S-MATRIX AND THE AHARONOV–BOHM EFFECT

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The S-matrix for the Aharonov–Bohm scattering is considered, and the optical theorem is derived. The persistence of the Fraunhofer diffraction in the short-wavelength limit is shown to be crucial for maintaining the optical theorem in the quasiclassical limit.

1. Introduction

Probability conservation and the unitarity of the scattering matrix are the basic elements of quantum theory, which have significant physical consequences. One of them is the optical theorem relating the imaginary part of the scattering amplitude in the forward direction to the total cross section of the interaction processes. The quantum-mechanical scattering of a charged particle by an impenetrable magnetic vortex is studied for a more than half-century, starting from the seminal paper by Aharonov and Bohm [1]. The theory of this process has been successfully confirmed in experiments, promising important practical applications, see review [2]. However, some theoretical issues still remain to be unclear. They include the question how the optical theorem should be formulated for the Aharonov–Bohm scattering. Several authors [3, 4] addressed this problem but without the decisive conclusions (see also [5]); they traced the encountered difficulties either to a subtlety in the choice of an incident wave [3], or to a divergent behavior of the scattering amplitude in the forward direction, which needs the yet unspecified regularization [4]. In our opinion, the following two circumstances should be taken into account: a) the long-range nature of the interaction of a scattered particle with the magnetic vortex and b) the nonvanishing transverse size of the magnetic vortex. Due to the first circumstance, although the unitarity of the S-matrix is undoubted, the standard scattering theory applicable to the case of short-range interactions has to be modified, which results in a rather unexpected form of the optical theorem. The second circumstance signifies that the limit of the vanishing transverse size of the vortex is an undue idealization which has to be avoided as physically irrelevant. Then, as it is shown in the present paper, the above problem can be treated properly and resolved.

The magnetic field configuration in the form of an infinitely long vortex possesses the cylindrical symmetry. The S-matrix in the cylindrically symmetric case takes form

$$S(k,\,\varphi,k_z;\,k',\,\varphi',k'_z) = \Big[I(k,\,\varphi;\,k',\,\varphi')+$$

$$+\delta(k-k')\frac{\mathrm{i}}{\sqrt{2\pi k}}f(k,\,\varphi-\varphi')\Big]\delta(k_z-k'_z), \qquad (1)$$

where the symmetry axis coincides with the z-axis, $I(k,\,\varphi;\,k',\,\varphi')$ is the identity matrix in polar coordinates in a two-dimensional space, and $f(k,\,\varphi-\varphi')$ is the scattering amplitude. The condition of the unitarity of the S-matrix, $S^\dagger S = SS^\dagger = I$, results in a conventional way in the optical theorem

$$2\sqrt{\frac{2\pi}{k}}\mathrm{Im}f(k,\,0)=\sigma, \qquad (2)$$

where σ is the total cross section per unit length along the symmetry axis, and, thus, σ has the dimension of length. We will prove that, namely in the case of the Aharonov–Bohm scattering, the optical theorem takes form which is different from the conventional one given by (2). Our consideration is based on earlier works [6–9], where important results concerning the S-matrix, scattering amplitude, and scattering wave function have been obtained.

In the next section, we review an auxiliary problem of scattering by an impenetrable tube; the roles of the Fraunhofer diffraction and the appropriate forward peak are exposed. In Section 3, we consider the same problem in the case where the tube is filled with the magnetic flux lines, i.e. in the case of an impenetrable magnetic vortex; the main results are derived here. The summary and the discussion of the results are given in Section 4.

