GAUSS APPROXIMATION IN THE OPTIMIZATION PROBLEM OF THE MINORITY GAME MODEL

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The optimization problem of the well-known minority game model is studied by methods of statistical physics. The problem is reduced to the study of the ground state of some effective system with continuous spin described by a replica Hamiltonian with random parameters. With the use of the central limit theorem of probability theory, the representations of the distribution function for parameters of the Hamiltonian are obtained, and the transition to the Gauss distribution in the case of large P is realized. Within approximations 1RSB and 2RSB in the replica method, the dependence of the minimum of the quantity under study on the parameter α is determined. It is shown that, in the region of applicability, the proposed method gives a less value of the minimum than that obtained in the cited works.

1. Introduction

As is well known [1–3], the application of methods and approaches of the statistical physics of disordered systems allowed one to obtain a number of important results concerning the classical optimization problems in studies of operations, models of neuron networks, and a number of other systems. In the last years, a trend named econophysics [4], where the methods and approaches of statistical physics are used in the simulation of economic systems, is developed. In particular, the physicists proposed several models for the description of financial and share markets [4, 5]. We will pay attention to the minority game model which was studied in [6–12]. This model concerns with diverse studies. But here, we will consider an optimization problem arisen in the model. The foundations of the model are presented in the literature quite completely (see, e.g., [11]), so we mention them briefly.

In the minority game, a market is modeled by the game interaction of N agents. At every time moment t, the i-th agent $(i = 1, \ldots, N)$ realizes an action $a_i(t) = 1$ (purchase) or $a_i(t) = -1$ (sale). The gain of the i-th agent $u_i(t)$ with regard for the actions of all agents is determined by the formula

$$u_i(t) = -a_i(t)\,A(t), \text{ where } A(t) = \sum_{i=1}^{N} a_i(t). \qquad (1)$$

Formula (1) models the interaction between agents on the market in terms of the global quantity $A(t)$. It is obvious that, at every time moment, all agents on the market can be divided into two groups by the chosen action (purchase or sale). The definition (1) yields the minority rule: the winner is an agent who belongs to the minority. The gained action of the agent who is in minority can be written in the form $a_i(t) = -\text{sign}(A(t))$, and his/her gain is equal to $|A(t)|$. Respectively, the action of an agent in majority is $a_i(t) = \text{sign}(A(t))$, and his/her loss is $-|A(t)|$). The total gain of all agents is $\sum_i u_i = -A^2$ and is always negative. All agents have access to the common information which is described at the time moment t by an integer $\mu(t) = 1, \ldots, P$. Since the behavior of agents affects the market, this action is denoted by $A^{\mu(t)}(t)$. Generally, there exist 2^P strategies, but we assume that each agent chooses only S ones from the total amount. Let the action of the i-th agent, who follows the strategy s and uses the information $\mu(t)$, be $a^\mu_{s,i}$ (in the stationary case, the time variable t is omitted). We consider that the number P is rather large, of the same order as N, and the value of $\alpha = P/N$ is finite as $N \to \infty$ and $P \to \infty$. The quantity μ is described by a certain distribution ρ^μ regardless of each time step (in what follows, we take $\rho^\mu = 1/P$). In the model, it is also given that the actions of agents $a^\mu_{s,i}$ for each time step

are random and independent with the probabilities

$$P(a_{s,i}^{\mu} = +1) = P(a_{s,i}^{\mu} = -1) = \frac{1}{2},$$

$$i = 1, \ldots, N, \ s = 1, \ldots, S, \ \mu = 1, \ldots, P.$$

For the analysis of a game, we introduce the mixed strategies (see, e.g., [13]) that are characterized by the probabilities, with which the i-th agent uses a given strategy (the condition $\sum_s \pi_{s,i} = 1$ is satisfied). The set of variables $\pi_{s,i}$ determines the phase space of the model [11]. In this space, the mean values of quantities are defined as follows:

$$\langle A^{\mu} \rangle = \sum_{i=1}^{N} \sum_{s=1}^{S} \pi_{s,i} \, a_{s,i}^{\mu} . \tag{2}$$

In works [9–11], the collective properties of the model are described with the use of the quantity

$$H = \sum_{\mu=1}^{P} \rho^{\mu} \left(\sum_{i=1}^{N} \sum_{s=1}^{S} \pi_{s,i} \, a_{s,i}^{\mu} \right)^2 \tag{3}$$

which determines fluctuations (2). It was also established [11] that (3) possesses properties of the Hamiltonian of a system. In the case where $H = 0$, the game is symmetric, i.e. $\langle A^{\mu} \rangle = 0$ for $\forall \mu$ (2), and the mean gain is zero, according to the above-written. But if $H > 0$, then the game is asymmetric (the symmetry of the actions of agents concerning the purchase and the sale is broken). Hence, there exists μ, for which $\langle A^{\mu} \rangle > 0$), and the gained strategy is realized.

In the above-cited works, the minimum of (3) was studied on the set of variables $\{\pi_{s,i}\}$. The investigation of the minimum is reduced to the calculation of the "free energy" of a system which is described by Hamiltonian (3) at "zero temperature" or at the unbounded reciprocal temperature $\beta \to \infty$. Since (3) depends on the actions of players $a_{s,i}^{\mu}$, the derivation of substantive results requires to average over all possible actions of agents $a_{s,i}^{\mu}$. As a result, we are faced with a problem of the theory of disordered systems with "frozen" disorder which is solved within the method of replicas. In works [6–12], the asymptotically exact solution was obtained for the minimum of (3) as $N, P \to \infty$ with $P/N = \alpha \sim 1$. This solution describes a phase transition with the symmetry breaking at the point $\alpha_c \approx 0.3374$. In other words, $H = 0$ for $\alpha \leqslant \alpha_c$, and $H > 0$ for $\alpha > \alpha_c$. Since the parameter α is connected with the quantity P that is a measure of information, the mentioned solution is also interpreted

as the influence of information on a market. Namely, if no information is available, then the market is characterized by the complete equilibrium. The presence of information breaks the equilibrium on the market, i.e., the gained strategy exists.

The asymptotic solution was obtained in [9–11] with the use of the expansion in the parameter $\sim \beta/P$ which is considered small for large values of P. In fact, a relation of the type

$$\cos\left(\sqrt{\frac{\beta}{P}} x\right) \simeq \exp\left(-\frac{\beta}{2P} x^2\right) \tag{4}$$

was applied. In work [14], it was noted that approximation (4) is valid for finite values of β and cannot be substantiated at $\beta \to \infty$. In addition, the mentioned solution involves some contradiction. In particular, the quantity $\chi = \beta(Q - q)/\alpha$ diverges for the obtained solution as $\alpha \to \alpha_c$. The divergence is obviously related to the fact that $Q \neq q$ as $\beta \to \infty$ ($0 \leq Q, q \leq 1$). At the same time, the quantity $\beta \sum_{i,s} \pi_{s,i} \pi_{s,i} \sim \chi$ is considered to be bounded in the input relations at the use of approximation (4).

In this connection, a procedure of construction of the asymptotic solution which is based on ideas of the limit theorems of probability theory [17] was considered in [14]. In particular, the transition to the Gauss variables (distributed by the normal law) was realized in the limit $P \to \infty$ for a sum of random variables of the form $\frac{1}{\sqrt{P}} \sum_{\mu} (\ldots)$. As a result, the study of the minimum of (3) is reduced to the study of the ground state of the Hamiltonian with parameters distributed by the Gauss law. The calculations executed in the approximation of symmetric replicas have revealed different dependences of (3) on the parameter α as compared with the literature data [9–11].

In the present work, we will construct the distribution function (characteristic function) for parameters of the Hamiltonian and carry out more successively the limit transition to the Gauss approximation. We will also study the ground state of the Hamiltonian with the replica symmetry breaking on one (1RSB) and two steps (2RSB) [15].

