DYNAMICAL PROPERTIES OF A BOSE GAS WITH δ-LIKE INTERACTION BETWEEN PARTICLES AT TEMPERATURES ABOVE THE PHASE TRANSITION POINT AND IN THE LIMIT OF STRONG INTERPARTICLE REPULSION

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Exact equations for the one-particle Green's function and for the irreducible part of the two-particle Green's function of a three-dimensional Bose gas with point-like interaction between particles have been derived in the framework of the functional integral approach. The two-particle spectrum of the system has been analyzed in detail in the simplest approximation, which makes allowance for all two-particle scattering processes. The leading asymptotics of the single-particle spectrum in the long-wave range was shown to remain quadratic. The critical temperature was found in the limit of strong repulsion between particles.

1. Introduction

Almost 70 years passed since the first successful description had been made for the weakly nonideal Bose gas. The details of the theory of interacting bosons compose a section in every textbook on quantum-mechanical statistics. Tens of reviews are devoted to this problem (one of the latest is work [1]). Nevertheless, the disputes concerning the properties of this system, which are rather simple at first glance, have been continuing till now. For instance, a model describing the behavior of the critical temperature T_c, when a weak interaction between particles is switched on, has been proposed relatively recently. It was found [2] that the shift of T_c with respect to the critical temperature of the ideal Bose gas is linear in the s-scattering length a:

$$\frac{T_c - T_0}{T_0} = ca\rho^{1/3}, \qquad (1)$$

where ρ is the equilibrium density, and c is a constant. Even the sign on this constant was not known for a long time, and the dependence on the scattering length was thought to be the root one (see works [3, 4] and discussion therein). The simplest calculation making use of the $1/N$-expansion [5] gives the value $c = 2.33$, which agrees quite well with the simulation results $c = 1.29 \pm 0.05$ [6] and $c = 1.32 \pm 0.02$ [7] for the classical φ^4-model. Theoretical calculations in higher approximations [8] improve the result to $c = 1.71$. Here, the difference from the results obtained by numerical methods does not exceed 30%. An excellent agreement was attained only in the six- ($c = 1.25 \pm 0.13$) and seven-loop ($c = 1.27 \pm 0.11$) approximations [9]. It is of interest that the next term in the expansion in formula (1) is nonanalytic in the gas parameter [10, 11].

In fact, for finite temperatures, the perturbation theory is constructed on the basis of the ratio a/Λ, where $\Lambda = \sqrt{2\pi\hbar^2/mT}$ is the length of a de Broglie thermal wave, rather than the gas parameter. It is clear that, for a weakly interacting gas, expansions in those parameters are identical in a vicinity of the critical temperature. In this work, we use another dimensionless parameter $g = 4\pi\hbar^2\rho a/mT$, the ratio between the characteristic potential energy of a particle and the temperature of the system. Our attention is mainly concentrated on the analysis of the limiting case where the values of this parameter are large. The application of this parameter is more convenient, but the results are expressed in terms of the gas parameter. In a vicinity of the critical temperature T_c, large g-values correspond to large $\rho^{1/3}a$-values, and, for a weakly interacting Bose gas, the perturbation theory in the parameter g is completely equivalent to the expansion in the parameter a/Λ.

The structure of the paper is as follows. In Section 2, the model is formulated, and the calculation method is described. For the sake of completeness, Section 2 also contains general relations, which couple the one-particle Green's function with parameters of the quasiparticle spectrum. In Section 3, an approximation, which is the key point of this work, is described. In particular, the choice of the structure for a one-particle Green's function is discussed. It turns out that, for the Green's function to be calculated in the limit $\rho^{1/3}a \gg 1$, one has to analyze the two-particle spectrum of a system of interacting Bose particles. In Section 4, the parameters inherent to the one-particle spectrum of the system are analyzed, and the critical temperature is calculated in the limiting case $g \gg 1$. Sections 3 and 4 contain the principal results of this work.

2. General Relations

Consider a collection of N spinless particles with a point-like potential of pair interaction, which are embedded into the volume V. In other words, the potential energy of interaction between two particles is chosen proportional to the delta-function,

$$\Phi(r) = \widetilde{\lambda}\delta(\mathbf{r}),$$

where r is the distance between particles, and the constant $\widetilde{\lambda}$ characterizes the magnitude of particle–particle interaction. This model is known to be stable only in the one-dimensional case, whereas, for higher space dimensions, this constant must be "regularized" in final expressions. In particular, in our case,

$$\widetilde{\lambda}^{-1} = \lambda^{-1} - \frac{1}{V}\sum_{\mathbf{p}} \frac{1}{2\varepsilon_p}, \tag{2}$$

where $\lambda = 4\pi\hbar^2 a/m$, a is the s-scattering length, and $\varepsilon_p = \hbar^2p^2/2m$ is the spectrum of the ideal gas. An alternative way consists in using the pseudopotential method [12],

$$\Phi(r) = \lambda\delta(\mathbf{r})\frac{\partial}{\partial r}r,$$

where the operator on the right-hand side of this equality eliminates the features of the $1/r$-type at short distances.

Let us write down the model partition function in terms of the functional integral [13]

$$Z = \int D\psi^* D\psi \exp\{S\}, \tag{3}$$

where

$$S = \sum_{\omega_n}\sum_{\mathbf{p}}\{i\omega_n - \xi_p\}\psi^*_{\mathbf{p}}(\omega_n)\psi_{\mathbf{p}}(\omega_n) -$$

$$-\frac{1}{2V\beta}\sum_{\omega''_n,\omega'_n,\omega_n}\sum_{\mathbf{k},\mathbf{q},\mathbf{p}}\widetilde{\lambda}\psi^*_{\mathbf{p}}(\omega_n)\psi^*_{\mathbf{q}}(\omega'_n)\times$$

$$\times\psi_{\mathbf{q}+\mathbf{k}}(\omega'_n+\omega''_n)\psi_{\mathbf{p}-\mathbf{k}}(\omega_n-\omega''_n). \tag{4}$$

is the Euclidean action, $\xi_p = \varepsilon_p - \mu$, μ is the chemical potential, $\omega_n = 2\pi n\beta$ $(n = 0, \pm1, \ldots)$ are the Matsubara frequencies, and $\beta = 1/T$ is the inverse temperature.