2. Scattering by an Impenetrable Tube

A plane wave propagating in the direction that is orthogonal to the z-axis can be presented as

$$\psi_{\mathbf{k}}^{(0)}(\mathbf{r}) = e^{\mathrm{i}kr\cos\varphi} = \sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} e^{\frac{\mathrm{i}}{2}|n|\pi} J_{|n|}(kr) =$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{in\varphi} e^{\frac{\mathrm{i}}{2}|n|\pi}\left[H_{|n|}^{(1)}(kr) + H_{|n|}^{(2)}(kr)\right], \qquad (3)$$

where $\mathbf{r}$ and $\mathbf{k}$ are the two-dimensional vectors, φ is the angle between them, $J_\alpha(u)$, $H_\alpha^{(1)}(u)$, and $H_\alpha^{(2)}(u)$ are the Bessel, first- and second-kind Hankel functions of order α, and $\mathbb{Z}$ is the set of integer numbers. Using the appropriate asymptotics of Hankel functions as $r \to \infty$, we get

$$\psi_{\mathbf{k}}^{(0)}(\mathbf{r}) \underset{r\to\infty}{=} \frac{1}{\sqrt{2\pi kr}} \sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} e^{\frac{\mathrm{i}}{2}|n|\pi}\left[e^{\mathrm{i}\left(kr-\frac{1}{2}|n|\pi-\frac{1}{4}\pi\right)} + \right.$$

$$\left. + e^{-\mathrm{i}\left(kr-\frac{1}{2}|n|\pi-\frac{1}{4}\pi\right)}\right] = \sqrt{\frac{2\pi}{kr}} e^{\mathrm{i}(kr-\pi/4)}\Delta(\varphi) +$$

$$+\sqrt{\frac{2\pi}{kr}} e^{-\mathrm{i}(kr-\pi/4)}\Delta(\varphi-\pi), \qquad (4)$$

where

$$\Delta(\varphi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} \qquad (5)$$

is the delta-function for the azimuthal angle, $\Delta(\varphi + 2\pi) = \Delta(\varphi)$. Thus, we see that the plane wave passing through the origin ($r = 0$) can be interpreted naturally at large distances from the origin as a superposition of two cylindrical waves: the diverging one, $e^{\mathrm{i}kr}$, in the forward, $\varphi = 0$, direction and the converging one, $e^{-\mathrm{i}kr}$, from the backward, $\varphi = \pi$, direction.

Now, let us place an obstacle in the form of an impenetrable tube along the z-axis. If the wave function obeys the Dirichlet boundary condition at the edge of the tube,

$$\psi_{\mathbf{k}}(\mathbf{r})|_{r=r_c} = 0, \qquad (6)$$

then, instead of (3), we obtain

$$\psi_{\mathbf{k}}(\mathbf{r}) =$$

$$= \sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} e^{\frac{\mathrm{i}}{2}|n|\pi}\left[J_{|n|}(kr) - \frac{J_{|n|}(kr_c)}{H_{|n|}^{(1)}(kr_c)} H_{|n|}^{(1)}(kr)\right]. \qquad (7)$$

At large distances from the origin, $r \gg k^{-1}$, we get

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{kr\gg 1}{=} -\frac{1}{\sqrt{2\pi kr}} e^{\mathrm{i}(kr-\pi/4)} \sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} \frac{H_{|n|}^{(2)}(kr_c)}{H_{|n|}^{(1)}(kr_c)} +$$

$$+\sqrt{\frac{2\pi}{kr}} e^{-\mathrm{i}(kr-\pi/4)}\Delta(\varphi-\pi). \qquad (8)$$

The sum over n in (8) yields the forward angular delta-function, $\Delta(\varphi)$, in the case of long wavelengths, $k \to 0$. In the opposite short-wavelength limit, $k \to \infty$, when $1 \ll kr_c < kr$, by substituting the large-argument asymptotics of the Hankel functions in (8), we obtain

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{kr>kr_c\gg 1}{=} -\sqrt{\frac{2\pi}{kr}} e^{\mathrm{i}\pi/4}\left[e^{\mathrm{i}k(r-2r_c)} - e^{-\mathrm{i}kr}\right]\Delta(\varphi-\pi). \qquad (9)$$

This result, which is actually given in [10], is quite understandable from the classical point of view: the obstacle forms a shadow in the forward direction, which is not accessible to waves, and, thus, both diverging and converging cylindrical waves are in the backward direction. However, this conclusion is wrong.