2. Gauss Approximation

As was noted above, the problem consists in the determination of the minimum of the Hamiltonian H on the set of variables $\{\pi_{s,i}\}$. To this end, we define the partition function of the system as

$$Z(\beta) = \mathrm{Sp}_{\pi} \exp(-\beta H(\pi)),$$

where β is the "reciprocal temperature", Sp_π stands for the integration over the variables $\pi_{s,i} \in [0,1]$ with regard for the condition $\sum_s \pi_{s,i} = 1$ for each $i = 1, \ldots, N$:

$$\mathrm{Sp}_\pi(\ldots) = \int_0^1 \prod_{s,i} d\pi_{s,i}(\ldots). \tag{5}$$

The notation $H(\pi)$ indicates the dependence on the probabilities of mixed strategies $\{\pi_{s,i}\}$. Like the study of a ground state of physical systems (see, e.g., [16]), the determination of the minimum can be reduced to the study of the ground state of the Hamiltonian at the "zero temperature". As a result, we obtain the relation

$$\min_\pi H(\pi) = -\lim_{\beta \to \infty} \frac{1}{\beta} \ln Z(\beta). \tag{6}$$

After the averaging of (6) over all possible actions of players $a_{s,i}^\mu$, we obtain

$$\mathcal{H} = \langle \min_\pi H(\pi) \rangle_a = -\lim_{\beta \to \infty} \frac{1}{\beta} \langle \ln Z(\beta) \rangle_a. \tag{7}$$

For simplicity, we consider the case with two states $S = 2$ [11] and represent (3) in the form of a Hamiltonian given on the set of variables π_i with random parameters [14]. The parameters of the Hamiltonian are sums (over the index μ) of independent random quantities. For large values of $P \sim N$, the parameters are random values that are approximately set by the Gauss distribution according to the limit theorems of probability theory [17]. Thus, we can pass from the averaging of the partition function (7) over distributions of the quantities $a_{s,i}^\mu$ to the averaging over the Gauss distributions of parameters of the Hamiltonian. After the obvious transformations, Hamiltonian (3) can be written in the form

$$H = \hat{C}_0 + \sum_i \hat{d}_i \pi_i^2 + \sum_{i<j} \hat{J}_{ij} \pi_i \pi_j + \sum_{i \neq j} \hat{k}_{ij} \pi_i, \tag{8}$$

where we introduced the notation

$$\hat{C}_0 = \frac{1}{4P} \sum_\mu \Big(\sum_i a_{i+}^\mu \Big)^2,$$

$$\hat{d}_i = \frac{1}{4P} \sum_\mu (a_{i-}^\mu)^2,$$

$$\hat{J}_{ij} = \frac{1}{2P} \sum_\mu a_{i-}^\mu a_{j-}^\mu,$$

$$\hat{k}_{ij} = \frac{1}{2P} \sum_\mu a_{i-}^\mu a_{j+}^\mu,$$

$$a_{i\pm}^\mu = a_{1,i}^\mu \pm a_{2,i}^\mu. \tag{9}$$

After the substitution of (8) in (7), it is easy to see that $\langle \hat{C}_0 \rangle_a = N/2$. The remaining terms in (8) have the structure of the Hamiltonian of a system with continuous spin ($\pi_i \in [-1, 1]$, $i = 1, \ldots, N$). We note that formula (8) contains no approximations and is only another form of Hamiltonian (3). We now pass to the normalized parameters $\{\tilde{d}_i, \tilde{J}_{ij}, \text{and } \tilde{k}_{ij}\}$, whose mean values and variances are equal to 0 and 1, respectively. We have

$$H = \frac{1}{2} \sum_i \Big(\frac{\tilde{d}_i}{\sqrt{P}} + 1 \Big) \pi_i^2 + \frac{1}{\sqrt{P}} \sum_{i<j} \tilde{J}_{ij} \pi_i \pi_j +$$

$$+ \frac{1}{\sqrt{P}} \sum_{i \neq j} \tilde{k}_{ij} \pi_i. \tag{10}$$

The normalized quantities are connected with the previous ones (9) by the relations

$$\hat{d}_i = \frac{1}{2} \Big(\frac{\tilde{d}_i}{\sqrt{P}} + 1 \Big), \quad \hat{J}_{ij} = \frac{\tilde{J}_{ij}}{\sqrt{P}}, \quad \hat{k}_{ij} = \frac{\tilde{k}_{ij}}{\sqrt{P}}. \tag{11}$$

We now introduce the distribution function $\Psi(d, J, k)$ for parameters (11) and represent the relation for the minimum of (7) in the form

$$\mathcal{H} = \frac{N}{2} - \lim_{\beta \to \infty} \frac{1}{\beta} \langle \ln \mathcal{Z}(\beta) \rangle_\Psi, \tag{12}$$

where

$$\mathcal{Z}(\beta) = \int_{-1}^1 \prod_i \frac{d\pi_i}{2} \exp(-\beta H).$$

The calculation of the distribution function $\Psi(d, J, k)$ (its characteristic function) is given in Appendix A, where it is shown that the function $\Psi(d, J, k)$ in the principal order in the parameter $1/\sqrt{P}$ tends to the Gauss distribution function (A.11). With the use of (A.11), the averaging in (12) within the method of replicas can be performed in the closed form, which gives the asymptotically exact solution.

3. Method of Replicas

In the method of replicas [3, 15], we set

$$\langle \ln \mathcal{Z}(\beta) \rangle_\Psi = \lim_{n \to 0} \frac{1}{n} \ln \langle \mathcal{Z}(\beta)^n \rangle_\Psi. \tag{13}$$

As is known, the averaging in (13) is carried out for integer n, and then the analytic continuation for real n is realized. As a result, the minimum of (7) can be written in the form

$$\mathcal{H} = \frac{N}{2} - \lim_{\beta \to \infty} \frac{1}{\beta} \lim_{n \to 0} \frac{1}{n} \ln \langle \mathcal{Z}(\beta)^n \rangle_\Psi. \tag{14}$$

The partition function of the system of n replicas is given by the formula

$$\mathcal{Z}(\beta)^n = \mathrm{Sp}_\pi \exp(-\beta H^{(n)}), \tag{15}$$

where the Hamiltonian of the system is as follows:

$$H^{(n)} = \frac{1}{2} \sum_{i,a} \left(\frac{d_i}{\sqrt{P}} + 1 \right) \pi_i^{a\,2} + \sum_{i<j,a} \frac{J_{ij}}{\sqrt{P}} \pi_i^a \pi_j^a + $$

$$+ \frac{1}{\sqrt{P}} \left(\sum_{i \neq j,a} k_{ij} \pi_i^a \right), \tag{16}$$

where the index a numbers replicas. In (15), we used the notation

$$\mathrm{Sp}_\pi(\dots) = \int\limits_{-1}^{1} \prod_{i,a} \frac{d\pi_i^a}{2} (\dots). \tag{17}$$

After the averaging of the partition function of the system of n replicas (15) with the Gauss distribution, we obtain

$$\langle \mathcal{Z}(\beta)^n \rangle_\Psi = \mathrm{Sp}_\pi \exp \left(-\beta H_0^{(n)} - \beta H_{\mathrm{int}}^{(n)} \right), \tag{18}$$

where the following notation is introduced:

$$H_0^{(n)} = \frac{1}{2} \sum_{i,a} \pi_i^{a\,2},$$

$$H_{\mathrm{int}}^{(n)} = -\frac{\beta}{8P} \sum_i \left(\sum_a \pi_i^{a\,2} \right)^2 - \frac{\beta N}{2P} \sum_i \left(\sum_a \pi_i^a \right)^2 - $$

$$- \frac{\beta}{2P} \sum_{i<j} \left(\sum_a \pi_i^a \pi_j^a \right)^2. \tag{19}$$

