\rho = \lambda \frac{x^2 - 1}{x\sqrt{3}}, \quad q = \lambda \frac{y^2 - 1}{y\sqrt{3}}. \tag{2}$$

Then we obtain

$$\lambda^2 + \frac{3}{4}q^2 = \lambda^2 \frac{(y^2+1)^2}{4y^2}, \quad d\rho = \lambda \frac{1+\gamma^2}{\sqrt{3}y^2} dy,$$
$$(0,\infty) \to (1,\infty), \tag{3}$$

and the equation takes the form

$$\varphi(x) = \varphi_0(x) -$$
$$- \frac{2\mu}{\sqrt{3}} \int_1^\infty \ln \frac{(x^2+xy+y^2)(x^2y^2-xy+1)}{(x^2-xy+y^2)(x^2y^2+xy+1)} \times$$
$$\times \frac{1}{1 - \frac{\alpha}{\lambda}\frac{y}{1+y^2}} \varphi(y) \frac{dy}{y}. \tag{4}$$

Consider the case where $\varphi_0(x)$ with $x < 1$ is continued analytically:

$$\varphi_0\left(\frac{1}{x}\right) = -\varphi_0(x). \tag{5}$$

Then the solution φ can be also continued. The equation becomes simplified:

$$\varphi(x) = \varphi_0(x) - \frac{2\mu}{3} \int_0^\infty \ln \frac{x^2+xy+y^2}{x^2-xy+y^2} \times$$
$$\times \frac{1}{1 - \frac{\alpha}{\lambda}\frac{y}{1-y^2}} \varphi(y) \frac{dy}{y}. \tag{6}$$

Using now the Mellin transformation, we get

$$\Phi(\xi) = \int\limits_0^{\infty} \varphi(x) x^{i\xi - 1} dx,$$
$$\varphi(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \Phi(\xi) x^{-i\xi} d\xi. \tag{7}$$

The equation for $\varphi(\xi)$ takes the form

$$\Phi(\xi) = \Phi_0(\xi) + L(\xi) \int\limits_{-\infty}^{+\infty} M(\xi - \eta))\Phi(\eta) d\eta, \tag{8}$$

where

$$L(\xi) = -\frac{2\mu}{\sqrt{3}} \int\limits_0^{\infty} \ln \frac{1 + t + t^2}{1 - t + t^2} t^{-\xi - 1} dt,$$
$$M(\xi) = \frac{1}{2\pi} \int\limits_0^{\infty} y^{i\xi} \frac{1}{1 - \frac{\alpha}{\lambda} \frac{y}{1+y^2}} \frac{dy}{y}. \tag{9}$$

Both integrals can be calculated using the contour integration. In the first one, it is necessary to integrate by parts. It is worth to mention that

$$L(\xi) = -\frac{2\mu}{3} \frac{2\pi}{\xi} \frac{\sinh \frac{\pi\xi}{6}}{\cosh \pi\xi/2} \tag{10}$$

is known as the Danilov factor [2], and

$$M(\xi) = \delta(\xi_) + M_1(\xi), \tag{11}$$

where M_1 is a smooth function which depends on λ analytically on the plane with the cut from $-\alpha^2$ to $+\infty$ (see [4]). In this way, we reduce the equation to the form [3, 4]

$$[1 - L(\xi)]\Phi(\xi) = \Phi_0(\xi) +$$
$$+ L(\xi) \int\limits_{-\infty}^{+\infty} M_1(\xi - \eta)\Phi(\eta) d\eta. \tag{12}$$

In the quartet case, $\mu = \frac{1}{\pi}$, and, in the doublet case, $\mu = -\frac{2}{\pi}$, and the factor $1 - L(\xi)$ does not vanish. Dividing by it, we obtain the Fredholm equation with the smooth kernel. In this kernel, we must do the analytic continuation in the negative λ^2.

It should be noted that the Mellin transformation can be useful also for other models, in particular, for the Yamaguchi model.

It should be also noted that the integral for $M_1(\xi)$ is

$$M_1(\xi) = \frac{1}{1 - \exp -2\pi\xi} \frac{\alpha}{\lambda} \frac{1}{2\sqrt{\frac{\alpha^2}{\lambda^2} - 1}} \times$$
$$\times \left[\left(\frac{\alpha}{\lambda} + \sqrt{\frac{\alpha^2}{\lambda^2} - 1} \right)^{i\xi} - \left(\frac{\alpha}{\lambda} - \sqrt{\frac{\alpha^2}{\lambda^2} - 1} \right)^{i\xi} \right]. \tag{13}$$

Here, it is possible to pass to the negative λ^2. The corresponding $M_1(\xi)$ will be rapidly decreasing. Thus, the recipe for the energy above the threshold is as follows:

1. To solve Eq. (12) for $\Phi(\xi)$ supposing λ^2 to be negative.

2. To construct the answer using the formula

$$\varphi(p) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \Phi(\xi) \left(\frac{\rho\sqrt{\xi}}{2\lambda} + \sqrt{\frac{\xi}{4\lambda^2} p + 1} \right)^{i\xi} d\xi. \tag{14}$$

If λ is negative, the integrand (the second factor) increases, if $|\xi| \to \infty$. However, apparently Eq. (12) implies in this case that the solution $\Phi(xi)$ decreases quickly. Therefore, the integrals must be convergent. This method is suitable for other kernels with logarithm.

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В.Ф. Харченко

ЩОДО АНАЛІТИЧНОГО РОЗВ'ЯЗУВАННЯ
РІВНЯННЯ ТЕР-МАРТИРОСЯНА–СКОРНЯКОВА
ДЛЯ ТРЬОХ ЧАСТИНОК
ПРИ НЕГАТИВНИХ ЕНЕРГІЯХ

Отримано простий аналітичний вираз для розв'язку рівняння Тер-Мартиросяна–Скорнякова для трьох частинок при негативній енергії.

Ключові слова: рівняння Тер-Мартиросяна–Скорнякова, перетворення Мелліна.