To find a loophole in the arguments leading to (9), one has to note the following property of the asymptotic behavior of the Bessel function at large values of its argument: it vanishes effectively when its order exceeds its large argument. Really, using the integral representation (see, e.g., [11])

$$J_{|n|}(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \exp\left[\mathrm{i}(|n|\theta - u\sin\theta)\right],$$

one notes that the integrand at large $|n|$ is vigorously oscillating, and its mean value is small almost everywhere with the exception of points, where the phase is stationary. This means that the prevailing contribution to the integral in the case of $|n| \gg 1$ is given by a vicinity of the point, where $\cos\theta = |n|/u$. Consequently, the integral is vanishingly small in the case of $|n|/u \gg 1$. The more is the value of u, the more abrupt is the decrease of the integral, when $|n|$ exceeds u (see, e.g., [12]).

Therefore, at $kr > kr_c \gg 1$, the sum in (7) containing $J_{|n|}(kr)$ is cut at $|n| = kr$, while the sum containing

$J_{|n|}(kr_c)$ is cut at $|n| = kr_c$. Instead of (9), we get the correct expression:

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{kr>kr_c\gg 1}{=} \sqrt{\frac{2\pi}{kr}} e^{\mathrm{i}(kr-\pi/4)} \left[\Delta_{kr}(\varphi) - \Delta_{kr_c}(\varphi)\right] +$$

$$+\sqrt{\frac{2\pi}{kr}} e^{\mathrm{i}\pi/4} \left[e^{-\mathrm{i}kr}\Delta_{kr}(\varphi-\pi) - e^{\mathrm{i}k(r-2r_c)}\Delta_{kr_c}(\varphi-\pi)\right], \tag{10}$$

where

$$\Delta_x(\varphi) = \frac{1}{2\pi}\sum_{|n|\le x} e^{\mathrm{i}n\varphi} \tag{11}$$

is the regularized (smoothed) angular delta-function,

$$\lim_{x\to\infty}\Delta_x(\varphi) = \Delta(\varphi), \qquad \Delta_x(0) = \frac{x}{\pi}. \tag{12}$$

We see that the cancellation of the diverging wave in the forward direction is not complete, as well as is that between the diverging and the converging waves in the backward direction; the complete cancellation is achieved at $r = r_c$ only, which is consistent with condition (6). In general, the diverging wave in the forward direction is suppressed by the factor $1 - r_c r^{-1}$, as compared with the case where the obstacle is absent. Thus, contrary to the classical anticipations, the wave penetrates to the region behind the obstacle even in the case where the wavelength is much less than the transverse size of the obstacle, $kr_c \gg 1$.

Turning now from the qualitative analysis to the quantitative one, we note, firstly, with regard for (7) that the asymptotics of the wave function at large distances from the obstacle is

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi^{(0)}_{\mathbf{k}}(\mathbf{r}) + f(k,\varphi)\frac{e^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}} + O(r^{-3/2}), \tag{13}$$

where

$$f(k,\varphi) = \mathrm{i}\sqrt{\frac{2}{k\pi}}\sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi}\frac{J_{|n|}(kr_c)}{H^{(1)}_{|n|}(kr_c)} \tag{14}$$

is the scattering amplitude which enters S-matrix (1), while the identity matrix there is evidently $I(k,\varphi;k',\varphi') = k^{-1}\delta(k-k')\Delta(\varphi-\varphi')$.

The S-matrix is unitary:

$$\int\limits_{-\infty}^{\infty} dk_z \int\limits_0^{\infty} dk\,k \int\limits_{-\pi}^{\pi} d\varphi\, S^*(k,\varphi,k_z;k',\varphi',k'_z)\times$$

$$\times S(k,\varphi,k_z;k'',\varphi'',k''_z) =$$

$$= \frac{1}{k'}\delta(k'-k'')\Delta(\varphi'-\varphi'')\delta(k'_z-k''_z), \tag{15}$$

and the latter relation can be recast into the form

$$\frac{1}{\mathrm{i}}\sqrt{\frac{k}{2\pi}}\left[f(k,\varphi'-\varphi'') - f^*(k,\varphi''-\varphi')\right] =$$

$$= \frac{k}{2\pi}\int\limits_{-\pi}^{\pi} d\varphi\, f^*(k,\varphi-\varphi')f(k,\varphi-\varphi''). \tag{16}$$

In particular, at $\varphi' = \varphi'' = 0$, one gets the optical theorem (2) with $\sigma = \int\limits_{-\pi}^{\pi} d\varphi |f(k,\varphi)|^2$ being the total cross section for the elastic scattering. Although relation (16) is valid for all wavelengths, it is related to vanishingly small quantities in the long-wavelength limit and to extremely large quantities in the short-wavelength limit.