The component of the Hamiltonian, $H_{\mathrm{int}}^{(n)}$, in (19) describes the effective interaction between replicas. The first term in (19) can be omitted, since its contribution $\sim N^0$. The further calculations consist in the factorization of integrals by the variables of particles. By introducing

the overlapping matrix for replicas $\hat{Q}_{ab} = \frac{1}{\sqrt{N}} \sum_i \pi_i^a \pi_i^b$, we write $H_{\mathrm{int}}^{(n)}$ (19) in the form

$$H_{\mathrm{int}}^{(n)} = -\frac{\beta N}{4P} \sum_{a,b} \hat{Q}_{ab}^2 - \frac{\beta N}{2P} \sqrt{N} \sum_{a,b} \hat{Q}_{ab}. \tag{20}$$

Using the substitution $P = \alpha N$ for $\beta \to \sqrt{\alpha}\beta$, we obtain the following formula for the minimum:

$$\mathcal{H} = \frac{N}{2} - \lim_{\beta \to \infty} \frac{1}{\sqrt{\alpha}\beta} \lim_{n \to 0} \frac{1}{n} \ln \bar{\mathcal{Z}}(\beta)^{(n)}. \tag{21}$$

After the above transformations, the partition function for n replicas takes the form

$$\bar{\mathcal{Z}}(\beta)^{(n)} = \mathrm{Sp}_\pi \exp\left(-\beta \bar{H}_0^{(n)} - \beta \bar{H}_{\mathrm{int}}^{(n)} \right), \tag{22}$$

where

$$\bar{H}_0^{(n)} = \frac{\sqrt{\alpha}}{2} \sum_{i,a} \pi_i^{a\,2},$$

$$\bar{H}_{\mathrm{int}}^{(n)} = -\frac{\beta}{4} \sum_{a,b} \hat{Q}_{ab}^2 - \frac{\beta}{2} \sqrt{N} \sum_{a,b} \hat{Q}_{ab}. \tag{23}$$

Formula (23) determines the components of the effective Hamiltonian after the averaging over the Gauss distribution. The dependence on the parameter α is given only by the term $\bar{H}_0^{(n)}$. By analogy with spin systems, it can be interpreted as the one-site anisotropy. Formulas (23) and (21) imply also that the quantity $\mathcal{H}$ depends monotonically on α. As α decreases, the second component in (21) increases due to the factor $1/\sqrt{\alpha}$, which causes the decrease of $\mathcal{H}$. At some α_0, $\mathcal{H}$ becomes zero. Thus, we consider that approximation (23) is suitable for $\alpha > \alpha_0$.

By using the integral transformation

$$e^{\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dz \, e^{-\frac{1}{2}z^2} e^{zx}$$

we linearize the component $\sim \hat{Q}_{ab}^2$ in (22) and factorize the partition function (22) by the variables of particles. As a result, we obtain the representation of the partition function in the space of replica variables as

$$\bar{\mathcal{Z}}(\beta)^{(n)} = L \int DQ \exp(N\Phi(Q)), \tag{24}$$

where we denote

$$L = \left(\frac{\beta}{\sqrt{2}} \right)^{n^2}, \quad DQ = \prod_{a,b} \frac{dQ_{ab}}{\sqrt{2\pi}},$$

$$\Phi(Q) = -\frac{\beta^2}{4}\sum_{a,b}Q_{ab}^2 + \ln Z_1(Q),$$

$$Z_1(Q) = \int\limits_{-1}^{1}\prod_a\frac{d\pi_a}{2}\exp(\Phi_1(Q)),$$

$$\Phi_1(Q) = -\beta\frac{\sqrt{\alpha}}{2}\sum_a\pi_a^2 + \frac{\beta^2}{2}\sum_{a,b}Q_{ab}\pi_a\pi_b +$$

$$+\frac{\beta^2}{2}\Big(\sum_a\pi_a\Big)^2. \tag{25}$$

In the limit $N\to\infty$, we calculate the integrals in (24) by the Laplace method and obtain

$$\bar{\mathcal{Z}}(\beta)^{(n)} = \exp(N\Phi_{\text{ext}}(\bar{Q})), \tag{26}$$

where $\bar{Q}$ is the extreme value which is determined from the steady-state equations

$$\frac{\partial\Phi(Q)}{\partial Q_{ab}} = 0 \to \bar{Q}_{ab} = \langle\pi_a\pi_b\rangle_\pi. \tag{27}$$

The content of the averaging in (27) is given by the formula

$$\langle(\dots)\rangle_\pi = \frac{1}{Z_1(\bar{Q})}\int\limits_{-1}^{1}\prod_a\frac{d\pi_a}{2}\exp\big[\Phi_1(\bar{Q})\big](\dots). \tag{28}$$

As a result, the minimum of the quantity $\mathcal{H}$ is as follows:

$$\mathcal{H} = \frac{N}{2} + \frac{N}{\sqrt{\alpha}}\lim_{\beta\to\infty}\lim_{n\to0}\Big(\frac{1}{4}\frac{\beta}{n}\sum_{a,b}Q_{ab}^2 - \frac{1}{\beta n}\ln Z_1(Q)\Big). \tag{29}$$

It is obvious that the factor L in (24) gives no contribution in the limit $n\to0$. Thus, formula (29) presents a solution of the problem within the method of replicas. The further calculations consist in a specific choice of the overlapping matrix $\hat{Q}$. In the approximation of symmetric replicas (RS) that was considered in [14], the matrix $\hat{Q}$ is set in the form

$$Q_{ab} = \begin{cases} Q, & a = b; \\ q, & a \neq b. \end{cases}$$

The approximations with the replica symmetry breaking, 1RSB and 2RSB, are considered in Appendices B and C.

4. Solution at $\beta\to\infty$

4.1. Approximation 1RSB

The general relations of approximation 1RSB for $\forall\beta$ are given in Appendix B. Let us consider the limit for $\mathcal{H}$ as $\beta\to\infty$. We introduce the quantities $\chi = \beta(Q - q_1)$ and $\chi_1 = \beta(q_1 - q)$ which are named susceptibilities, by analogy with spin systems. First, we consider the case where these quantities are bounded as $\beta\to\infty$. By performing the subsequent transformations of the variable $z_1 \to \sqrt{\beta}\sqrt{1+q}/\sqrt{\chi_1}\,z_1$ and $z_1 \to z_1 - z$, we represent quantities (B.6) in the form

$$Z_1 = \frac{\sqrt{\beta}\sqrt{1+q}}{\sqrt{\chi_1}}\int\limits_{-\infty}^{\infty}\frac{dz_1}{\sqrt{2\pi}}\exp(-\beta\frac{1+q}{2\chi_1}(z_1 - z)^2)\tilde{Z}_1^m,$$

$$Z_1 = \frac{\sqrt{\beta}\sqrt{1+q}}{\sqrt{\chi_1}}\int\limits_{-\infty}^{\infty}\frac{dz_1}{\sqrt{2\pi}}\exp\Big(-\beta\frac{1+q}{2\chi_1}(z_1 - z)^2\Big)\tilde{Z}_1^m,$$

$$\tilde{Z}_1 = \int\limits_{-1}^{1}\frac{d\pi}{2}\exp\big(\beta\phi(\pi)\big),$$

$$\phi(\pi) = -\frac{\sqrt{\alpha}}{2}\pi^2 + \frac{1}{2}\chi\pi^2 + \sqrt{1+q}\pi z_1. \tag{30}$$

It is easy to see that, as $\beta\to\infty$,

$$\frac{\sqrt{\beta}\sqrt{1+q}}{\sqrt{2\pi\chi_1}}\exp\big[-\beta\frac{1+q}{2\chi_1}(z_1 - z)^2\big] \to \delta(z_1 - z). \tag{31}$$

This implies that, at the substitution of (31) in formulas (B.7) and (B.8), the integrals over the variable z_1 are calculated explicitly, and we obtain approximation RS [14]. Obviously, it is a consequence of the assumption that the quantity $\chi_1 = \beta(q_1 - q)$ is bounded as $\beta\to\infty$. In other words, in the case where $\beta\to\infty$, we have that $q_1 \to q$, and, respectively, approximation 1RSB passes into RS (see also the remarks in Appendices B and C). We note that this property of approximation 1RSB was also indicated in [20, 21], where it was observed at numerical calculations.