For the term in formula (4), which describes the interaction, let us execute the Stratonovich–Hubbard transformation by additionally introducing a collection of complex-valued variables $\eta_{\mathbf{q}}(\omega_n)$. Then, the effective action, which is a basis for the subsequent consideration, looks like

$$S_{\text{eff}} = \sum_{\omega_n}\sum_{\mathbf{p}}\{i\omega_n - \xi_p\}\psi^*_{\mathbf{p}}(\omega_n)\psi_{\mathbf{p}}(\omega_n) -$$

$$-(2\widetilde{\lambda})^{-1}\sum_{\omega_n}\sum_{\mathbf{q}}|\eta_{\mathbf{q}}(\omega_n)|^2 + \frac{i}{2\sqrt{\beta V}}\sum_{\omega_n,\omega'_n}\sum_{\mathbf{p},\mathbf{q}}\{\eta^*_{\mathbf{q}+\mathbf{p}}\times$$

$$\times(\omega'_n+\omega_n)\psi_{\mathbf{p}}(\omega_n)\psi_{\mathbf{q}}(\omega'_n) + \text{c.c.}\}. \tag{5}$$

It is clear that functional integration should be carried out over the real and imaginary parts of the fields ψ and η:

$$Z = \int D\psi^* D\psi D\eta^* D\eta \exp\{S_{\text{eff}}\}. \tag{6}$$

Changing over from S to S_{eff}, we partially reconstruct the perturbation theory series, taking the direct processes of particle-particle scattering as the basis.

The one-particle temperature Green's function is

$$G(\omega_n, p) = -\langle\psi_{\mathbf{p}}(\omega_n)\psi^*_{\mathbf{p}}(\omega_n)\rangle, \tag{7}$$

where the angle brackets mean the averaging with action (4) or (5). Let us introduce a pair correlator $\langle\eta_{\mathbf{q}}(\omega_n)\eta^*_{\mathbf{q}}(\omega_n)\rangle$ into consideration. Since our system undergoes a phase transition, it is convenient for our calculations to construct exact equations that couple the one-particle Green's function and the η-correlator. The corresponding equations can be obtained, if one takes into account that "classical" trajectories–they can be obtained by solving the corresponding equations, $\delta(-S_{\text{eff}})/\delta\psi^*_{\mathbf{p}}(\omega_n) = 0$ and $\delta(-S_{\text{eff}})/\delta\eta^*_{\mathbf{q}}(\omega_n) = 0$—give the main contribution to the partition function. Statistically averaging those equalities and taking the jump in the corresponding functions into account, we obtain the relations

$$G^{-1}(\omega_n, p) = i\omega_p - \xi_p + \frac{1}{\beta V}\sum_{\omega'_n}\sum_{\mathbf{q}}\Gamma(\omega_n, \mathbf{p}|\omega'_n, \mathbf{q})\times$$

$$\times\langle\eta_{\mathbf{p}+\mathbf{q}}(\omega_n+\omega'_n)\eta^*_{\mathbf{p}+\mathbf{q}}(\omega_n+\omega'_n)\rangle G(\omega'_n, q), \tag{8}$$

$$\langle \eta_{\mathbf{q}}(\omega_n)\eta^*_{\mathbf{q}}(\omega_n)\rangle^{-1} = (2\widetilde{\lambda})^{-1} +$$

$$+\frac{1}{2\beta V}\sum_{\omega'_n}\sum_{\mathbf{p}}\Gamma(\omega'_n,\mathbf{p}|\omega_n-\omega'_n,\mathbf{q}-\mathbf{p})\times$$

$$\times G(\omega'_n,p)G(\omega_n-\omega'_n,|\mathbf{q}-\mathbf{p}|), \tag{9}$$

where the vertex function [14]

$$\langle \eta^*_{\mathbf{p}+\mathbf{q}}(\omega_n+\omega'_n)\psi_{\mathbf{p}}(\omega_n)\psi_{\mathbf{q}}(\omega'_n)\rangle = \frac{i}{\sqrt{\beta V}}\Gamma(\omega_n,\mathbf{p}|\omega'_n,\mathbf{q})\times$$

$$\times G(\omega_n,p)G(\omega'_n,q)\langle \eta_{\mathbf{p}+\mathbf{q}}(\omega_n+\omega'_n)\eta^*_{\mathbf{p}+\mathbf{q}}(\omega_n+\omega'_n)\rangle$$

is introduced as usual. It is a complicated system of nonlinear integral equations with two unknown functions; the approximation is constructed for the vertex function. The search for the solution should be proceeded from the one-particle Green's function, for which

$$G^{-1}(\omega_n,p) = i\omega_n - \xi_p - \Sigma(\omega_n,p), \tag{10}$$

where $\Sigma(\omega_n,p)$ is the self-energy part (the mass operator). Let us execute an analytical continuation into the upper half-plane of the complex frequency plane and introduce the notations

$$\mathrm{Re}\,\Sigma(\omega_n,p)|_{i\omega_n\to\omega+i0} = \Sigma_R(\omega,p), \tag{11}$$

$$\mathrm{Im}\,\Sigma(\omega_n,p)|_{i\omega_n\to\omega+i0} = \Sigma_I(\omega,p). \tag{12}$$