Really, the scattering amplitude in the long-wavelength limit is

$$f(k,\varphi) = -\sqrt{\frac{\pi}{2k}}|\ln(kr_c)|^{-1}\left[1 + \left(\gamma - \mathrm{i}\frac{\pi}{2}\right)|\ln(kr_c)|^{-1}\right] +$$

$$+k^{-1/2}O\left[|\ln(kr_c)|^{-3}\right], \; kr_c \ll 1 \tag{17}$$

(γ is the Euler constant), and formula (16) is the relation between quantities of order $O\left[|\ln(kr_c)|^{-2}\right]$. The scattering amplitude in the short-wavelength limit is

$$f(k,\varphi) = \mathrm{i}\sqrt{\frac{2\pi}{k}}\Delta_{kr_c}(\varphi) - e^{-\mathrm{i}\pi/4}\sqrt{\frac{r_c}{2}\cos\left[\frac{\varphi}{2} - \mathrm{sgn}(\varphi)\frac{\pi}{2}\right]}\times$$

$$\times\exp\left\{-2\mathrm{i}kr_c\cos\left[\frac{\varphi}{2} - \mathrm{sgn}(\varphi)\frac{\pi}{2}\right]\right\}, \quad kr_c \gg 1, \tag{18}$$

where $\mathrm{sgn}(u) = \begin{cases} 1, & u > 0 \\ -1, & u < 0 \end{cases}$, and it is implied that $-\pi < \varphi < \pi$. The first term in (18) represents the forward peak of the Fraunhofer diffraction on an obstacle, whereas the second term describes the reflection from the obstacle according to the laws of geometric (ray) optics. Hence, formula (16) in this case is the relation between quantities of order $O(kr_c)$ at $\varphi' = \varphi''$ or of order $O(\sqrt{kr_c})$ at $\varphi' \neq \varphi''$, and the optical theorem (2) is the relation between finite quantities of order r_c. It should be noted that the Fraunhofer diffraction is crucial for ensuring the optical theorem in the short-wavelength limit, since the second term in (18) vanishes at $\varphi = 0$.

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3. Scattering by an Impenetrable Magnetic Vortex

Let us consider the scattering of a charged particle by an obstacle in the form of an impenetrable tube which is filled with a magnetic field with total flux Φ. The particle wave function obeys condition (6) at the edge of the tube. However, the Schrödinger Hamiltonian out of the tube is no longer free but takes form

$$H = -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\left(\partial_\varphi - \mathrm{i}\frac{\Phi}{\Phi_0}\right)^2 + \partial_z^2\right], \quad (19)$$

where $\Phi_0 = 2\pi\hbar c e^{-1}$ is the London flux quantum.

The scattering wave solution in this case is (cf. (7))

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} e^{\mathrm{i}(|n|-\frac{1}{2}|n-\mu|)\pi} \times$$

$$\times \left[J_{|n-\mu|}(kr) - \frac{J_{|n-\mu|}(kr_c)}{H^{(1)}_{|n-\mu|}(kr_c)} H^{(1)}_{|n-\mu|}(kr)\right], \quad (20)$$

where $\mu = \Phi\Phi_0^{-1}$. In the long-wavelength limit, we get the asymptotics

