SUPPRESSION OF OSCILLATIONS BY LÉVY NOISE

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We find the analytic solution of a pair of stochastic equations with arbitrary forces and multiplicative Lévy noises in a steady-state nonequilibrium case. This solution shows that Lévy flights always suppress a quasiperiodic motion related to the limit cycle. We prove that such suppression is caused by that the Lévy variation $\Delta L \sim (\Delta t)^{1/\alpha}$ with the exponent $\alpha < 2$ is always negligible in comparison with the Gaussian variation $\Delta W \sim (\Delta t)^{1/2}$ in the $\Delta t \to 0$ limit.

1. Introduction

It is known that the crucial change in a behavior of the systems, which display noise-induced [1, 2] and recurrence [3–5] phase transitions, stochastic resonance [6, 7], noise-induced pattern formation [8, 9], noise-induced transport [2, 10], *etc.*, is caused by the interplay between noise and a nonlinearity (see Ref. [11] for review). Noises of different origin can play a constructive role in the dynamical behavior such as the hopping between multiple stable attractors [12, 13] and the stabilization of the Lorenz attractor near the threshold of its formation [14, 15]. This type of behavior is inherent in finite systems where the examples of a substantial alteration under the effect of intrinsic noises are the epidemics [16–18], predator-prey population dynamics [19, 20], opinion dynamics [21], biochemical clocks [22, 23], genetic networks [24], cyclic trapping reactions [25], *etc.*

The above-indicated phase transitions represent the simplest case where the joint effect of both noise and a nonlinearity arrives at the nontrivial fixed point appearance only on the phase-plane of system states. In this consideration, we are interested in studying a much more complicated situation, when the stochastic system can display the oscillatory behavior related to the limit cycle appearing as a result of the Hopf bifurcation [26, 27]. It was conjectured for a long time [28] that, in some situations, the influence of noise would be sufficient to produce the cyclic behavior [29]. Moreover, it was shown that the excitable [30] and bistable [31] systems and the systems close to bifurcations [32] display the oscillatory behavior, whose adjacency to an ideally periodic signal depends resonantly on the noise intensity [33] (due to this reason, such oscillations were called coherence resonance [30] or stochastic coherence [11]).

A characteristic peculiarity of the mentioned considerations is that all of them are restricted by studying the Gaussian noise effect, while such a noise is a special case of the Lévy stable process (the principal difference of these noises is known [34] to consist in the form of the probability distribution which exhibits the asymptotic power-law decay in the latter case and decays exponentially in the former one). Nowadays, the anomalous diffusion processes associated with the Lévy stable noise are attracting much attention in a vast variety of fields not only of natural sciences (physics, biology, earth science, and so on), but also in social sciences such as risk management, finances, *etc.*

In the context of physics, the recent investigation [35] has shown that the joint effect of both a nonlinearity and the Lévy noise can cause the occurrence of genuine phase transitions which is related to a fixed point on the phase-plane of system states. In this connection, the natural question arises: Can a self-organized quasiperiodic behavior related to the limit cycle be displayed by a system driven by the Lévy stable noises? Our work is devoted to the answer to this question within the analytic study of a two-dimensional stochastic system.

The paper is organized as follows. Since the equations, governing a behavior of a stochastic system driven by the multiplicative Lévy stable noise, are so complicated [36] that possess very nontrivial solutions [37] and, moreover, their derivation is now in progress [38], we start by Section 2 containing a derivation scheme of the Fokker–Planck equation for the Lévy multiplicative noises. In Section 3, we consider a pair of stochastic equations with arbitrary forces and multiplicative Lévy noises to obtain their analytic solution in a steady-state nonequilibrium case. This allows us to conclude in Section 4 that the Lévy flights opposite to the Gaussian noises always suppress a quasiperiodic motion related to the limit cycle. Appendix A completes our consideration to demonstrate that a closed consideration of the Lévy processes is achieved only within the Fourier representation.

2. Preliminaries: Statistical Picture of Lévy Multiplicative Noises

We start by considering the one-dimensional α-stable Lévy process $X(t)$, whose statistical description is achieved by the use of both single- and double-point distribution functions: the probability density $P(x,t)$ defines the rate of particle finding in the infinitesimal neighborhood of a point x at a time moment t, while the conditional density $p(x,t|x_0,t_0)$ gives the probability of the same event under condition that, at a previous time moment t_0, the particle was in the neighborhood of a point x_0. For Markov processes, which are related to statistically independent particle motions, the pointed out functions are connected by the equality

$$P(x,t) = \int dx_0\ p(x,t|x_0,t_0)P(x_0,t_0), \tag{1}$$

which means that the enumeration of all initial states of the conditional probability leads to its total value. On the other hand, the conditional probabilities of Markov processes are connected by the Chapman–Kolmogorov equation

$$p(x,t+dt|x_0,t_0) = \int dy\ p(x,t+dt|y,t)p(y,t|x_0,t_0). \tag{2}$$

The multiplicativity property of independent processes' probabilities inherently underlies this equality. However, in contrast to (1), the integration means here the enumeration of not initial but transitional states y, which are realized at the time moment t, being previous to the following time moment $t+dt$.