The roots of the equation [15],

$$\widetilde{\xi}_p = \xi_p + \Sigma_R(\widetilde{\xi}_p,p), \tag{13}$$

determine a new one-particle spectrum $\tilde{\xi}_p$, provided that the damping

$$\gamma_p = Z(p)\Sigma_I(\widetilde{\xi}_p,p), \tag{14}$$

$$Z^{-1}(p) = 1 - \frac{\partial\Sigma_R(\omega,p)}{\partial\omega}\Big|_{\omega=\widetilde{\xi}_p} \tag{15}$$

is low enough, of course. Then we choose, in a certain approximation, the vertex function $\Gamma(\omega_n,\mathbf{p}|\omega'_n,\mathbf{q})$ and write down Eq. (9). It is clear that the described calculation procedure is difficult to be carried out in the general case. Therefore, we make certain simplifications. The Green's function (10) is replaced by its value in the pole vicinity at small wave vectors,

$$G^{-1}(\omega_n,p) = Z^{-1}\{i\omega_n - \widetilde{\xi}_p\}, \tag{16}$$

where $\tilde{\xi}_p = \tilde{\varepsilon}_p - \tilde{\mu}$; $Z = Z(0)$; the one-particle spectrum $\tilde{\varepsilon}_p = \hbar^2p^2/2\tilde{m}$ is chosen to be quadratic in the wave vector, which reminds the spectrum of the ideal Bose gas, but with the renormalized mass; and $\tilde{\mu}$ is the renormalized chemical potential. At the critical point, either $G^{-1}(0,0) = 0$ or $\tilde{\mu} = 0$. After those simplifications, the problem is reduced to a self-consistent determination of the leading asymptotics for the corresponding functions at small p-values. Note also that the structure of the one-particle Green's function completely changes at the Bose-condensation point, where expression (16) becomes inapplicable. This means that, in what follows, we deal with temperatures higher than the critical one, although arbitrarily close to it.

3. Account of Direct Particle–Particle Scattering Processes. The Two-particle Spectrum

The simplest approximation of our theory, which involves all direct processes of two-particle scattering, is $\Gamma(\omega_n,\mathbf{p}|\omega'_n,\mathbf{q}) = 1$. Pay attention at once that the scope of applicability of this simplification is confined to low densities, because we neglect ternary and higher-order processes of scattering. In this approximation, the system of nonlinear integral equations (8), (9) for the single one-particle Green's function becomes closed.

First, let us analyze the result of perturbation theory, and then generalize it to our case (16) using a trivial redesignation of constants. For this purpose, the Green's function of the ideal gas $G_0(\omega_n,p) = \{i\omega_n - \xi_p\}^{-1}$ has to be substituted into the right-hand side of Eq. (9). It is easy to calculate the corresponding sum over the frequencies ($n(x) = \{e^x - 1\}^{-1}$),

$$\frac{1}{\beta}\sum_{\omega'_n} G_0(\omega'_n,p)G_0(\omega_n-\omega'_n,|\mathbf{p}-\mathbf{q}|) =$$

$$= \frac{1 + n(\beta\xi_{|\mathbf{p}-\mathbf{q}|}) + n(\beta\xi_p)}{\xi_{|\mathbf{p}-\mathbf{q}|} + \xi_p - i\omega_n}. \tag{17}$$

After making the indicated substitution into the sum over the wave vector and renormalizing the interaction constant at zero values of the external frequency and the wave vector, we arrive at the equality

$$\langle \eta_{\mathbf{q}}(\omega_n)\eta^*_{\mathbf{q}}(\omega_n)\rangle = 2\lambda/\{1 + \lambda t(\omega_n,q)\}, \tag{18}$$

where

$$t(\omega_n,q) = \frac{1}{V}\sum_{\mathbf{p}}\frac{n(\beta\xi_{|\mathbf{p}-\mathbf{q}|}) + n(\beta\xi_p)}{\xi_{|\mathbf{p}-\mathbf{q}|} + \xi_p - i\omega_n}. \tag{19}$$

It is important that renormalization (2) completely eliminates density-independent terms in the function $t(\omega_n, q)$. Let us study its properties. However, it is worth noting first that the same result (with certain technical features) can be obtained in the framework of the pseudo-potential method [12]. Let us make an analytical continuation of $t(\omega_n, q)$ into the upper half-plane and designate

$$\mathrm{Re}\, t(\omega_n, q)|_{i\omega_n \to \omega + i0} = t_R(\omega, q), \quad (20)$$

$$\mathrm{Im}\, t(\omega_n, q)|_{i\omega_n \to \omega + i0} = t_I(\omega, q). \quad (21)$$

The presence of the δ-function in the integrand allows the imaginary part to be calculated at once:

$$t_I(\omega, q) = \frac{1}{8\pi}\beta q_0^3 \frac{q_0}{q} \ln\left|\frac{1-\exp\{\beta\mu - \varepsilon_+^2\}}{1-\exp\{\beta\mu - \varepsilon_-^2\}}\right| \times$$

$$\times\theta(\omega - \varepsilon_q/2 + 2\mu), \quad (22)$$

where the notations $\varepsilon_\pm = q/2q_0 \pm \sqrt{\beta\omega/2 - q^2/4q_0^2 + \beta\mu}$ and $q_0 = \sqrt{2mT}/\hbar$ were introduced to simplify the formulas. The real part (20) cannot be integrated to the end:

$$t_R(\omega, q) = \beta\rho\frac{q_0}{q}\{f(\varepsilon_+, \beta\mu) + f(\varepsilon_-, \beta\mu)\}, \quad (23)$$

where the function $f(\varepsilon, y)$ stands for the integral

$$f(\varepsilon, y) = \varepsilon\int\limits_0^1 \frac{dx}{\sqrt{1-x}} g_{1/2}(e^{y - x\varepsilon^2})/g_{3/2}(e^y), \quad (24)$$

$$g_l(e^y) = \sum_{n\geq 1}\frac{e^{yn}}{n^l},$$

and $\rho = N/V$ is the equilibrium density, which is assumed to be a function of the chemical potential.