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{\substack{kr\gg 1\\ kr_c\ll 1}}{=} \sqrt{\frac{2\pi}{kr}}\, e^{\mathrm{i}(kr-\pi/4)} \times$$

$$\times \left[\cos(\mu\pi)\Delta(\varphi) - \sin(\mu\pi)\Gamma^{(\nu)}(\varphi)\right] +$$

$$+ \sqrt{\frac{2\pi}{kr}}\, e^{-\mathrm{i}(kr-\pi/4)} \Delta(\varphi - \pi), \quad (21)$$

where $\nu = [\![\mu]\!]$ and $[\![u]\!]$ denotes the integer part of u (i.e. the integer which is less than or equal to u), and

$$\Gamma^{(\nu)}(\varphi) = \frac{1}{2\pi \mathrm{i}} \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n-\mu) e^{\mathrm{i}n\varphi}. \quad (22)$$

In the short-wavelength limit, we get the asymptotics

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{kr > kr_c \gg 1}{=} \sqrt{\frac{2\pi}{kr}}\, e^{\mathrm{i}(kr-\pi/4)} \times$$

$$\times \left\{\cos(\mu\pi)\left[\Delta^{(\nu)}_{kr}(\varphi) - \Delta^{(\nu)}_{kr_c}(\varphi)\right] - \right.$$

$$\left. - \sin(\mu\pi)\left[\Gamma^{(\nu)}_{kr}(\varphi) - \Gamma^{(\nu)}_{kr_c}(\varphi)\right]\right\} + \sqrt{\frac{2\pi}{kr}}\, e^{\mathrm{i}\pi/4} \times$$

$$\times \left[e^{-\mathrm{i}kr}\Delta^{(\nu)}_{kr}(\varphi - \pi) - e^{\mathrm{i}k(r-2r_c)}\Delta^{(\nu)}_{kr_c}(\varphi - \pi)\right], \quad (23)$$

where

$$\Delta^{(\nu)}_x(\varphi) = \frac{1}{2\pi} \sum_{|n-\mu|\le x} e^{in\varphi},$$

$$\Gamma^{(\nu)}_x(\varphi) = \frac{1}{2\pi \mathrm{i}} \sum_{|n-\mu|\le x} \mathrm{sgn}(n-\mu) e^{in\varphi}; \quad (24)$$

we note that $\Delta^{(\nu)}_x(\varphi)$ can be regarded as a regularization for $\Delta(\varphi)$ (5), since $\Delta^{(\nu)}_x(\varphi)$, as well as $\Delta_x(\varphi)$ (11), satisfies conditions (12). It should be noted that, since it is a merely qualitative analysis, the long-wavelength asymptotics of the wave function can be estimated as well as (21) with $\Delta^{(\nu)}_{kr}$ and $\Gamma^{(\nu)}_{kr}$ substituted for Δ and $\Gamma^{(\nu)}$.

Thus, we conclude that, both in the long- and short-wavelength limits, the particle wave penetrating in the forward direction behind the obstacle depends periodically as cosine on the enclosed magnetic flux with the period equal to the London flux quantum. This periodic dependence, as well as the sine periodic dependence of the reflected wave in other directions, is due to the fact that the interaction with the scatterer is neither of the potential type, nor sufficiently decreases at large distances from the scatterer, see Hamiltonian (19).

Turning now from the qualitative analysis to the quantitative one, we note, firstly, that, owing to the long-range nature of the interaction, the identity matrix in (1) is distorted:

$$I(k,\varphi;k',\varphi') = \cos(\mu\pi)\frac{1}{k}\delta(k-k')\Delta(\varphi-\varphi'), \quad (25)$$

while the scattering amplitude is

$$f(k,\varphi) = f_0(k,\varphi) + f_c(k,\varphi), \quad (26)$$

where

$$f_0(k,\varphi) = \mathrm{i}\sqrt{\frac{2\pi}{k}}\sin(\mu\pi)\Gamma^{(\nu)}(\varphi) \quad (27)$$

and

$$f_c(k,\varphi) = \mathrm{i}\sqrt{\frac{2}{k\pi}} \sum_{n\in\mathbb{Z}} e^{\mathrm{i}n\varphi} e^{\mathrm{i}(|n|-|n-\mu|)\pi} \frac{J_{|n-\mu|}(kr_c)}{H^{(1)}_{|n-\mu|}(kr_c)}. \quad (28)$$