Solutions different from approximation RS arise if we set the dependence $m = m_0/\beta$ for the parameter m. Then the quantity $\Delta q = q_1 - q$ is finite as $\beta\to\infty$. At the next step, we will calculate the asymptotics of the quantity $\tilde{Z}_1$ (B.6). In formulas (B.7), (B.8), and

(B.11), we perform preliminarily the successive change of variables:

$$z \to \frac{z}{1+q}, z_1 \to \frac{z_1}{\Delta q}, z_1 \to z_1 - z.$$

As a result, we obtain

$$\tilde{Z}_1 = \int\limits_{-1}^{1} \frac{d\pi}{2} \exp\big(\beta\phi(\pi)\big),$$

$$\phi(\pi) = -\frac{\sqrt{\alpha}}{2}\pi^2 + \frac{1}{2}\chi\pi^2 + \pi z_1. \qquad (32)$$

As $\beta \to \infty$, the integral over π can be calculated by the Laplace method. The maximum of $\phi(\pi)$ is attained at the point

$$\pi_{\max} = \begin{cases} -1, & \text{if } z_1 < -z_0, \\ z_1/z_0, & \text{if } -z_0 < z_1 < z_0, \\ 1, & \text{if } z_1 > z_0, \end{cases} \qquad (33)$$

$$z_0 = \sqrt{\alpha} - \chi.$$

Respectively, for $\tilde{Z}_1$ and Z_1, we obtain

$$\tilde{Z}_1 \simeq \exp\big(m_0\phi_{\max}\big), \ Z_1 \simeq \big\langle \exp\big(m_0\phi_{\max}\big)\big\rangle_{z_1},$$

where

$$\phi_{\max} = \begin{cases} -z_0/2 - z_1, & \text{if } z_1 < -z_0, \\ z_1^2/(2z_0), & \text{if } -z_0 < z_1 < z_0, \\ -z_0/2 + z_1, & \text{if } z_1 > z_0. \end{cases} \qquad (34)$$

After simple algebraic transformations in the limit $\beta \to \infty$, we get

$$\mathcal{H}/N = \frac{1}{2} + \frac{1}{4\sqrt{\alpha}}[2\chi(\Delta q + q) + m_0\Delta q(\Delta q + 2q)] -$$

$$- \frac{1}{\sqrt{\alpha}}\frac{1}{m_0}\langle\ln\langle e^{m_0\phi_{\max}}\rangle_{z_1}\rangle_z. \qquad (35)$$

On the basis of (B.8) and (B.11), we obtain the system of equations for the parameters χ, Δq, q, and m_0:

$$\chi = \frac{1}{\sqrt{1+q}}\left\langle \frac{\langle e^{m_0\phi_{\max}}\frac{d}{dz_1}\pi_{\max}\rangle_{z_1}}{\langle e^{m_0\phi_{\max}}\rangle_{z_1}}\right\rangle_z,$$

$$\Delta q + q = \langle\frac{\langle \pi_{\max}^2 e^{m_0\phi_{\max}}\rangle_{z_1}}{\langle e^{m_0\phi_{\max}}\rangle_{z_1}}\rangle_z,$$

$$\Delta q = \langle\frac{\langle \pi_{\max}e^{m_0\phi_{\max}}\rangle_{z_1}^2}{\langle e^{m_0\phi_{\max}}\rangle_{z_1}^2}\rangle_z,$$

$$\frac{1}{4}\Delta q(\Delta q + 2q) + \frac{1}{m_0^2}\langle\ln\langle e^{m_0\phi_{\max}}\rangle_{z_1}\rangle_z -$$

$$-\frac{1}{m_0}\langle\frac{\langle\phi_{\max}e^{m_0\phi_{\max}}\rangle_{z_1}}{\langle e^{m_0\phi_{\max}}\rangle_{z_1}}\rangle_z = 0. \qquad (36)$$

The averaging over the variables z and z_1 is determined by the formulas

$$\langle\ldots\rangle_z = \int\limits_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi(1+q)}} \exp\Big(-\frac{z^2}{2(1+q)}\Big)(\ldots),$$

$$\langle\ldots\rangle_{z_1} = \int\limits_{-\infty}^{\infty} \frac{dz_1}{\sqrt{2\pi\Delta q}} \exp\Big(-\frac{(z_1-z)^2}{2\Delta q}\Big)(\ldots). \qquad (37)$$

We studied the system of equations (36) numerically. In order to compare with the results of RS [14], we consider the dependence of Δq on α. It turns out that nonzero values of Δq are observed in the region where the parameter $\alpha \lesssim 1.32$. Moreover, Δq decreases monotonically to zero, as α increases. As was noted above, this approximation is not suitable in the region, where the parameter $\alpha \lesssim 1.32$, since $\mathcal{H} < 0$. In turn, the results of 1RSB at $\Delta q \to 0$ coincide with those in the approximation of symmetric replicas, which is indicated in Appendix B. This circumstance can be also illustrated in the following way. As $\Delta q \to 0$, the distribution function for the variable z_1 in (37) tends to the δ-function

$$\frac{1}{\sqrt{2\pi\Delta q}} \exp\Big(-\frac{(z_1-z)^2}{2\Delta q}\Big) \to \delta(z_1 - z),$$

and the integration in formulas (35) and (36) over the variable z_1 is performed explicitly, and we arrive at approximation RS [14]. Hence, at $\alpha \gtrsim 1.32$, the results of approximations 1RSB and RS are practically identical. In this case, the value $\mathcal{H} = 0$ is attained at $\alpha_0 \approx 1.345$ (Figure, curve 1).

4.2. Approximation 2RSB

By analogy with approximation 1RSB, we now change the variables $m_1 \to m_1/\beta$, $m_2 \to m_2/\beta$. The susceptibility $\chi = \beta(Q - q_2)$ is bounded as $\beta \to \infty$. We now define the variables $\Delta q_1 = q_1 - q$ and $\Delta q_2 = q_2 - q_1$ and perform
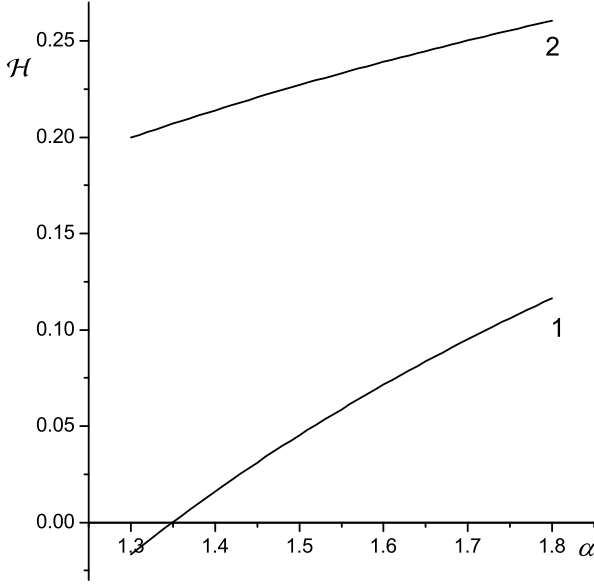
Curve 1 – dependence of $\mathcal{H}$ on the parameter α; the curve crosses the axis at the point $\alpha_0 \approx 1.345$. Curve 2 – segment of the corresponding function by work [11]