To make the next step, we recall that, according to representation (5), η-correlator (9) is, to an accuracy of its sign, the irreducible part of the two-particle Green's function. Therefore, the probable roots of the equation

$$1 + \lambda t_R(\omega(q), q) = 0 \quad (25)$$

determine the spectrum of particle pairs. It is clear that those excitation are stable only provided that the damping is low:

$$\Gamma(q) = t_I(\omega(q), q)\Big/\frac{\partial t_R(\omega(q), q)}{\partial\omega(q)} \ll \omega(q). \quad (26)$$

Note that, for an attractive interaction, $\lambda < 0$, and Eq. (25) determines the spectrum of particle pairs in the bound state. We confine ourselves to the case of a repulsive interaction. Let

$$\omega(q) = \frac{1}{2}\varepsilon_q - 2\mu + 2\Delta(q). \quad (27)$$

The explicit form of Eq. (25) is

$$1 + g\frac{q_0}{q}\{f(\sqrt{\beta\Delta(q)} + q/2q_0, \beta\mu) -$$

$$-f(\sqrt{\beta\Delta(q)} - q/2q_0, \beta\mu)\} = 0, \quad (28)$$

where $g = \beta\rho\lambda$. Taking into account that the function $f(\varepsilon, y)$ is positive, we draw conclusion that the real roots of this equation exist only if $\beta\Delta(q) > q^2/4q_0^2$. In addition, it is necessary that the derivative of the function $f(\varepsilon, y)$ with respect to the first variable have a negative sign, at least within a certain interval of the argument. Since the function $f(\varepsilon, y)$ is finite, whereas the ratio gq_0/q can vanish, the spectrum $\omega(q)$ has an end point. It turns out that this boundary value of the wave vector, q_c, can be estimated in the general case. For this purpose, it is necessary to specify that the function $f(\varepsilon, y)$ is confined from above and, consequently, the inequality $q_c/q_0 \leq g\max(f(\varepsilon, y))$ always holds true.

An important derivative $\frac{\partial t_R(\omega,q)}{\partial\omega}|_{\omega=\omega(q)}$ appeared in formula (26) for the first time. Its straightforward calculation is not a simple task; however, this problem can be bypassed. For this purpose, let us differentiate t_R with respect to ω and immediately substitute $\omega(q)$:

$$[\partial t_R(\omega, q)/\partial\omega]\,|_{\omega=\omega(q)} =$$

$$= \frac{\beta\rho}{2\sqrt{\beta\Delta(q)}}\frac{q_0}{q}\{f'(\sqrt{\beta\Delta(q)} + q/2q_0, \beta\mu) -$$

$$-f'(\sqrt{\beta\Delta(q)} - q/2q_0, \beta\mu)\}. \quad (29)$$

Then, we differentiate Eq. (28) with respect to the dimensionless interaction parameter, bearing in mind (though we do not write it explicitly) that the quantity $\Delta(q)$ also depends on g:

$$\frac{\beta g}{2\sqrt{\beta\Delta(q)}}\frac{\partial\Delta(q)}{\partial g}\frac{q_0}{q}\{f'(\sqrt{\beta\Delta(q)} + q/2q_0, \beta\mu) -$$

$$-f'(\sqrt{\beta\Delta(q)} - q/2q_0, \beta\mu)\} +$$

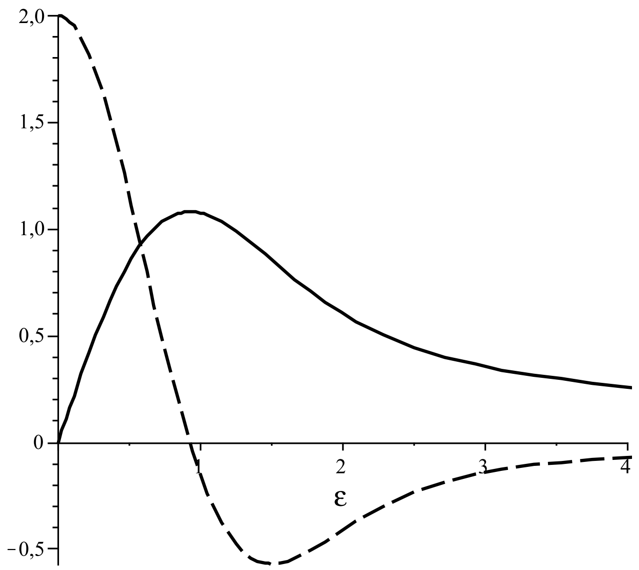

Fig. 1. Plots of the function $f(\varepsilon)$ (solid curve) and its derivative (dashed curve)

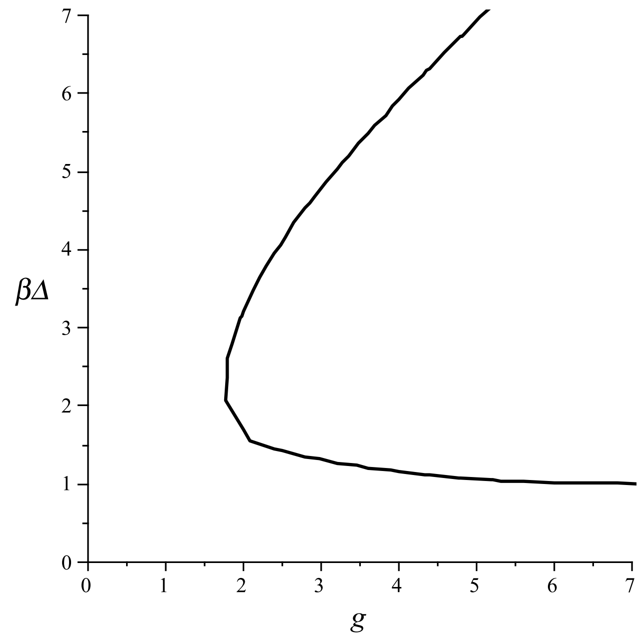

Fig. 2. Dependences of the gaps $\beta\Delta$ (upper branch) and $\beta\Delta_s$ (lower branch) in the two-particle spectrum on the dimensionless interaction parameter