The asymptotics of the wave function at large distances from the vortex is

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi^{(0)}_{\mathbf{k}}(\mathbf{r}) e^{\mathrm{i}\mu[\varphi - \mathrm{sgn}(\varphi)\pi]} +$$

$$+f(k,\,\varphi)\frac{e^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}}+O(r^{-3/2}). \tag{29}$$

The unitarity condition for the S-matrix (15) is certainly valid for all wavelengths. However, due to the long-range nature of the interaction, the consequent condition in terms of the scattering amplitude takes forms that differ from (16).

The long-wavelength limit, $k \to 0$, is the same as the $r_c \to 0$ limit corresponding to the idealized case of a singular vortex of zero thickness. Amplitude (28), f_c, can be neglected in this case, and the S-matrix unitarity condition involves amplitude (27), f_0, only. In view of (25) and relation

$$\Gamma^{(\nu)}(\varphi)+\left[\Gamma^{(\nu)}(-\varphi)\right]^{*}=0, \tag{30}$$

we get, instead of (16), the following relation:

$$\sin^2(\mu\pi)\Delta(\varphi'-\varphi'')=$$

$$=\frac{k}{2\pi}\int\limits_{-\pi}^{\pi}d\varphi\, f_0^{*}(k,\,\varphi-\varphi')f_0(k,\,\varphi-\varphi''). \tag{31}$$

Thus, we see that the optical theorem which should be derived from (31) by putting $\varphi'=\varphi''=0$ is hardly informative, being a relation between infinite quantities, $\Delta(0)$.

The failure with the optical theorem for the Aharonov–Bohm scattering in the long-wavelength limit is due to an unphysical idealization inherent in the treatment of this case. As long as the transverse size of the vortex is taken into account, the optical theorem emerges as a relation between finite quantities. Really, retaining f_c in the unitarity relation (15), we get

$$\frac{1}{\mathrm{i}}\sqrt{\frac{k}{2\pi}}\Big\{\cos(\mu\pi)\left[f_c(k,\,\varphi'-\varphi'')-f_c^{*}(k,\,\varphi''-\varphi')\right]+$$

$$+\sin(\mu\pi)\int\limits_{-\pi}^{\pi}d\varphi\left[\Gamma^{(\nu)}(\varphi'-\varphi)f_c(k,\,\varphi-\varphi'')+\right.$$

$$\left.+f_c^{*}(k,\,\varphi-\varphi')\Gamma^{(\nu)}(\varphi-\varphi'')\right]\Big\}=$$

$$=\frac{k}{2\pi}\int\limits_{-\pi}^{\pi}d\varphi\, f_c^{*}(k,\,\varphi-\varphi')f_c(k,\,\varphi-\varphi''), \tag{32}$$

which yields, at $\varphi'=\varphi''=0$, the optical theorem

$$\sqrt{\frac{2\pi}{k}}\Big\{2\cos(\mu\pi)\mathrm{Im}\, f_c(k,\,0)-\mathrm{i}\sin(\mu\pi)\int\limits_{-\pi}^{\pi}d\varphi\times$$

$$\times\left[\Gamma^{(\nu)}(-\varphi)f_c(k,\,\varphi)+\Gamma^{(\nu)}(\varphi)f_c^{*}(k,\,\varphi)\right]\Big\}=$$

$$=\int\limits_{-\pi}^{\pi}d\varphi|f_c(k,\,\varphi)|^2. \tag{33}$$

As the wavelength decreases, f_0 decreases as $O(k^{-1/2})$, see (27), becoming negligible as compared with f_c. Thus, the right-hand side of (33) tends to the total cross section in the short-wavelength limit. Further, estimating appropriately the integral on the left-hand side of (32), we find that this relation in the short-wavelength limit turns out to be

$$\frac{1}{\mathrm{i}}\sqrt{\frac{k}{2\pi}}\cos(\mu\pi)\left[f_c(k,\,\varphi'-\varphi'')-f_c^{*}(k,\,\varphi''-\varphi')\right]+$$