The description of stochastic processes is achieved in the simplest way if the characteristic function, being a Fourier transform

$$p(k,t+dt|y,t) = \int dx e^{ik(x-y)}p(x,t+dt|y,t), \tag{3}$$

is used. Since the initial and final moments of time are split here with an infinitesimal interval dt, we may assume that the characteristic function (3) slightly differs from one, and this difference comes to an infinitesimal increment $dK_X(k,dt|y,t)$ of the cumulant generating function of the stochastic process $X(t)$:

$$p(k,t+dt|y,t) := e^{dK_X(k,dt|y,t)}, \tag{4}$$

where the value of $dK_X(k,dt|y,t)$ should be determined.

With this aim, we use Eq. (2) in the limit $dt\to 0$ and the identity

$$p(x,t|x_0,t_0) = \int dy\ p(y,t|x_0,t_0)\int \frac{dk}{2\pi}e^{-ik(x-y)} \tag{5}$$

to write the chain of equalities

$$p(x,t+dt|x_0,t_0) - p(x,t|x_0,t_0) = \int dy\ p(y,t|x_0,t_0)\times$$

$$\times\int\frac{dk}{2\pi}e^{-ik(x-y)}[e^{dK_X(k,dt|y,t)}-1] \simeq$$

$$\simeq \int dy\ p(y,t|x_0,t_0)\int\frac{dk}{2\pi}e^{-ik(x-y)}dK_X(k,dt|y,t) =$$

$$= \int dy\ dK_X(x-y,t)p(y,t|x_0,t_0) \equiv$$

$$\equiv dK_X(x,t)\star p(x,t|x_0,t_0), \tag{6}$$

where the star $\star$ denotes the convolution of the inverse Fourier transform

$$dK_X(x-y,t) = \int\frac{dk}{2\pi}e^{-ik(x-y)}dK_X(k,dt|y,t),$$

$$p(x,t+dt|y,t) = \int\frac{dk}{2\pi}e^{-ik(x-y)}p(k,t+dt|y,t). \tag{7}$$

As a result, taking the definition

$$\mathcal{L}(x) := \frac{dK_X(x,t)}{dt} \tag{8}$$

into account, equalities (6) yield a symbolic representation of the Fokker–Planck equation

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \mathcal{L}(x)\star p(x,t|x_0,t_0). \tag{9}$$

The explicit form of the increment $\mathcal{L}(x)$ follows from the Langevin equation [39]

$$dX = fdt + gdL, \tag{10}$$

where the force $f = f(x)$ and the noise amplitude $g = g(x)$ are functions of the stochastic variable x related to the α-stable Lévy process $L = L(t)$. Within the

Itô calculus, this process is defined with the elementary characteristic function

$$\left\langle e^{ikdL}\right\rangle := e^{dK_L(k,dt|y,t)}, \tag{11}$$

where the Lévy increment reads [41]

$$\Lambda(k) := \frac{dK_L(k,dt|y,t)}{dt} = ik\gamma - D|mk|^{\alpha}e^{-i\varphi(\alpha)}. \tag{12}$$

Here, the asymmetry angle φ and the modulus m are defined as

$$\tan\left[\varphi(\alpha)\right] = \beta \mathrm{sgn}(gk)\tan(\pi\alpha/2),$$

$$m^{\alpha} = \sqrt{1+\beta^2\tan^2(\pi\alpha/2)}, \tag{13}$$

Lévy index $\alpha \in (0,2)$ characterizes the asymptotic tail $x^{-(\alpha+1)}$ of the Lévy stable distribution at $1 \neq \alpha < 2$ (the case $\alpha = 2$ is related to the Gaussian distribution), the parameter $\beta \in [-1,+1]$ defines a distribution asymmetry, a value of $-\infty < \gamma < +\infty$ defines the mean value of the stochastic variable X at $\alpha > 1$, and the angle brackets denote averaging over Lévy noises dL.

A Lévy increment of the cumulant generating function (12) defines a stochastic process in the absence of a force f and at a constant noise amplitude $g = 1$. In order to find the total increment related to process (10), we rewrite the corresponding characteristic function in the limits $dt \to 0$ and $dL(t) \to 0$:

$$e^{dK_X(k,dt|y,t)} := \left\langle e^{ikdX}\right\rangle = \left\langle e^{ik(fdt+gdL)}\right\rangle \simeq$$

$$\simeq 1 + ik\langle fdt + gdL\rangle = 1 + ik\left(\bar{f}dt + \langle gdL\rangle\right) \simeq$$

$$\simeq e^{ik\left(\bar{f}dt+\langle gdL\rangle\right)} \simeq e^{ik\bar{f}dt}\left\langle e^{i(kg)dL}\right\rangle :=$$

$$:= e^{ik\bar{f}dt}e^{dK_L(\bar{g}k,dt|y,t)}, \tag{14}$$

where $\bar{f} \equiv \langle f(X,t)\rangle$ and $\bar{g} \equiv \langle g(X,t)\rangle$ are, respectively, the mean values of the force and the noise amplitude, and Eq. (11) is taken into account. Similarly to definition (12), the elementary increment of the cumulant generating function

$$dK_X(k,dt|x,t) := \mathcal{L}(k,x)dt \tag{15}$$

is determined by the increment

$$\mathcal{L}(k,x) = ik\bar{f}(x) + \Lambda\big(\bar{g}(x)k\big) \tag{16}$$

whose explicit form reads [36, 38]

$$\mathcal{L}(k,x) = ik\left[f(x)+\gamma g(x)\right] - |mg(x)k|^{\alpha}e^{-i\varphi(\alpha)}. \tag{17}$$

Hereafter, we renormalize the noise amplitude $g(x)$ to suppress the scale factor D and skip the bar in the notations $\bar{f}$ and $\bar{g}$.