$$+\frac{q_0}{q}\{f(\sqrt{\beta\Delta(q)}+q/2q_0,\beta\mu)-$$

$$-f(\sqrt{\beta\Delta(q)}-q/2q_0,\beta\mu)\}=0.$$

Comparing those two expressions and taking Eq. (28) into account, we obtain the useful relation

$$[\partial t_R(\omega,q)/\partial\omega]^{-1}\,|_{\omega=\omega(q)}=\frac{2}{\beta\rho}g^2\frac{\partial}{\partial g}\Delta(q). \quad (30)$$

Let us study the long-wave asymptotics at the poles of the two-particle Green's function by expanding Eq. (28) in a power series of q/q_0. At small wave vectors, we obtain

$$\Delta(q)=\Delta+\frac{1}{2}\Delta''\beta\varepsilon_q \quad (31)$$

where Δ is defined by the equation

$$1+gf'(\sqrt{\beta\Delta},\beta\mu)=0, \quad (32)$$

the quantity $\beta\Delta''$ is

$$\beta\Delta''=-\frac{1}{3!}\sqrt{\beta\Delta}f^{(3)}(\sqrt{\beta\Delta},\beta\mu)/f^{(2)}(\sqrt{\beta\Delta},\beta\mu), \quad (33)$$

and $f'(\varepsilon,y)$, $f^{(2)}(\varepsilon,y)$, and $f^{(3)}(\varepsilon,y)$ are the corresponding derivatives of function (24) with respect to its first variable (the properties of this function will be examined below). At a point, where the second derivative $f^{(2)}(\sqrt{\beta\Delta},\beta\mu)$ does not exist, the spectrum becomes nonanalytic at small q/q_0. Actually, expansion (31) becomes impossible, although the value of $\Delta(0)$ is finite. Below, we demonstrate that there are certain restrictions on the quantity $\beta\Delta''$ as well ($\beta\Delta''\geq 1/2$).

At first, consider the classical limit. For this purpose, we formally tend the absolute value of chemical potential to infinity. Then, the function

$$f(\varepsilon)=f(\varepsilon,y\to-\infty)=\varepsilon\int\limits_0^1\frac{dx}{\sqrt{1-x}}\exp\{-x\varepsilon^2\} \quad (34)$$

with the leading asymptotics acquires values

$$f(\varepsilon\to 0)=2\varepsilon,\quad f(\varepsilon\to\infty)=1/\varepsilon$$

at small and large arguments. An evident simplification of this limit is the disappearance of the dependence on the chemical potential. The plots of the function $f(\varepsilon)$ and its derivative are depicted in Fig. 1. The form of the function plot testifies that, at $g>g_{\min}=1.757$, Eq. (32) has two roots (see Fig. 2). One of them increases almost linearly with the growth of the interaction constant, tending to the asymptotics $\beta\Delta=g$; the other saturates at the level $\beta\Delta_s=0.854$. Accordingly, we obtain two spectral branches: the former will be called main, and we preserve the notations $\omega(q)$ and $\Delta(q)$ for it; the latter will be called soft and be characterized by the corresponding index: $\omega_s(q)$ and $\Delta_s(q)$. Note, by the way, that there are no real-valued gapless solutions–i.e. solutions obeying the condition $\beta\Delta(q \to 0) \to 0$–of Eq. (25) in the classical case.

The location of spectral end point is governed by the parameter g. In a vicinity of the point $g = 1.757$, the limiting value $q_{\rm c}$ for the wave vector of the two-particle spectrum is small, whereas, at large values of the dimensionless interaction parameter, the ratio $q_{\rm c}/q_0$ is also large. Then, only the second term in the braces in Eq. (28) is important, and, accordingly, the difference $\sqrt{\beta\Delta(q)} - q/2q_0$ shifts into the maximum point $\varepsilon_{\max} = 0.924$ of the function $f(\varepsilon)$. Under these conditions, we obtain $q_{\rm c}/q_0 = gf(\varepsilon_{\max}) = 1.082g$ in the limit $g \gg 1$.

Now, let us analyze the damping of the two-particle spectrum. Let the damping of branches $\omega(q)$ and $\omega_s(q)$ be designated as $\Gamma(q)$ and $\Gamma_s(q)$, respectively. Our task is to determine in which regions the product $\lambda t_I(\omega(q), q)$ is small. It is easy to show that, in the classical limit, this product can be written down as follows:

$$\lambda t_I(\omega(q), q) = 2\sqrt{\pi} g \frac{q_0}{q} \exp\left\{-q^2/4q_0^2 - \beta\Delta(q)\right\} \times$$

$$\times \sinh\left[\sqrt{\beta\Delta(q)}q/q_0\right]. \qquad (35)$$

The corresponding equality, to within the replacement $\Delta(q) \to \Delta_s(q)$, is valid for the branch $\omega_s(q)$ as well. Quantity (35) is small at large wave vectors in both branches. A different situation takes place in the long-wave region,

$$\lambda t_I(\omega(0), 0) = 2\sqrt{\pi} g \sqrt{\beta\Delta} e^{-\beta\Delta}, \qquad (36)$$

(and similarly for the other branch), whence it follows that the quantity $\lambda t_I(\omega_s(0), 0)$ is not small in the classical case for any interaction parameter value, growing linearly as g increases. In turn, this means that the damping of the branch with the spectrum $\omega_s(q)$ is large; in other words, there is no s-branch at all in the classical limit. For the main branch of the two-particle spectrum, quantity (36) is very small in the limit $g \gg 1$. The damping $\Gamma(q)$ can be calculated by formula (26). In the long-wave range, it approaches a constant, and–it is important–the ratio

$$\Gamma(0)/\omega(0) = 2\sqrt{\pi} g^{3/2} e^{-g}, \quad g \gg 1, \qquad (37)$$

falls down exponentially.