the following changes in the relations of approximation 2RSB (formulas (C.4), (C.5), (C.6), and (C.8)):

$$z \to \frac{z}{1+q},\; z_1 \to \frac{z_1}{\Delta q_1},\; z_2 \to \frac{z_2}{\Delta q_2},$$

$$z_1 \to z_1 - z,\; z_2 \to z_2 - z_1\,.$$

After these transformations, we obtain the minimum of $\mathcal{H}$ in the form

$$\mathcal{H}/N = \frac{1}{2} + \frac{1}{4\sqrt{\alpha}}\Big[2\chi(\Delta q_1 + \Delta q_2 + q)+$$

$$+m_2\Delta q_2(2\Delta q_1 + \Delta q_2 + q) + m_1\Delta q_1(\Delta q_1 + 2q)\Big]-$$

$$-\frac{1}{\sqrt{\alpha}}\frac{1}{m_1}\langle \ln Z_1\rangle_z\,. \tag{38}$$

Formulas (C.5) take the form

$$Z_1 = \langle\langle \tilde{Z}_2^{m_2}\rangle_{z_2}^{\frac{m_1}{m_2}}\rangle_{z_1},\; \tilde{Z}_2 = \int\limits_{-1}^{1} \frac{d\pi}{2}\exp(\beta\phi(\pi)),$$

$$\phi(\pi) = -\frac{\sqrt{\alpha}}{2}\pi^2 + \frac{1}{2}\chi\pi^2 + \pi z_2\,. \tag{39}$$

In this case, the averaging over the variables z_1 and z_2 is realized with the Gauss functions

$$\langle\ldots\rangle_{z_2} = \int\limits_{-\infty}^{\infty} \frac{dz_2}{\sqrt{2\pi\Delta q_2}}\exp\Big(-\frac{(z_2 - z_1)^2}{2\Delta q_2}\Big)(\ldots),$$

$$\langle\ldots\rangle_{z_1} = \int\limits_{-\infty}^{\infty} \frac{dz_1}{\sqrt{2\pi\Delta q_1}}\exp\Big(-\frac{(z_1 - z)^2}{2\Delta q_1}\Big)(\ldots),$$

and the averaging over z is carried out by the first formula in (37). The subsequent scheme of calculations is the same as for approximation 1RSB. First, we calculate the asymptotics of the integral over π (quantity $\tilde{Z}_2$ in (39)). As a result, we obtain

$$Z_1 \simeq \langle\langle\exp(m_2\phi_{\max})\rangle_{z_2}^{\frac{m_1}{m_2}}\rangle_{z_1},\; \tilde{Z}_2 \simeq \langle\exp(m_2\phi_{\max})\rangle_{z_2}.$$

By comparing formulas (32) and (39), it is easy to see that $\phi_{\max}$ is set by the same formula, as in (34), only with the change $z_1 \to z_2$. In this case, $\pi_{\max}$ is also determined by formula (33) with the analogous change $z_1 \to z_2$. Eventually on the basis of Eqs. (C.6) and (C.8) in the limit $\beta \to \infty$, we get the system of equations for the parameters $\chi, \Delta q_2, \Delta q_1, q, m_1$, and m_2:

$$\chi = \frac{1}{\sqrt{1+q}}\langle\frac{1}{Z_1}\langle\langle e^{m_2\phi_{\max}}\frac{d\pi_{\max}}{dz_2}\rangle_{z_2}\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}-1}\rangle_{z_1}\rangle_z,$$

$$\Delta q_2 + \Delta q_1 + q =$$

$$= \langle\frac{1}{Z_1}\langle\langle \pi_{\max}^2 e^{m_2\phi_{\max}}\rangle_{z_2}\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}-1}\rangle_{z_1}\rangle_z,$$

$$\Delta q_1 + q = \langle\frac{1}{Z_1}\langle\langle \pi_{\max} e^{m_2\phi_{\max}}\rangle_{z_2}^2\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}-2}\rangle_{z_1}\rangle_z,$$

$$q = \langle\langle\Big(\frac{1}{Z_1}\langle\langle \pi_{\max} e^{m_2\phi_{\max}}\rangle_{z_2}\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}-1}\rangle_{z_1}\Big)^2\rangle_z, \tag{40}$$

$$\frac{1}{4}\Delta q_1(\Delta q_1 + 2q) + \frac{1}{m_1^2}\langle \ln(Z_1)\rangle_z-$$

$$-\frac{1}{m_1 m_2}\langle\frac{1}{Z_1}\langle\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}}\ln\langle e^{m_2\phi_{\max}}\rangle_{z_2}\rangle_{z_1}\rangle_z = 0, \tag{41}$$

$$\frac{1}{4}\Delta q_2(\Delta q_2 + 2\Delta q_1 + 2q)+$$

$$+\frac{1}{m_2^2}\langle\frac{1}{Z_1}\langle\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}}\ln\langle e^{m_2\phi_{\max}}\rangle_{z_2}\rangle_{z_1}\rangle_z-$$

$$-\frac{1}{m_2}\langle\frac{1}{Z_1}\langle\langle e^{m_2\phi_{\max}}\rangle_{z_2}^{\frac{m_1}{m_2}-1}\langle e^{m_2\phi_{\max}}\phi_{\max}\rangle_{z_2}\rangle_{z_1}\rangle_z=0.$$

$$(42)$$

According to the input positions of the method of replicas [15], the condition $m_1/m_2 < 1$ is also satisfied.

The numerical solution indicates that the parameter Δq_2 decreases monotonically with increase in α and becomes 0 at $\alpha \simeq 0.84$. As shown in Appendix C for $\Delta q_2 \to 0$, approximation 2RSB passes into 1RSB. This can be observe in such a way. In formulas (38) and (40), we perform the substitution

$$\frac{1}{\sqrt{2\pi\Delta q_2}}\exp\Big(-\frac{(z_2-z_1)^2}{2\Delta q_2}\Big) \to \delta(z_2-z_1).$$

Then the integration over the variable z_2 can be carried out explicitly, and we arrive at approximation 1RSB. Hence, for $\alpha \gtrsim 0.84$, the results coincide with those in approximation 1RSB considered in the previous section.

By summarizing, we note that, in the region of applicability of the Gauss approximation $\alpha \geq \alpha_0$ ($\alpha_0 \approx 1.345$), the results with approximation RS [14] are corroborated, α_0 being slightly increased. The numerical studies reveal that approximation 2RSB tends to 1RSB, and approximation 1RSB goes to RS, as the parameter α increases. Obviously, such a behavior is caused by the domination of the term $\bar{H}_0^{(n)}$ in the effective replica Hamiltonian (23) at $\alpha \sim 1$.