According to Eq. (9), the characteristic function $p(k,x;t) \equiv p(k,t|x,t)$, being the $dt \to 0$ limit of expression (3), is determined by the simplified Fokker–Planck equation

$$\frac{\partial}{\partial t}p(k,x;t) = \mathcal{L}(k,x)p(k,x;t), \tag{18}$$

where the kernel

$$\mathcal{L}(k,x) \equiv \mathcal{F}\{\mathcal{L}(x-y,x)\}(k,x) =$$

$$= \int d(x-y)\ \mathcal{L}(x-y,x)e^{ik(x-y)} \tag{19}$$

is determined by Eq. (17). Hereafter, the symbolic notation $\mathcal{F}$ is used for a direct Fourier transform and $\mathcal{F}^{-1}$ for an inverse one. It is fundamentally important that both the kernel $\mathcal{L}(k,x)$ and the characteristic function $p(k,x;t)$, being defined with the inverse Fourier tranforms (7) over the difference $x-y$, depends on the coordinate x through both the force $f(x)$ and the noise amplitude $g(x)$ [36].

In the simplest case of a stationary state, the behavior of the system is determined by the equation

$$\left[f(x)+\gamma g(x)\right]p(k,x) =$$

$$= -i\ \mathrm{sgn}(k)|k|^{\alpha-1}e^{-i\varphi(\alpha)}|mg(x)|^{\alpha}p(k,x) \tag{20}$$

following from Eqs. (18) and (17). The solution of this equation is achieved with the use of transformations of the direct space representation into the Fourier one which are expressed by the formal relations

$$\mathcal{F}\left\{\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}h(x)\right\} = -|k|^{\alpha}\tilde{h}(k),$$

$$\mathcal{F}\left\{|x|^{2m+1}h(x)\right\} = i(-1)^m\frac{\partial^{2m+1}}{\partial|k|^{2m+1}}\tilde{h}(k) \tag{21}$$

for an arbitrary function $h(x)$ and the Riesz derivative of both integer $2m+1$, $m = 0,1,\ldots$, and fractional α orders. For symmetric α-stable Lévy processes ($\gamma = 0$, $\varphi(\alpha) = 0$, and $m = 1$) under the effect of the force $f =$

x^{2m+1}, using the second equation (21) in Eq. (20) gives the asymptotics $p(k,x) \propto x^{-(\alpha+2m+1)}$ obtained first in Ref. [37].

Making use of Eqs. (18) and (17) yields the fractional Fokker–Planck equation for the one-point probability distribution function (1) [36, 38]

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[f(x)+\gamma g(x)\right]P(x,t)+$$

$$+\left[\frac{\partial^\alpha}{\partial|x|^\alpha}+\beta\tan\left(\frac{\pi\alpha}{2}\right)\frac{\partial}{\partial x}\frac{\partial^{\alpha-1}}{\partial|x|^{\alpha-1}}\right]\times$$

$$\times|g(x)|^\alpha P(x,t). \tag{22}$$

In symbolic form, a many-dimensional generalization of this equation for a symmetric Lévy flight reads

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = -\nabla\left[\mathbf{f}(\mathbf{x})+\hat{g}(\mathbf{x})\cdot\gamma\right]P(\mathbf{x},t)-$$

$$-\left[-\hat{\Delta}:\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})\right]^{\alpha/2}P(\mathbf{x},t). \tag{23}$$

Here, each of the dots denotes the summation over indices $i=1,2$, and the axes x_1, x_2 forming the pseudovector $\mathbf{x}$ are chosen in such a way that the noise amplitude matrix $\hat{g}$ takes diagonal form $g_{ij}=g_i\delta_{ij}$; and the components g_i, x_i form pseudovectors $\mathbf{g}$, $\mathbf{x}$. In the component representation, Eq. (23) has the continuity equation form

$$\frac{\partial}{\partial t}P(\mathbf{x},t)+\sum_i\frac{\partial}{\partial x_i}J_i(\mathbf{x})=0 \tag{24}$$

with the probability fluxes

$$J_i(\mathbf{x}) = \left\{\left[f_i(\mathbf{x})+g_i(\mathbf{x})\gamma_i\right]+\right.$$

$$\left.+\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_i^{\frac{\alpha}{2}-1}}\sum_j\left(-\frac{\partial}{\partial x_j}\right)^{\frac{\alpha}{2}}\left[g_i(\mathbf{x})g_j(\mathbf{x})\right]^{\frac{\alpha}{2}}\right\}P(\mathbf{x}). \tag{25}$$