Qualitatively, the main features in the behavior of the two-particle spectrum survive at low temperatures. It is clear that a dependence on the chemical potential, which grows as the temperature decreases, must appear. The general picture is as follows. As the critical temperature is approached and provided that the chemical potential is fixed, the maximum of the function $f(\varepsilon, \beta\mu)$ moves to the left, and its derivative at zero grows. In turn, this results in a reduction of the gap Δ_s in the corresponding branch of the two-particle spectrum (this branch disappears at the point $T = T_{\rm c}$). It is of importance that, for the main branch of the spectrum, $\omega(q)$, and in the range of its existence, all leading asymptotics remain the same as they are in the classical case.

An interesting feature of the proposed theory is both the very fact that the strong interaction limit does exist and a possibility to analyze it analytically. Let us determine the gap in the soft-mode branch in a vicinity of the critical temperature and for large values of dimensionless interaction parameter. Consider Eq. (32) once more. In the case $g \to \infty$, its root coincides with the maximum point of the first variable in the function $f(\varepsilon, \beta\mu)$, provided that the chemical potential is fixed. Therefore, to obtain the leading asymptotics for Δ_s, it is enough to analyze the behavior of this function at small values of both arguments. In formula (24), the function $g_{1/2}(e^y)$ is nonanalytic at the point $y = 0$. Extracting this singularity, we obtain the leading asymptotics for the integral

$$f(\varepsilon, y) = 2\{\sqrt{\pi}\arctan(\varepsilon/\sqrt{y}) +$$

$$+\varepsilon\zeta(1/2) + o(\varepsilon^3, \varepsilon y)\}/\zeta(3/2), \qquad (38)$$

where $\zeta(3/2) = 2.6124$ and $\zeta(1/2) = -1.4604$. It is easy to see that the limits

$$f(0, y) = 0, \quad f(\varepsilon \to 0, 0) = \frac{\pi^{3/2}}{\zeta(3/2)}\mathrm{sign}\,(\varepsilon)$$

do not "commute". Differentiating function (38) with respect to the first argument and substituting the result into Eq. (32), we obtain

$$\beta\Delta_s = \frac{\sqrt{\pi\beta|\mu|}}{-\zeta(1/2)}\left\{1 + \frac{\zeta(3/2)}{-\zeta(1/2)}\frac{1}{2g}\right\}. \qquad (39)$$

We emphasize once more that the applicability range for expression (39) is confined by the temperatures in a vicinity of the critical one and by large values of the parameter g, although Eq. (32) has solutions at $g > -\zeta(3/2)/2\zeta(1/2) = 0.894$. In fact, this value of the dimensionless interaction constant defines the ranges of weak ($g \ll 0.894$) and strong ($g \gg 0.894$) interactions, or the ranges of high and low temperatures, respectively.

It is not difficult to find the first term of the expansion series of quantity (33) in the wave vector in the range $q/2q_0 < \sqrt{\beta\Delta_s}$ of the $\omega_s(q)$-spectrum and in a vicinity of the critical temperature to obtain $\beta\Delta_s'' = 1/2$. For the soft mode, in a vicinity of the critical temperature, the damping is not low, similarly to the classical limit case. This fact can be easily verified by analyzing expression (22). Then, the product

$$\lambda t_I(\omega_s(q), q)|_{q\to 0} = \frac{2\sqrt{\pi}g}{\zeta(3/2)} \frac{\sqrt{\beta\Delta_s}}{\beta\Delta_s + q^2/2q_0^2} \tag{40}$$

grows linearly with the dimensionless interaction constant and depends substantially on which parameter, the gap Δ_s or the wave vector, goes to zero first. This behavior is typical, if we approach the phase transition point. Now, we can go beyond the framework of perturbation theory and generalize the result obtained to case (16). It is sufficient to make the substitutions $m \to \tilde{m}$, $\mu \to \tilde{\mu}$, and $\lambda \to Z\lambda$ in every sum over the wave vector. The general conclusion is as follows. There exist two branches in the two-particle spectrum of the system above the Bose condensation temperature. As the interaction constant grows, the damping increases in one of them and quickly tends to zero in another one. As follows from the next section, this fact is very important for the determination of a quasiparticle spectrum.

4. One-particle Spectrum. Critical Temperature

Using Eq. (8), let us rewrite the self-energy part in the adopted approximation as follows:

$$\Sigma(\omega_n, p) = 2\rho\lambda + \frac{1}{\beta V} \sum_{\omega_n'} \sum_{\mathbf{q}} \frac{2\lambda^2 t(\omega_n', q)}{1 + \lambda t(\omega_n', q)} \times$$

$$\times G(\omega_n' - \omega_n, |\mathbf{q} - \mathbf{p}|), \tag{41}$$

where the exchange term is singled out. We are going to obtain a result for the self-energy part in the framework of perturbation theory. Then the generalization on case (16) is obvious.