$$+2\sin^2(\mu\pi)\Delta_{kr_c}^{(\nu)}(\varphi'-\varphi'')+O(\sqrt{kr_c})=$$

$$=\frac{k}{2\pi}\int\limits_{-\pi}^{\pi}d\varphi\, f_c^{*}(k,\,\varphi-\varphi')f_c(k,\,\varphi-\varphi''), \tag{34}$$

and the optical theorem in this limit takes form

$$2\sqrt{\frac{2\pi}{k}}\cos(\mu\pi)\mathrm{Im}\, f_c(k,\,0)+$$

$$+\frac{4\pi}{k}\sin^2(\mu\pi)\Delta_{kr_c}^{(\nu)}(0)+O(k^{-1})=\sigma. \tag{35}$$

The scattering amplitude in the short-wavelength limit was considered in [13], and it can be presented in the form (cf. (18)):

$$f_c(k,\,\varphi)=\mathrm{i}\sqrt{\frac{2\pi}{k}}\left[\cos(\mu\pi)\Delta_{kr_c}^{(\nu)}(\varphi)-\sin(\mu\pi)\Gamma_{kr_c}^{(\nu)}(\varphi)\right]-$$

$$-e^{-\mathrm{i}\pi/4}\sqrt{\frac{r_c}{2}\cos\left[\frac{\varphi}{2}-\mathrm{sgn}(\varphi)\frac{\pi}{2}\right]}\times$$

$$\times\exp\left\{-2\mathrm{i}kr_c\cos\left[\frac{\varphi}{2}-\mathrm{sgn}(\varphi)\frac{\pi}{2}\right]+\mathrm{i}\mu[\varphi-\mathrm{sgn}(\varphi)\pi]\right\},$$

$$kr_c \gg 1. \tag{36}$$

The Fraunhofer diffraction on the vortex is described by the first term, while the classical reflection from the vortex is described by the second term, which, apart from the phase factor, is the same as in (18). It should be noted that the left-hand side of (35) in the nonvanishing order involves the contribution of the diffraction peak only, whereas the right-hand side of (35) includes the contribution of the classical reflection as well. Moreover, the contribution of the diffraction peak to the total cross section is flux-independent and is equal to that of the classical reflection, see [13]. Thus, the total cross section in the short-wavelength limit is equal to $4r_c$ that is twice the classical total cross section.

The explicit form of $\Delta^{(\nu)}_{kr_c}(\varphi)$ and $\Gamma^{(\nu)}_{kr_c}(\varphi)$ is as follows:

$$\Delta^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2})\varphi}}{2\pi}\,\frac{\sin(s_c\varphi)}{\sin(\varphi/2)} \tag{37}$$

and

$$\Gamma^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2})\varphi}}{2\pi}\,\frac{1-\cos(s_c\varphi)}{\sin(\varphi/2)} \tag{38}$$

in the case

$$[\![kr_c+\mu]\!]-\nu = [\![kr_c-\mu]\!]+\nu+1 = s_c \tag{39}$$

or

$$\Delta^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2}\mp\frac{1}{2})\varphi}}{2\pi}\,\frac{\sin\left[\left(s_c+\frac{1}{2}\right)\varphi\right]}{\sin(\varphi/2)} \tag{40}$$

and

$$\Gamma^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2}\mp\frac{1}{2})\varphi}}{2\pi}\left\{\frac{1-\cos\left[\left(s_c+\frac{1}{2}\right)\varphi\right]}{\sin(\varphi/2)} - \right.$$

$$\left. -\mathrm{tg}(\varphi/4)\pm\mathrm{i}\right\} \tag{41}$$

in the case

$$[\![kr_c+\mu]\!]-\nu-\frac{1}{2}\pm\frac{1}{2} = [\![kr_c-\mu]\!]+\nu+\frac{1}{2}\mp\frac{1}{2} = s_c. \tag{42}$$

Thus, in the short-wavelength limit for the strictly forward direction, we get

$$f_c(k,\,0) = \mathrm{i}\sqrt{\frac{2k}{\pi}}\,r_c\cos(\mu\pi)+O(k^{-1/2}). \tag{43}$$