In this connection, we note the following. We define the Gauss field

$$h_i = \frac{1}{\sqrt{N}}\sum_{j(\neq i)}k_{ji}$$

with $\langle h_i\rangle = 0$, $\langle h_i^2\rangle = 1$. Then the effective Hamiltonian (16) can be written in the equivalent form,

$$H^{(n)} \approx \frac{1}{2}\sum_{i,a}\pi_i^{a\,2} + \sum_{i<j,a}\frac{J_{ij}}{\sqrt{P}}\pi_i^a\pi_j^a + \sqrt{N/P}\sum_{i,a}h_i\pi_i^a.$$

$$(43)$$

After the substitution $P = \alpha N$, we write Hamiltonian (43) in the form

$$H^{(n)}/\sqrt{\alpha} \approx \frac{\sqrt{\alpha}}{2}\sum_{i,a}\pi_i^{a\,2} + \sum_{i<j,a}\frac{J_{ij}}{\sqrt{N}}\pi_i^a\pi_j^a + \sum_{i,a}h_i\pi_i^a. \quad (44)$$

Hamiltonian (44) can be considered as an analog to the Ghatak–Sherrington Hamiltonian [22–24] for a system with continuous spin $\pi = 1$ in Gauss fields h_i. As far as we know, no works, where the ground state of the Hamiltonian corresponding to (44) was studied, are available. We note that the ground state of a system with the the Ghatak–Sherrington Hamiltonian for the spin $S = 1$,

$$H = -\sum_{i,j}J_{i,j}S_iS_j + D\sum_i S_i,$$

was studied in work [24] with the Gauss distribution of interactions $J_{i,j}$ ($\langle J_{i,j}\rangle = 0$, $\langle J_{i,j}^2\rangle = J/N$). It is worth noting that the dependence of the free energy on the parameter D/J manifests a characteristic monotonic behavior like that in Figure (curve 1).

5. Conclusions

In the present work, we have considered the problem of minimization within the minority game model in the Gauss approximation. The transition from the averaging over discrete variables to that over continuous Gauss variables in the limit of large N and P and for bounded $\alpha = P/N$ is realized. In this case, the optimization problem is reduced to the study for the ground state of the system, whose Hamiltonian includes parameters distributed by the Gauss law. In the proposed approach, as was noted in Introduction, there is no expansion in the parameter β/P that was used in the literature sources. Since the Gauss distribution of parameters of Hamiltonian (A.11) is the limiting one for $P \gg 1$, this imposes restrictions on the applicability of results in the region of variation of the parameter α. In our case, we determined the domain of applicability of the approximation by values of the parameter α, for which $\mathcal{H} > 0$. The numerical calculations indicate that, in this region, it is sufficient to restrict ourselves by approximation RS, which is related to the structure of the effective Hamiltonian (16). By summarizing, we note that though it is impossible to compare our results and those of works [6, 7, 9–11] for $\alpha < 1.345$, the Gauss approximation gives significantly less values of $\mathcal{H}$ (see Figure) for $\alpha > 1.345$. The improvement of the method, according to the central limit theorem, can be realized by the construction of asymptotic expansions for the Gauss distribution function (see, e.g., [17]), which is a complicated computational problem and requires the further studies.

APPENDICES

A. Distribution Function for Parameters of the Hamiltonian

The parameters of Hamiltonian (10) depend on the variables $a_{s,i}^{\mu}$, $s = (1, 2)$. We perform the transition from the averaging of some function $\Phi(\tilde{d}, \tilde{J}, \tilde{k})$ of parameters (11) over the variables $a_{s,i}^{\mu}$ to the averaging with the distribution function of these parameters with the help of the relation

$$\langle \Phi(\tilde{d}, \tilde{J}, \tilde{k}) \rangle_a = \int Dd\, DJ\, Dk\, \Psi(d, J, k) \Phi(d, J, k), \tag{A.1}$$

where $\Psi(d, J, k) = \langle \delta(d - \tilde{d}) \delta(J - \tilde{J}) \delta(k - \tilde{k}) \rangle_a$ has the sense of the probability density for the variables $\{d_i\}$, $\{J_{ij}\}$, $\{k_{ij}\}$. In formula (A.1), we introduced the definitions of the product of differentials

$$Dd = \prod_i dd_i,\; DJ = \prod_{i<j} dJ_{ij},\; Dk = \prod_{i\neq j} dk_{ij}$$

and δ-functions

$$\delta(d - \tilde{d}) \delta(J - \tilde{J}) \delta(k - \tilde{k}) =$$
$$= \prod_i \delta(d_i - \tilde{d}_i) \prod_{i<j} \delta(J_{ij} - \tilde{J}_{ij}) \prod_{i\neq j} \delta(k_{ij} - \tilde{k}_{ij}).$$

In the standard way (see, e.g., [17]), the characteristic function is defined as

$$\varphi(y, z, x) = \int Dd\, DJ\, Dk\, Dl\, \Psi(d, J, k) \times$$

$$\times \exp(i\mathbf{z}\mathbf{d} + i\mathbf{z}\mathbf{J} + i\mathbf{x}\mathbf{k}) = \langle \exp(i\mathbf{y}\tilde{\mathbf{d}} + i\mathbf{z}\tilde{\mathbf{J}} + i\mathbf{x}\tilde{\mathbf{k}}) \rangle_a, \tag{A.2}$$

where

$$\mathbf{y}\mathbf{d} = \sum_i y_i d_i,\; \mathbf{z}\mathbf{J} = \sum_{i<j} z_{ij} J_{ij},\; \mathbf{v}\mathbf{k} = \sum_{i\neq j} x_{ij} k_{ij}. \tag{A.3}$$

In the exponential function in (A.2), the content of notations is given by formulas (A.3), but it is necessary to replace the quantities $\{d_i\}$, $\{J_{ij}\}$, and $\{k_{ij}\}$ by (11). The characteristic function (A.2) is factorized by the index μ. As a result, we get

$$\varphi(y, z, x) = \varphi_1(y, z, x)^P,$$
$$\varphi_1(y, z, x) = \langle \exp(i\mathbf{y}\tilde{\mathbf{d}}_1 + i\mathbf{z}\tilde{\mathbf{J}}_1 + i\mathbf{x}\tilde{\mathbf{k}}_1) \rangle_a. \tag{A.4}$$

Index 1 in (A.4) indicates that a single term from the sum over μ should be taken in the relevant formulas determining $\tilde{d}$, $\tilde{J}$, and $\tilde{k}$. In particular,

$$\mathbf{y}\tilde{\mathbf{d}}_1 = -\sum_i y_i a_{1,i} a_{2,i}/\sqrt{P},\; \mathbf{z}\tilde{\mathbf{J}}_1 + \mathbf{x}\tilde{\mathbf{k}}_1 = \sum_i a_{i-} B_i,$$

$$B_i = \frac{1}{2\sqrt{P}} \sum_{j(>i)} a_{j-} z_{ij} + \frac{1}{2\sqrt{P}} \sum_{j(\neq i)} a_{j+} x_{ij}. \tag{A.5}$$

We recall that the variables $a_{j\pm}$ are defined in (9). To factorize the terms in (A.4) by index i of particles, we use the integral identity

$$F\Big(\sum_i a_{i-} B_i\Big) = \iint \frac{d\mathbf{u} d\mathbf{v}}{2\pi} \exp(-i\mathbf{u}\mathbf{v}) \exp(i\mathbf{v}\mathbf{a}_-) F\Big(\sum_i u_i B_i\Big),$$

where we introduced the notation $\mathbf{u}\mathbf{v} = \sum_i u_i v_i$, $\mathbf{v}\mathbf{a}_- = \sum_i v_i a_{i-}$. After simple transformations, we obtain

$$\varphi_1(y, z, x) = \int d\mathbf{u} \prod_j F_j(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}),$$

$$F_j(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}) = \frac{1}{2}\big[\exp(iL_j)\cos(2\omega_j^x)\delta(u_j)+$$

$$+ \frac{1}{2}\exp(-iL_j)\exp(-2i\omega_j^z)\delta(u_j + 1/\sqrt{P})+$$

$$+ \frac{1}{2}\exp(-iL_j)\exp(2i\omega_j^z)\delta(u_j - 1/\sqrt{P})\big], \tag{A.6}$$

where

$$L_j = -\frac{y_j}{\sqrt{P}},\; \omega_j^z = \sum_{i(<j)} u_i z_{ij},\; \omega_j^x = \sum_{i(\neq j)} u_i x_{ij}. \tag{A.7}$$