To calculate the sum over frequencies, it is convenient to write down the relevant fraction in expression (41), making use of the spectral relation

$$\frac{\lambda t(\omega_n, q)}{1 + \lambda t(\omega_n, q)} = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{d\omega}{\omega - i\omega_n} \times$$

$$\times \frac{\lambda t_I(\omega, q)}{(1 + \lambda t_R(\omega, q))^2 + \lambda^2 t_I^2(\omega, q)},$$

where the second factor in the integrand is the imaginary part of the left-hand side of expression (41) obtained after the substitution $i\omega_n \to \omega + i0$. Then, the procedure of summation over the frequencies becomes simple, and we obtain the following formula for the mass operator:

$$\Sigma(\omega_n, p) = 2\rho\lambda + \frac{1}{V} \sum_{\mathbf{q}} \int\limits_{-\infty}^{\infty} \frac{d\omega}{\pi} 2\lambda \times$$

$$\times \frac{\lambda t_I(\omega, q)}{(1 + \lambda t_R(\omega, q))^2 + \lambda^2 t_I^2(\omega, q)} \frac{n(\beta\omega) - n(\beta\xi_{|\mathbf{q}-\mathbf{p}|})}{\omega - \xi_{|\mathbf{q}-\mathbf{p}|} - i\omega_n}. \tag{42}$$

The straightforward calculation of this expression in the general case needs the application of numerical procedures. Only the limits of the small ($g \ll 1$) and, as we shall demonstrate below, large ($g \gg 1$) dimensionless interaction parameter can be analyzed analytically. We will not discuss the case of small g in detail, because this limit has been well studied in the literature. We only notice that, from the technical viewpoint, it is more convenient to analyze expression (41) rather than to consider formula (42). Nevertheless, if we are to work with formula (42) at small g, it is easy to see that the ω-intervals in a vicinity of the function $t_I(\omega, q)$ maxima (at a fixed q) give the main contribution to the integral over ω. This situation changes cardinally in the opposite case $g \gg 1$. An important underlying reason for that is the fact that it is this region, where the branch of the two-particle spectrum $\omega(q)$ is well determined and where the product $\lambda t_I(\omega(q), q)$ rapidly goes to zero. This situation is realized in a vicinity of all well-determined (the damping is low) branches of the two-particle spectrum. Then, the quasi-Lorentz contour "stretches" (in the integrand) into a delta-like peak,

$$\frac{1}{\pi} \frac{\lambda t_I(\omega, q)}{(1 + \lambda t_R(\omega, q))^2 + \lambda^2 t_I^2(\omega, q)} \to$$

$$\to \operatorname{sign}(t_I(\omega, q)) \delta(1 + \lambda t_R(\omega, q)). \tag{43}$$

Now, using properties of the δ-function, we can calculate the integral in formula (42),

$$\Sigma(\omega_n, p) = 2\rho\lambda + \frac{2}{V} \sum_{\mathbf{q}} \left[\partial t_R(\omega, q)/\partial\omega\right]^{-1} |_{\omega=\omega(q)} \times$$

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$$\times \frac{n(\beta\omega(q)) - n(\beta\xi_{|\mathbf{q}-\mathbf{p}|})}{\omega(q) - \xi_{|\mathbf{q}-\mathbf{p}|} - i\omega_n}. \tag{44}$$

This expression obtained for the self-energy part is an important result of this paper, because it allows the damping of the quasiparticle spectrum to be analyzed and the critical temperature of the system to be calculated. Formula (44) is valid in the range of large g's. Nevertheless, we should expect that it would provide a satisfactory description for intermediate values of interaction parameter as well.

Let us analyze the imaginary part of the mass operator (44) after its analytical continuation into the upper half-plane, $i\omega_n \to \xi_p + i0$:

$$\Sigma_I(\xi_p, p) = \frac{4\pi g^2}{\beta N} \sum_{\mathbf{q}} \frac{\partial \Delta(q)}{\partial g} [n(\beta\omega(q)) - n(\beta\xi_{|\mathbf{q}-\mathbf{p}|})] \times$$

$$\times \delta(\omega(q) - \xi_{|\mathbf{q}-\mathbf{p}|} - \xi_p). \tag{45}$$

For simplification (it does not affect the estimation accuracy), we substitute the quantity $\beta\Delta''$ in the integral over the wave vector by its value at a large interaction parameter ($\beta\Delta'' \to 1/2$). Then, after a simple calculation, we obtain

$$\Sigma_I(\xi_p, p) = \frac{1}{4\pi} \frac{p_0^3}{\rho} g^2 \frac{\partial \Delta}{\partial g} \frac{p_0}{p} \times$$

$$\times \left[\ln \left| 1 - \exp\left\{ \beta\mu - \left(\frac{\beta\Delta}{p/p_0} \right)^2 \right\} \right| - \right.$$

$$\left. - \ln \left| 1 - \exp\left\{ 2\beta\mu - \left(\frac{\beta\Delta}{p/p_0} \right)^2 - p^2/p_0^2 \right\} \right| \right], \tag{46}$$

where $p_0 = q_0 = \sqrt{2mT}/\hbar$. We see that the damping is very low at any wave vector. This fact is an important and necessary confirmation that choice (16) for the structure of the one-particle Green's function is valid.

Now, let us analyze the real part of the mass operator. Our purpose is to find the leading asymptotics for $Z(p)$ and the new one-particle spectrum $\tilde{\xi}_p$ at small p and large g. In this limit, a simple differentiation gives rise to

$$[\partial \Sigma_R(\omega, p)/\partial \omega]\,|_{\omega=\xi_p} = \frac{1}{N} \sum_{\mathbf{q}} n(\beta\xi_q) \times$$

$$\times g^2 \frac{\partial}{\partial g} \frac{1}{\beta\Delta} \left\{ 1 - \frac{\beta\Delta'' - 1/2}{\Delta} \varepsilon_q + \dots \right\}, \tag{47}$$

$$\frac{\widetilde{m}}{m} = 1 - \frac{1}{N} \sum_{\mathbf{q}} n(\beta\xi_q) \times$$

$$\times g^2 \frac{\partial}{\partial g} \left\{ \frac{\beta\Delta'' - 1/2}{\beta\Delta} - \frac{1}{3} \beta\varepsilon_q \frac{(\beta\Delta'' + 1/2)^2}{(\beta\Delta)^2} + \dots \right\}. \tag{48}$$