In the generalized case of nonsymmetric Lévy flights, the two-dimensional flux components (25) are written as the Fourier transforms

$$J_1=\left\{(f_1+g_1\gamma_1)+i|m_1g_1|^{\frac{\alpha}{2}}e^{-i\varphi_1\left(\frac{\alpha}{2}\right)}|k_1|^{\frac{\alpha}{2}-2}k_1\times\right.$$

$$\left.\times\left[|m_1g_1k_1|^{\frac{\alpha}{2}}e^{-i\varphi_1\left(\frac{\alpha}{2}\right)}+|m_2g_2k_2|^{\frac{\alpha}{2}}e^{-i\varphi_2\left(\frac{\alpha}{2}\right)}\right]\right\}\widetilde{P},$$

$$J_2=\left\{(f_2+g_2\gamma_2)+i|m_2g_2|^{\frac{\alpha}{2}}e^{-i\varphi_2\left(\frac{\alpha}{2}\right)}|k_2|^{\frac{\alpha}{2}-2}k_2\times\right.$$

$$\left.\times\left[|m_1g_1k_1|^{\frac{\alpha}{2}}e^{-i\varphi_1\left(\frac{\alpha}{2}\right)}+|m_2g_2k_2|^{\frac{\alpha}{2}}e^{-i\varphi_2\left(\frac{\alpha}{2}\right)}\right]\right\}\widetilde{P}, \tag{26}$$

where the asymmetry parameters (13) are used.

3. Statistical Picture of Limit Cycle

According to the theorem of central manifold [26], in order to achieve a closed description of a limit cycle, it is enough to consider only two degrees of freedom related to some stochastic variables X_i, $i=1,2$. In this way, the stochastic evolution of the system under investigation is defined by the Langevin equations

$$dX_i = f_i dt + g_i dL_i, \quad i=1,2 \tag{27}$$

with arbitrary forces $f_i=f_i(x_1,x_2)$ and noise amplitudes $g_i=g_i(x_1,x_2)$, being functions of the variables x_i, $i=1,2$; and stochastic terms are related to the α-stable Lévy processes $L_i=L_i(t)$. Within the Itô calculus, these processes are determined by the elementary characteristic function

$$\left\langle e^{ik_i dX_i}\right\rangle := e^{\mathcal{L}_i dt} \tag{28}$$

with increments $\mathcal{L}_i=\mathcal{L}_i(k_1,k_2;x_1,x_2)$, whose expression [36]

$$\mathcal{L}_i = ik_i\left(f_i+\gamma_i g_i\right)-$$

$$-|m_ig_ik_i|^{\frac{\alpha}{2}}e^{-i\varphi_i\left(\frac{\alpha}{2}\right)}\sum_{j=1}^{2}|m_jg_jk_j|^{\frac{\alpha}{2}}e^{-i\varphi_j\left(\frac{\alpha}{2}\right)} \tag{29}$$

follows from Eq. (26), where the asymmetry parameters (13) are used.

As is shown in Section 2, the Fourier transform of the probability distribution function

$$\widetilde{P}(k_1,k_2;t)\equiv\mathcal{F}\{P(x_1,x_2)\}(k_1,k_2;t):=$$

$$:=\iint\limits_{-\infty}^{+\infty}dx_1dx_2\ P(x_1,x_2;t)e^{i(k_1x_1+k_2x_2)} \tag{30}$$

is governed by the Fokker–Planck equation

$$\frac{\partial\widetilde{P}}{\partial t}=\sum_{i=1}^{2}\left[i\left(f_i+\gamma_ig_i\right)k_i-\right.$$

$$-|m_i g_i k_i|^{\frac{\alpha}{2}} e^{-i\varphi_i\left(\frac{\alpha}{2}\right)} \sum_{j=1}^{2} |m_j g_j k_j|^{\frac{\alpha}{2}} e^{-i\varphi_j\left(\frac{\alpha}{2}\right)}\Bigg] \widetilde{P}. \quad (31)$$

Characteristically, being Fourier transformed, the r.h.s. of this equation depends on the wave vector components k_1 and k_2, while both forces $f_i = f_i(x_1, x_2)$ and multiplicative noise amplitudes $g_i = g_i(x_1, x_2)$ are dependent on the coordinate components x_1 and x_2.

According to the continuity equation (24), components of the steady-state probability flux obey the condition $\sum_i \partial J_i/\partial x_i = 0$, which means that the first component $J_1 = J_1(x_2)$ is a function of only the variable x_2, whereas the second component $J_2 = J_2(x_1)$. Then, in the Fourier space, the system's behavior is determined by the equations

$$\Big\{(f_1 + g_1\gamma_1) + i|m_1 g_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)} |k_1|^{\frac{\alpha}{2}-2} k_1 \times$$

$$\times \left[|m_1 g_1 k_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)} + |m_2 g_2 k_2|^{\frac{\alpha}{2}} e^{-i\varphi_2\left(\frac{\alpha}{2}\right)}\right]\Big\} \widetilde{P} =$$

$$= 2\pi J_1(k_2)\delta(k_1), \quad (32)$$

$$\Big\{(f_2 + g_2\gamma_2) + i|m_2 g_2|^{\frac{\alpha}{2}} e^{-i\varphi_2\left(\frac{\alpha}{2}\right)} |k_2|^{\frac{\alpha}{2}-2} k_2 \times$$

$$\times \left[|m_1 g_1 k_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)} + |m_2 g_2 k_2|^{\frac{\alpha}{2}-i\varphi_2} \left(\frac{\alpha}{2}\right)\right]\Big\} \widetilde{P} =$$

$$= 2\pi J_2(k_1)\delta(k_2). \quad (33)$$

Since the pair of these equations determines a single distribution function $\widetilde{P}(k_1, k_2)$, the consistency condition

$$\left[(f_1 + g_1\gamma_1) + ie^{-i\varphi_1(\alpha)}|m_1 g_1|^{\alpha}|k_1|^{\alpha-2}k_1\right] \times$$

$$\times \delta(k_2) J_2(k_1) =$$

$$= \left[(f_2 + g_2\gamma_2) + ie^{-i\varphi_2(\alpha)}|m_2 g_2|^{\alpha}|k_2|^{\alpha-2}k_2\right] \times$$

$$\times \delta(k_1) J_1(k_2) \quad (34)$$

should be fulfilled to restrict the choice of the probability flux components $J_1(k_2)$ and $J_2(k_1)$.