Formulas (A.6) and (A.4) set an exact representation of the characteristic function. In the limit of large P within the given procedure [17], we should take principal terms of the expansion of $\varphi_1(y, z, x)$ in $1/P$. This can be easily made, by expanding the function $\delta(u_j \pm 1/\sqrt{P})$ in a series in $1/\sqrt{P}$ at the first step. In particular, we obtain

$$F_j(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}) \approx F_{j0}(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x})+$$

$$+ \frac{1}{\sqrt{P}} F_{j1}(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}) + \frac{1}{2P} F_{j2}(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}), \tag{A.8}$$

where

$$F_{j0}(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}) = \frac{1}{2}[\exp(iL_j)\cos(2\omega_j^x) + \exp(-iL_j)\cos(2\omega_j^z)]\delta(u_j),$$

$$F_{j1}(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}) = -\frac{1}{2}\exp(-iL_j)\sin(2\omega_j^z)\delta'(u_j),$$

$$F_{j2}(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{x}) = \frac{1}{2}\exp(-iL_j)\cos(2\omega_j^z)\delta''(u_j). \tag{A.9}$$

The primes in (A.9) mean the derivatives of δ-functions. In the presence of δ-functions, the integrals over the variables u_j are simply taken after the substitution of (A.8) in (A.6). At the next step, we expand the factors $\exp(\pm iL_j)$ in degrees of $1/\sqrt{P}$. As a result, we obtain

$$\varphi_1(y, z, x) \approx 1 - \frac{1}{P}\sum_i y_i^2 - \frac{1}{2P}\sum_{i<j} z_{ij}^2 - \frac{1}{2P}\sum_{i\neq j} x_{ij}^2.$$

Respectively, for the characteristic function (A.4) in the limit $P \to \infty$, we get

$$\varphi(y, z, x) \approx \exp\Big(-\frac{1}{2}\big(\mathbf{y}^2 + \mathbf{z}^2 + \mathbf{x}^2\big)\Big). \tag{A.10}$$

Performing the Fourier transformation of the characteristic function (A.10), we obtain the asymptotics of the distribution function for parameters of the Hamiltonian $\Psi(d, J, k)$:

$$\Psi(d, J, k) = \prod_i \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}d_i^2\Big) \prod_{i<j} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}J_{ij}^2\Big) \times$$

$$\times \prod_{i\neq j} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}k_{ij}^2\Big). \tag{A.11}$$

B. Approximation 1RSB

Approximation 1RSB determines the replica symmetry breaking (approximation RS) on a single step (see, e.g., [3, 15]). In this approximation, the overlapping matrix $\hat{Q}$ is set in the form

$$Q_{ab} = \begin{cases} Q, & \text{if } a = b; \\ q, & \text{if } |a - b| > m; \\ q_1, & \text{if } |a - b| \leq m, \end{cases} \tag{B.1}$$

where m is the number of replicas in a group at the division of n into n/m groups. The numbers m and n/m are set as integers. It follows from definition (B.1) that if the indices of the matrix Q_{ab} belong to the same group, then $Q_{ab} = q$; if they belong to different groups, we have $Q_{ab} = q_1$. Relation (B.1) can be also written in the matrix form as

$$\hat{Q} = (Q - q_1)\hat{I}_n + (q_1 - q)\hat{I}_{\frac{n}{m}} \oplus \hat{E}_m + q\hat{E}_n. \tag{B.2}$$

The matrices $\hat{I}$ and $\hat{E}$ are, respectively, the identity matrix and a matrix, whose elements are equal to 1 (the index indicates the matrix order). In formula (B.2), we used also the definition of a direct product of matrices [19], which allows us to obtain

$$\sum_{a,b} Q_{ab}^2 = n\{(Q^2 - q_1^2) + m(q_1^2 - q^2)\}, \tag{B.3}$$

$$\sum_{a,b} \pi_a Q_{ab} \pi_b = (Q - q_1)\sum_a \pi_a^2 + q\left(\sum_a \pi_a\right)^2 +$$

$$+ (q_1 - q)\sum_{k=1}^{n/m}\left(\sum_{a \in A_k} \pi_a\right)^2, \tag{B.4}$$

where A_k in (B.4) stands for the collection of replicas belonging to the k-th group. The subsequent transformations are typical of the method of replicas (see [3, 15]). Therefore, we give them quite briefly. Let us substitute formulas (B.3) and (B.4) in the general relations (24), (27), and (29). We perform the factorization by replica variables in the calculation of $Z_1(Q)$ in (25) with the help of the integral transformation

$$e^{\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2}z^2} e^{zx}.$$

After a number of transformations, $Z_1(Q)$ can be written in the form

$$Z_1(Q) = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp(-z^2/2) Z_1^{n/m}, \tag{B.5}$$

where

$$Z_1 = \int_{-\infty}^{\infty} \frac{dz_1}{\sqrt{2\pi}} \exp(-z_1^2/2) \tilde{Z}_1^m,$$

$$\tilde{Z}_1 = \int_{-1}^{1} \frac{d\pi}{2} \exp\left(-\beta\frac{\sqrt{\alpha}}{2}\pi^2 + \frac{\beta^2}{2}(Q - q_1)\pi^2\right) \times$$

$$\times \exp(\beta\sqrt{1+q}\pi z)\exp(\beta\sqrt{q_1 - q}\pi z_1). \tag{B.6}$$

Passing to the limit $n \to 0$ in (29), we obtain

$$\mathcal{H} = \frac{N}{2} + \frac{N}{4\sqrt{\alpha}} \lim_{\beta \to \infty} \beta((Q^2 - q_1^2) + m(q_1^2 - q^2)) -$$

$$- \frac{N}{\sqrt{\alpha}}\frac{1}{m}\lim_{\beta \to \infty}\frac{1}{\beta}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\exp(-z^2/2)\ln Z_1. \tag{B.7}$$

The quantity (B.7) depends on the parameters Q, q, q_1, and m, the equations for which can be obtained by the simple differentiation of (B.7) and on the basis of the general steady-state equation (27). Simple calculations yield

$$Q = \left\langle \frac{\langle\langle\pi^2\rangle_\pi \tilde{Z}_1^m\rangle_{z_1}}{\langle \tilde{Z}_1^m\rangle_{z_1}}\right\rangle_z, \quad q_1 = \left\langle \frac{\langle\langle\pi\rangle_\pi^2 \tilde{Z}_1^m\rangle_{z_1}}{\langle \tilde{Z}_1^m\rangle_{z_1}}\right\rangle_z,$$

$$q = \left\langle \left(\frac{\langle\langle\pi\rangle_\pi \tilde{Z}_1^m\rangle_{z_1}}{\langle \tilde{Z}_1^m\rangle_{z_1}}\right)^2\right\rangle_z. \tag{B.8}$$

The averaging in (B.8) over the variables z and z_1 is realized by the formulas

$$\langle\ldots\rangle_z = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\exp\left(-z^2/2\right)(\ldots), \tag{B.9}$$

$$\langle\ldots\rangle_{z_1} = \int_{-\infty}^{\infty} \frac{dz_1}{\sqrt{2\pi}}\exp\left(-z_1^2/2\right)(\ldots), \tag{B.10}$$

and

$$\langle\ldots\rangle_\pi = \frac{1}{\tilde{Z}_1}\int_{-1}^{1}\frac{d\pi}{2}\exp\left(\beta\varphi(\pi)\right)(\ldots). \tag{B.11}$$

Equating the derivative of (B.7) with respect to the parameter m to zero, we get the equation for m:

$$\frac{1}{4}\beta(q_1^2 - q^2) + \frac{1}{m^2}\frac{1}{\beta}\langle\ln Z_1\rangle_z - \frac{1}{m}\frac{1}{\beta}\left\langle\frac{\langle\tilde{Z}_1^m \ln\tilde{Z}_1\rangle_{z_1}}{\langle\tilde{Z}_1^m\rangle_{z_1}}\right\rangle_z = 0. \tag{B.11}$$

Jointly, the system of equations (B.8), (B.11) and formula (B.7) yield a solution of the problem in approximation 1RSB. It follows from the above formulas that, for $q_1 = q$, approximation $1RSB$ transits into the approximation of symmetric replicas [3, 14].