Here, only those terms are included, which make the largest contribution to the integral. After simple calculations with the use of the leading asymptotics for $\beta\Delta = g$ and $\beta\Delta'' = 1/2$, we obtain that the effective mass of one-particle excitations tends to the particle mass, and $Z = 1/2$. From the condition

$$\frac{Z}{\rho} \left(\frac{\widetilde{m} T_c}{2\pi\hbar^2} \right)^{3/2} g_{3/2}(1) = 1, \tag{49}$$

which follows from formula (16), we can find the critical temperature,

$$\frac{T_c - T_0}{T_0} = 0.587, \quad g \gg 1. \tag{50}$$

This result can be compared with the result of calculations of the critical temperature in the recent work [16], namely $(T_c - T_0)/T_0 = 0.396$. A numerical discordance between the coefficients is evidently associated with different approximations used at calculations. It is important that the results coincide qualitatively, although we report the results of calculations for the temperature range above the critical temperature; i.e., strictly speaking, we deal with the limit $T \to T_c + 0$. At the same time, the authors of work [16] considered the Bose gas in the condensate phase, i.e. in the limit $T \to T_c - 0$. Hence, the critical temperature increases and saturates, if g grows infinitely. The result is unexpected, because the unconfined growth of the s-scattering length must be accompanied by an effective increase of the "own" particle volume and, as a consequence, by the growth of the density and, hence, the Bose condensation temperature. These qualitative speculations bring us to an idea that the unconfined growth of the interparticle interaction should result in an unlimited growth of the critical temperature of the system. We recall that this is a result of the conventional perturbation theory.

For self-consistent calculations, it is necessary to determine the following terms in the expansion of quantity (31) in the parameter $1/g$:

$$\beta\Delta = g + \frac{3}{2} g_{5/2}(e^{\beta\mu})/g_{3/2}(e^{\beta\mu}), \tag{51}$$

$$\beta\Delta'' = \frac{1}{2} + \frac{1}{\beta\Delta} g_{5/2}(e^{\beta\mu})/g_{3/2}(e^{\beta\mu}). \quad (52)$$

We shall briefly describe subsequent calculations. To determine the critical temperature (49), the quantity Z and the effective mass $\tilde{m}$ of quasiparticles must be calculated once more by formulas (47) and (48), respectively. The difference is that now we must use the exact value $\tilde{m}$ rather than m on the right-hand sides of those equalities and rescale the interaction constant, i.e. make the substitution $\lambda \to Z\lambda$. Not dwelling on the details of integral calculations and the solution of simple algebraic equations, we present the final result expressed in terms of the gas parameter ρa^3,

$$\widetilde{m}_{T\to T_c} = \frac{2}{3} m, \quad Z_{T\to T_c} = 0.830(\rho a^3)^{-1/8}. \quad (53)$$

Accordingly, for the critical temperature, we have

$$\frac{T_c}{T_0} = 0.472(\rho a^3)^{1/12}, \quad (54)$$

where T_0 is the Bose condensation temperature of the ideal Bose gas. A condition for those formulas to be applicable at the critical temperature point is $Z\rho\lambda/T_c = 1.855(\rho a^3)^{1/8} \gg 1$, although, as is seen from Fig. 1, the description is satisfactory for the values of this parameter larger than 3–4 (it is those argument values where the function $f(\varepsilon)$ approaches the $1/\varepsilon$-asymptotics). It should also be noticed that it is only the leading asymptotics for the critical temperature at large gas parameter values.

In the opposite case, i.e. at $g \to 0$, it is simpler to take advantage of a procedure proposed in work [5] rather than to calculate expression (42). It is important that a shift of the critical temperature in our case completely coincide with the result of work [17], where the same approximation was used.

At last, let us briefly discuss the extension of the results obtained on smaller g-values. First, it is necessary to calculate the quantity $\Delta(q)$ more precisely and take into account that the two-particle spectrum has an end point. Second, for smaller g, the accuracy of relation (44) becomes worse, and formula (42) must be used for calculations.

5. Conclusions

Hence, the proposed calculation technique, which, in the simplest approximation, makes allowance for all processes of two-particle scattering, describes well a Bose gas with point-like interaction between the particles in the whole range of the nonideality parameter. The idea (see Eq. (16)) of that this system does not differ in principle from the ideal Bose gas was substantiated. In particular, the interaction renormalizes constants rather than changes the leading asymptotics of dynamic quantities in the system, down to the temperature of Bose condensation inclusive. It is easy to show that the leading asymptotics of the self-energy part (41) at zero frequency is nonanalytic in the wave vector at the critical point, $\Sigma(0,p) - \Sigma(0,0) \sim p^2 \ln(p)$, which is a hint that the behavior of the one-particle Green's function is power-like, $G(0,p) \sim p^{\eta-2}$. The two-particle spectrum of the system above the critical temperature was analyzed in detail, and the temperature of Bose condensation was calculated in the limit of a strong repulsion between particles.

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ДИНАМІЧНІ ВЛАСТИВОСТІ БОЗЕ-ГАЗУ З δ-ПОДІБНОЮ ВЗАЄМОДІЄЮ ВИЩЕ ТЕМПЕРАТУРИ ФАЗОВОГО ПЕРЕХОДУ У ГРАНИЦІ СИЛЬНОГО ВІДШТОВХУВАННЯ МІЖ ЧАСТИНКАМИ

В.С. Пастухов

Р е з ю м е

За допомогою функціонального інтегрування побудовано точні рівняння для одночастинкової і незвідної частини двочастинкової функцій Гріна тривимірного бозе-газу з точковою взаємодією. У найпростішому наближенні теорії, яке враховує всі прямі попарні процеси розсіяння частинок, детально проаналізовано двочастинковий спектр системи. Показано, що ведуча асимптотика одночастинкового спектра залишається квадратичною в довгохвильовій області. Знайдено величину критичної температури у границі сильного відштовхування між частинками.