Multiplying Eq. (32) by the factor $|m_2 g_2|^{\frac{\alpha}{2}} e^{-i\varphi_2\left(\frac{\alpha}{2}\right)}$ and Eq. (33) by $|m_1 g_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)}$ and then subtracting results, we obtain

$$\Big\{F + i|m_1 m_2 g_1 g_2|^{\frac{\alpha}{2}} e^{-i\left[\varphi_1\left(\frac{\alpha}{2}\right)+\varphi_2\left(\frac{\alpha}{2}\right)\right]} \times$$

$$\times \left[|m_1 g_1 k_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)} + |m_2 g_2 k_2|^{\frac{\alpha}{2}} e^{-i\varphi_2\left(\frac{\alpha}{2}\right)}\right] \times$$

$$\times \left(|k_1|^{\frac{\alpha}{2}-2}k_1 - |k_2|^{\frac{\alpha}{2}-2}k_2\right)\Big\} \widetilde{P} =$$

$$= 2\pi \Big[J_1(k_2)\delta(k_1)|m_2 g_2|^{\frac{\alpha}{2}} e^{-i\varphi_2\left(\frac{\alpha}{2}\right)} -$$

$$- J_2(k_1)\delta(k_2)|m_1 g_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)}\Big], \quad (35)$$

where we have denoted

$$F \equiv (f_1 + \gamma_1 g_1)\, |m_2 g_2|^{\frac{\alpha}{2}} e^{-i\varphi_2\left(\frac{\alpha}{2}\right)} -$$

$$- (f_2 + \gamma_2 g_2)\, |m_1 g_1|^{\frac{\alpha}{2}} e^{-i\varphi_1\left(\frac{\alpha}{2}\right)}. \quad (36)$$

Equation (35) yields the explicit form of the probability distribution function

$$P(x_1, x_2) =$$

$$= \int_{-\infty}^{+\infty} \frac{dk_2}{2\pi} \frac{J_1(k_2)|m_2 g_2|^{\frac{\alpha}{2}} e^{-i\left[k_2 x_2 + \varphi_2\left(\frac{\alpha}{2}\right)\right]}}{F_2 - i|g_1|^{\frac{\alpha}{2}}|m_2 g_2|^{\alpha} e^{-i\varphi_2(\alpha)}|k_2|^{\alpha-2}k_2} -$$

$$- \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \frac{J_2(k_1)|m_1 g_1|^{\frac{\alpha}{2}} e^{-i\left[k_1 x_1 + \varphi_1\left(\frac{\alpha}{2}\right)\right]}}{F_1 + i|g_2|^{\frac{\alpha}{2}}|m_1 g_1|^{\alpha} e^{-i\varphi_1(\alpha)}|k_1|^{\alpha-2}k_1}, \quad (37)$$

where the effective forces $F_{1,2}$ are determined by Eq. (36) at $m_{2,1} = 1$ and $\varphi_{2,1} = 0$, respectively.

Before the analysis of Eq. (37), it is worth to note that the stochastic integration in case of multiplicative α-stable Lévy noises is based on the mathematical problem of integration of semimartingales (see Ref. [42], for example). In this work, we focus on the investigation of stability conditions of the limit cycle under the effect of multiplicative Lévy noises, leaving open subtle mathematical problems. However, from the physical-theoretic point of view, it is intuitively clear that the used method to obtain expression (37) is rather nonpathologic.

In the case of constant values of the probability flux within the state space x_1, x_2, the related Fourier transforms are $J_1(k_2) = 2\pi J_1^{(0)}\delta(k_2)$ and $J_2(k_1) = 2\pi J_2^{(0)}\delta(k_1)$ with factors $J_i^{(0)} = \text{const}$. Then the consistency condition (34) takes the form $(f_1 + g_1\gamma_1)\, J_2^{(0)} = (f_2 + g_2\gamma_2)\, J_1^{(0)}$, the effective force (36) is $F_0 =$

$(f_1+\gamma_1 g_1)\,|g_2|^{\frac{\alpha}{2}}-(f_2+\gamma_2 g_2)\,|g_1|^{\frac{\alpha}{2}}$, and the probability density (37) reads

$$P=\frac{J_1^{(0)}|g_2|^{\frac{\alpha}{2}}-J_2^{(0)}|g_1|^{\frac{\alpha}{2}}}{F_0}. \tag{38}$$

In order to create a limit cycle, this distribution function should diverge on a closed curve, so that the effective force should be equal to $F_0=0$. Together with the consistency condition, this equation gives

$$\frac{J_1^{(0)}}{J_2^{(0)}}=\frac{f_1+\gamma_1 g_1}{f_2+\gamma_2 g_2}=\left|\frac{g_1}{g_2}\right|^{\frac{\alpha}{2}}. \tag{39}$$

But these equalities mean that the numerator of the probability density (38) disappears together with its denominator. As a result, we conclude that the limit cycle creation is impossible for a stationary nonequilibrium state, both probability flux components $J_1(x_1,x_2)$ and $J_2(x_1,x_2)$ being constants.