C. Approximation 2RSB

In approximation 2RSB ([15]), the overlapping matrix for replicas is set in the form

$$\hat{Q} = (Q - q_2)\hat{I}_n + (q_2 - q_1)\hat{I}_{\frac{n}{m_2}} \oplus \hat{E}_{m_2}(q_1 - q)\hat{I}_{\frac{n}{m_1}} \oplus$$

$$\oplus\hat{E}_{m_1} + q\hat{E}_n, \tag{C.1}$$

where m_1 and m_2 determine, respectively, the number of replicas in a group at the division of n replicas into groups and m_1 replicas into m_1/m_2 groups. In this case, m_1, m_2, and m_1/m_2 are considered integers. The sense of the remaining notations was indicated above. Substituting matrix (C.1) in the general relations (25) and (29), we obtain

$$\sum_{a,b} Q_{ab}^2 = n\{(Q^2 - q_2^2) + m_2(q_2^2 - q_1^2) + m_1(q_1^2 - q^2)\}, \tag{C.2}$$

$$\sum_{a,b} \pi_a Q_{ab}\pi_b = (Q - q_2)\sum_a \pi_a^2 + (q_2 - q_1)\sum_{k_2=1}^{n/m_2}\left(\sum_{a \in A_{k_2}}\pi_a\right)^2 +$$

$$+ (q_1 - q)\sum_{k_1=1}^{n/m_1}\left(\sum_{a \in A_{k_1}}\pi_a\right)^2 + q\left(\sum_a \pi_a\right)^2, \tag{C.3}$$

where A_{k_2} and A_{k_1} in (C.3) stand for the collections of replicas belonging to the relevant groups. Like the case of 1RSB, the subsequent transformations consist in the factorization by replica variables at the calculation of $Z_1(Q)$ (29). The logic of calculations is the same as that for 1RSB. After the necessary transformations, we obtain

$$\mathcal{H}/N = \frac{1}{2} + \frac{1}{4\sqrt{\alpha}} \lim_{\beta \to \infty} \beta((Q^2 - q_2^2) + m_2(q_2^2 - q_1^2) +$$

$$+ m_1(q_1^2 - q^2)) - \frac{1}{\sqrt{\alpha}}\frac{1}{m_1}\lim_{\beta \to \infty}\frac{1}{\beta}\langle\ln Z_1\rangle_z, \tag{C.4}$$

where

$$Z_1 = \langle\langle \tilde{Z}_2^{m_2} \rangle_{z_2}^{\frac{m_1}{m_2}} \rangle_{z_1} \,, \; \tilde{Z}_2 = \int\limits_{-1}^{1} \frac{d\pi}{2} \exp[\beta\varphi(\pi)],$$

$$\varphi(\pi) = -\frac{\sqrt{\alpha}}{2}\pi^2 + \frac{\beta}{2}(Q - q_2)\pi^2 +$$
$$+ \sqrt{q_2 - q_1}\pi z_2 + \sqrt{q_1 - q}\pi z_1 + \sqrt{1 + q}\pi z. \qquad (C.5)$$

The free energy is a function of the parameters $Q, q_2, q_1, q, m_1,$ and m_2 which can be easily determined from the general steady-state equations (27). In particular,

$$Q = \langle \frac{1}{Z_1} \langle\langle\langle\pi^2\rangle \tilde{Z}_2^{m_2}\rangle_{z_2} \langle \tilde{Z}_2^{m_2} \rangle_{z_2}^{\frac{m_1}{m_2} - 1}\rangle_{z_1}\rangle_z,$$

$$q_2 = \langle \frac{1}{Z_1} \langle\langle\langle\pi\rangle^2 \tilde{Z}_2^{m_2}\rangle_{z_2} \langle \tilde{Z}_2^{m_2} \rangle_{z_2}^{\frac{m_1}{m_2} - 1}\rangle_{z_1}\rangle_z,$$

$$q_1 = \langle \frac{1}{Z_1} \langle\langle\langle\pi\rangle \tilde{Z}_2^{m_2}\rangle_{z_2}^2 \langle \tilde{Z}_2^{m_2} \rangle_{z_2}^{\frac{m_1}{m_2} - 2}\rangle_{z_1}\rangle_z,$$

$$q = \langle(\frac{1}{Z_1} \langle\langle\langle\pi\rangle \tilde{Z}_2^{m_2}\rangle_{z_2} \langle \tilde{Z}_2^{m_2} \rangle_{z_2}^{\frac{m_1}{m_2} - 1}\rangle_{z_1})^2\rangle_z. \qquad (C.6)$$

The averaging over the variables $z, z_1,$ and z_2 in formulas (C.5) and (C.6) is carried out with the Gauss function, as in formula (B.9). The equations for the parameters m_1 and m_2 are obtained by the differentiation of (C.4) with respect to m_1 and m_2:

$$\frac{\beta}{4}(q_1^2 - q^2) + \frac{1}{\beta m_1^2}\langle\ln(Z_1)\rangle_z -$$

$$-\frac{1}{\beta m_1 m_2}\langle \frac{1}{Z_1} \langle\langle \tilde{Z}_2^{m_2}\rangle_{z_2}^{\frac{m_1}{m_2}} \ln\langle \tilde{Z}_2^{m_2}\rangle_{z_2}\rangle_{z_1}\rangle_z = 0, \qquad (C.7)$$

$$\frac{\beta}{4}(q_2^2 - q_1^2) + \frac{1}{\beta m_2^2}\langle \frac{1}{Z_1} \langle\langle \tilde{Z}_2^{m_2}\rangle_{z_2}^{\frac{m_1}{m_2}} \ln\langle \tilde{Z}_2^{m_2}\rangle_{z_2}\rangle_{z_1}\rangle_z -$$

$$-\frac{1}{m_2}\langle \frac{1}{Z_1} \langle\langle \tilde{Z}_2^{m_2}\rangle_{z_2}^{\frac{m_1}{m_2} - 1} \langle \tilde{Z}_2^{m_2} \ln \tilde{Z}_2\rangle_{z_2}\rangle_{z_1}\rangle_z = 0. \qquad (C.8)$$

We note that all formulas are written prior to the transition to the limit $\beta \to \infty$. It follows from the structure of the above-presented solution that approximation 2RSB passes to 1RSB for $q_2 = q_1$ and, respectively, to RS for $q_2 = q_1$ and $q_1 = q$.

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ГАУССОВЕ НАБЛИЖЕННЯ В ОПТИМІЗАЦІЙНІЙ ЗАДАЧІ МОДЕЛІ ГРИ У МЕНШОСТІ

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Р е з ю м е

Методами статистичної фізики досліджено оптимізаційну задачу у відомій моделі гри у меншості. Оптимізаційну задачу зведено до вивчення основного стану реплічного гамільтоніана з випадковими параметрами деякої ефективної системи з неперервним спіном. Використовуючи ідеї центральних граничних теорем теорії ймовірностей, отримано представлення для функції розподілу параметрів гамільтоніана і виконано перехід до гауссового розподілу у випадку великих P. Застосовуючи наближення 1RSB та 2RSB в методі реплік, отримано залежність мінімуму досліджуваної величини від параметра α. Показано, що в області застосовності запропонований метод дає менші значення мінімуму, ніж в оригінальних роботах.