The subsequent consideration of the problem requires a calculation of integrals in Eq. (37) at arbitrary dependences $J_1(k_2)$ and $J_2(k_1)$. For this purpose, it is convenient to write $|k|=\mathrm{sgn}(k)k=e^{i\pi\theta(-k)}k$, where $\theta(k)$ denotes the Heaviside step function. Thus, we have $|k|^{\alpha-2}k=e^{-i\pi\theta(-k)(2-\alpha)}k^{\alpha-1}$, and the pole points of the integrands in Eq. (37) are expressed by the equality

$$K_{1,2}=\left(\frac{F_{1,2}}{|m_{1,2}g_{1,2}|^{\alpha}|g_{2,1}|^{\frac{\alpha}{2}}}\right)^{\frac{1}{\alpha-1}}\times$$

$$\times\exp\left\{i\frac{\varphi_{1,2}(\alpha)+(2-\alpha)\pi\theta(-\Re K_{1,2})}{\alpha-1}+\right.$$

$$\left.+\frac{(\pi/2)\mathrm{sgn}(\Im K_{1,2})}{\alpha-1}\right\}. \tag{40}$$

Due to the sign-changing term $(\pi/2)\mathrm{sgn}(\Im K_{1,2})$ in the exponent, the $K_{1,2}$ poles are located on the opposite half-planes of the complex variables $k_{1,2}$. Making use of the power series expansion

$$k^{\alpha-1}=K^{\alpha-1}\left(1+\frac{k-K}{K}\right)^{\alpha-1}\approx$$

$$\approx K^{\alpha-1}+(\alpha-1)K^{\alpha-2}(k-K) \tag{41}$$

allows us to reduce the integrands in Eq. (37) to a pole form. However, we cannot close the integration contours around both the upper and lower complex half-planes of the k variable, since the related integrands contain absolute magnitudes.

In order to find the required integrals, let us specify the contribution of a pole located on the upper half-plane of the complex number k. With this aim, we divide this half-plane into two parts related to the positive and negative values of the real part of k. As Figure shows, the integrals in Eq. (37) can be rewritten as follows:

$$\int_{-\infty}^{+\infty}\frac{f(k)}{k-K}dk\equiv\int_{AB}\frac{f(k)}{k-K}dk+\int_{DE}\frac{f(k)}{k-K}dk=$$

$$=\oint_{ABC}\frac{f(k)}{k-K}dk-\left[\int_{BC}\frac{f(k)}{k-K}dk+\int_{CA}\frac{f(k)}{k-K}dk\right]+$$

$$+\oint_{DEF}\frac{f(k)}{k-K}dk-\left[\int_{EF}\frac{f(k)}{k-K}dk+\int_{FD}\frac{f(k)}{k-K}dk\right]=$$

$$=\oint_{ABC}\frac{f(k)}{k-K}dk+\oint_{DEF}\frac{f(k)}{k-K}dk-$$

$$-\left[\int_{BC}\frac{f(k)}{k-K}dk+\int_{FD}\frac{f(k)}{k-K}dk\right]-$$

$$-\left[\int_{CA}\frac{f(k)}{k-K}dk+\int_{EF}\frac{f(k)}{k-K}dk\right]. \tag{42}$$

If the radii of arcs CA and EF tend to infinity, both integrals in the last square brackets disappear. On the other hand, when both half-axes BC and FD tend one to another, we have $\int_{BC}=-\int_{FD}$, so that the terms in the square brackets standing before are cancelled also. Moreover, the integral over contour DEF equals zero, because this contour does not envelop any pole. As a result, we obtain

$$\int_{-\infty}^{+\infty}\frac{f(k)}{k-K}dk=\oint_{ABC}\frac{f(k)}{k-K}dk=2\pi i\ \mathrm{sgn}(\Im K)f(K), \tag{43}$$

where the last equality is due to the residue theorem.

Finally, making use of the Cauchy integral (43) yields the probability distribution function (37) in the form

$$P\left(x_1, x_2\right) = F_1^{\frac{2-\alpha}{\alpha-1}} P_1 e^{-i(K_1 x_1 - \phi_1)} +$$

$$+ F_2^{\frac{2-\alpha}{\alpha-1}} P_2 e^{-i(K_2 x_2 - \phi_2)}, \quad (44)$$

where we have denoted

$$P_{1,2} \equiv \frac{J_{2,1}\left(K_{1,2}\right)}{(\alpha-1)|g_{2,1}|^{\frac{\alpha}{2(\alpha-1)}}|m_{1,2}g_{1,2}|^{\frac{\alpha(3-\alpha)}{2(\alpha-1)}}},$$

$$\phi_{1,2} \equiv \frac{3-\alpha}{\alpha-1}\varphi_{1,2}\left(\frac{\alpha}{2}\right) + \frac{\pi}{2}\frac{2-\alpha}{\alpha-1}\times$$

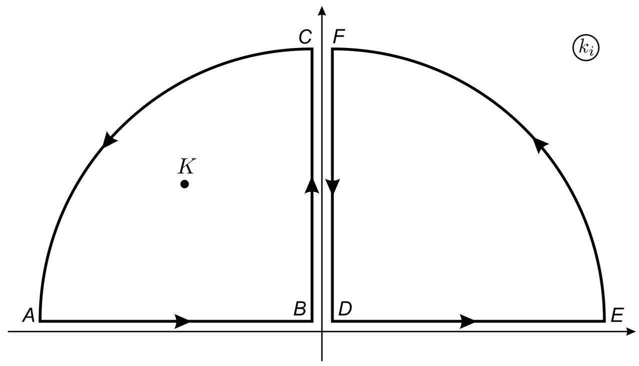
$$\times\left[\mathrm{sgn}\left(\Im K_{1,2}\right) + 2\theta\left(-\Re K_{1,2}\right)\right]. \quad (45)$$

4. Discussion

The analytic consideration developed in the previous section allowed us to obtain the probability distribution function (44) which describes the behavior of a nonequilibrium steady-state stochastic system driven by the Lévy multiplicative noise with two degrees of freedom. Recently, we have studied conditions for the limit cycle creation in stochastic Lorenz-type systems driven by Gaussian noises [40]. The noise-induced resonance has been found analytically to appear in a nonequilibrium steady state if the principal variable, which is coupled with two different degrees of freedom or more, displays the fastest variations. The condition for the appearance of this resonance is expressed formally in the divergence of the probability distribution function, being inversely proportional to an effective force of type (36). When this force vanishes on a closed curve of the phase plane, the system evolves along this cycle with the diverging probability density.

In opposite to such a dependence, the distribution function (44) contains the effective force (36) in the positive power $(2-\alpha)/(\alpha-1)$ only. To this end, we can conclude the Lévy flights suppress always a quasiperiodic motion related to the limit cycle in a nonequilibrium steady state. This is the main result of our consideration. The cornerstone of the difference between stochastic systems driven by the Lévy and Gaussian noises is that the Lévy variation $\Delta L \sim (\Delta t)^{1/\alpha}$ with the exponent $\alpha < 2$ is negligible in comparison with the Gaussian variation $\Delta W \sim (\Delta t)^{1/2}$ as $\Delta t \to 0$.

It is worth to note that the above difference removes the problem of the calculus choice [1, 39]. This problem



To the calculation of the integrals standing in Eqs. (42) and (43)

is known to be caused by the irregularity of the time dependence $X(t)$ of a stochastic variable (for the sake of simplicity, we return to the one-dimensional case). Keeping in mind all problems of the integration of semimartingales [42], we can write the formal integral of the equation of motion (10) as follows:

$$X(t) = \int_0^t f\big(x(t')\big)dt' + \int_{L(0)}^{L(t)} g\big(x(\tilde{t'})\big)dL(t'). \quad (46)$$

Here, one should take the noise amplitude $g\big(x(\tilde{t'})\big)$ at the time moment

$$\tilde{t'} = t' + \lambda\Delta t'; \quad \lambda \in [0,1],\ \Delta t' \to 0 \quad (47)$$

which does not coincide with the integration time t' due to a parameter $\lambda \in [0,1]$, whose value fixes the calculus choice (for example, the magnitude $\lambda = 1/2$ relates to the Stratonovich case) [1, 39]. Taking Eqs. (47) and (27) into account, we obtain

$$g\big(x(\tilde{t})\big) \simeq g\big(x(t)\big) + \lambda g'\big(x(t)\big)\Delta X(t) \simeq$$

$$\simeq g\big(x(t)\big) + \lambda g'\big(x(t)\big) f\big(x(t)\big)\Delta t +$$

$$+ \lambda g'\big(x(t)\big) g\big(x(t)\big)\Delta L(t), \quad (48)$$

where primes denote the differentiation with respect to the argument x. Being inserted into Eq. (46), the first term in the last line of Eq. (48) relates to the usual case of the Itô calculus. The corresponding insertion of the second term gives an addition, whose order $\Delta L\Delta t \sim (\Delta t)^{1+(1/\alpha)} \ll \Delta t$ is higher than one for the previous term (this situation is inherent in the Gaussian case as well). Finally, after the insertion of the last term of Eq.

(48), the last integrand in Eq. (46) obtains an addition of order $(\Delta L)^2 \sim (\Delta t)^{2/\alpha}$. In a special case of the Gaussian noise ($\alpha = 2$), the order $2/\alpha$ of this addition coincides with that of the first integrand of Eq. (46), which results in the addition $\lambda g(x) g'(x)$ to the physical force $f(x)$. A principally different situation is realized for the Lévy stable process, when the index $\alpha < 2$ and the above addition should be suppressed in comparison with the physical force because $(\Delta t)^{2/\alpha} \ll \Delta t$.

APPENDIX A.
Consideration of the Lévy stable processes within the direct stochastic space

After the inverse Fourier transformation, components (32) and (33) of the stationary probability flux are written as follows:

$$\left\{(f_1+g_1\gamma_1)+\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\left[\left(-\frac{\partial}{\partial x_1}\right)^{\frac{\alpha}{2}} g_1^{\alpha}+\right.\right.$$

$$\left.\left.+\left(-\frac{\partial}{\partial x_2}\right)^{\frac{\alpha}{2}}(g_1g_2)^{\frac{\alpha}{2}}\right]\right\}P=J_1^{(0)}(x_2),$$

$$\left\{(f_2+g_2\gamma_2)+\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}\left[\left(-\frac{\partial}{\partial x_1}\right)^{\frac{\alpha}{2}}(g_2g_1)^{\frac{\alpha}{2}}+\right.\right.$$

$$\left.\left.+\left(-\frac{\partial}{\partial x_2}\right)^{\frac{\alpha}{2}} g_2^{\alpha}\right]\right\}P=J_2^{(0)}(x_1). \qquad (A.1)$$

Acting by the $g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}$ operator on the first of these equations and by the $g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}$ operator on the second one, we obtain

$$g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\left[\left(-\frac{\partial}{\partial x_1}\right)^{\frac{\alpha}{2}} g_1^{\alpha}+\left(-\frac{\partial}{\partial x_2}\right)^{\frac{\alpha}{2}}(g_1g_2)^{\frac{\alpha}{2}}\right]P=$$

$$=g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}\left[J_1^{(0)}-(f_1+g_1\gamma_1)P\right],$$

$$g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}\left[\left(-\frac{\partial}{\partial x_1}\right)^{\frac{\alpha}{2}}(g_2g_1)^{\frac{\alpha}{2}}+\left(-\frac{\partial}{\partial x_2}\right)^{\frac{\alpha}{2}} g_2^{\alpha}\right]P=$$

$$=g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\left[J_2^{(0)}-(f_2+g_2\gamma_2)P\right]. \qquad (A.2)$$

Subtracting the above equalities term-by-term, we arrive at the fractional differential equation

$$\left[(f_1+g_1\gamma_1)g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}-(f_2+g_2\gamma_2)g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\right]P+$$

$$+G(x_1,x_2)P=g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}J_1^{(0)}(x_2)-g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}J_2^{(0)}(x_1), \quad (A.3)$$

where the function

$$G(x_1,x_2)\equiv g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\times$$

$$\times\left[\left(-\frac{\partial}{\partial x_1}\right)^{\frac{\alpha}{2}} g_1^{\alpha}+\left(-\frac{\partial}{\partial x_2}\right)^{\frac{\alpha}{2}}(g_1g_2)^{\frac{\alpha}{2}}\right]-$$

$$-g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}\times$$

$$\times\left[\left(-\frac{\partial}{\partial x_1}\right)^{\frac{\alpha}{2}}(g_2g_1)^{\frac{\alpha}{2}}+\left(-\frac{\partial}{\partial x_2}\right)^{\frac{\alpha}{2}} g_2^{\alpha}\right]+$$

$$+\left[g_2^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_2^{\frac{\alpha}{2}-1}}(f_1+g_1\gamma_1)\,g_1^{\frac{\alpha}{2}}\frac{\partial^{\frac{\alpha}{2}-1}}{\partial x_1^{\frac{\alpha}{2}-1}}(f_2+g_2\gamma_2)\right] \qquad (A.4)$$

is introduced. For the Gauss processes ($\alpha = 2$), the differential equation (A.3) is reduced to an algebraic one to give the probability distribution function that was found in our previous work [40]. However, in the general case $\alpha \le 2$, the solution of the fractional differential equation (A.3) arrives at a complicated problem, so that we are obliged to use the Fourier representation in Section 3.

Finally, it is worth to note that the consistency condition (34) takes the form

$$\left[\frac{\partial}{\partial x_1}(f_1+g_1\gamma_1)-\left(-\frac{\partial}{\partial x_1}\right)^{\alpha}|g_1|^{\alpha}\right]J_2^{(0)}(x_1)=$$

$$=\left[\frac{\partial}{\partial x_2}(f_2+g_2\gamma_2)-\left(-\frac{\partial}{\partial x_2}\right)^{\alpha}|g_2|^{\alpha}\right]J_1^{(0)}(x_2) \qquad (A.5)$$

within the inverse Fourier representation, where $\varphi_i = 0$ and $m_i = 1$ are taken for simplicity. Equation (A.5) connects explicitly the probability flux components $J_{2,1}(x_{1,2})$, being arbitrary functions, with given dependences of both forces $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ and multiplicative amplitudes $g_1(x_1, x_2)$, $g_2(x_1, x_2)$, respectively.

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ПРИГНІЧЕННЯ ОСЦИЛЯЦІЙ ШУМОМ ЛЕВІ

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Р е з ю м е

Знайдено аналітичний розв'язок пари стохастичних рівнянь з довільними силами та мультиплікативними шумами Леві у стаціонарному нерівноважному випадку. Це рішення показує, що польоти Леві завжди пригнічують квазіперіодичний рух, пов'язаний з граничним циклом. Доведено, що таке пригнічення викликано тим, що варіація Леві $\Delta L \sim (\Delta t)^{1/\alpha}$ зі ступенем $\alpha < 2$ завжди незначна порівняно з гаусівською варіацією $\Delta W \sim (\Delta t)^{1/2}$ при $\Delta t \to 0$.