It is instructive to derive the explicit form of $\Gamma^{(\nu)}(\varphi)$ (22) here. Using the elementary trigonometric relation

$$\cot(\varphi/2)\left\{\sin[(n+1)\varphi]-\sin(n\varphi)\right\} =$$

$$= \cos[(n+1)\varphi]+\cos(n\varphi),$$

one can get

$$\int\limits_0^\pi d\varphi\cot(\varphi/2)\sin(N\varphi) = \pi, \quad N = 1,\,2,\,3,\,\ldots,$$

which results in the relation

$$\cot(\varphi/2) = 2\sum_{\substack{n\in\mathbb{Z}\\ n\geq 1}}\sin(n\varphi).$$

The use of the latter along with definition (5) yields

$$\sum_{\substack{n\in\mathbb{Z}\\ n\geq N}} e^{in\varphi} = \pi\Delta(\varphi) - e^{\mathrm{i}N\varphi}\mathrm{PV}\frac{1}{e^{\mathrm{i}\varphi}-1}, \quad N = 1,\,2,\,3,\ldots, \tag{44}$$

where symbol PV denotes the principal-value prescription in treating the divergence at $\varphi = 2\pi l$ ($l\in\mathbb{Z}$). With the help of (44), one can easily get

$$\Gamma^{(\nu)}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2})\varphi}}{2\pi}\mathrm{PV}\frac{1}{\sin(\varphi/2)}. \tag{45}$$

Although amplitude (27), f_0, with $\Gamma^{(\nu)}$ (45) diverges in the forward direction, this divergence is cancelled with the discontinuity of the incident wave (see [9, 14–17]), and the wave function (29) is finite and continuous in the forward direction:

$$\psi_{\mathbf{k}}(\mathbf{r})|_{\varphi=0} = \cos(\mu\pi)e^{\mathrm{i}kr} + f_c(k,\,0)\frac{e^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}}. \tag{46}$$

The appearance of the factor $\cos(\mu\pi)$ in the transmitted wave (first term in (46)) can be intuitively understood as a result of the self-interference from different sides of the vortex [16,17]. As is shown in the present paper, the same factor appears also in the scattered wave (second term in (46)) due to the Fraunhofer diffraction, see (43). Our main result is the optical theorem (35) that relates the amplitude of the diffraction peak to the total cross section in the short-wavelength limit.

4. Summary and Discussion of Results

We have considered the S-matrix, its unitarity, and the consequent optical theorem for the case of the Aharonov–Bohm scattering. The standard scattering theory (see, e.g., [18]) is not applicable here, since the interaction with a scatterer is not of short-range type. That is why, although the S-matrix is evidently unitary, its unitarity condition in terms of the scattering amplitude takes a rather unusual form.

In the ultraquantum (long-wavelength, $k \to 0$) limit, where the vortex thickness is neglected, the unitarity condition takes the form (31) with no terms which are linear in the scattering amplitude. The scattering amplitude in the ultraquantum limit, see (27) with (45), was first obtained by Aharonov and Bohm [1] and then rederived in the framework of different approaches (perhaps, the one presented at the end of the preceding section here is the most simple and straightforward). As to the behavior of the amplitude in the forward direction, the only thing that should be borne in mind is that the divergence has to be understood in the principal-value sense. The total cross section in the ultraquantum limit diverges as well. Hence, the optical theorem in this limit seems to be hardly efficient, being merely a consistency relation between two divergent (infinite) quantities, $\Delta(0)$.

The divergence of the scattering amplitude and the total cross section, as well as the failure with the optical theorem, has no physical meaning, being an artefact of the approximation which neglects the vortex thickness: this is certainly an excessive idealization, whereas any realistic vortex is of finite nonzero thickness. As long as one departs from the ultraquantum limit and the vortex thickness ($2r_c$) is taken into account, the unitarity condition can be formulated as relation (32) involving the r_c-dependent part of the scattering amplitude, f_c, (28); the optical theorem relates f_c in the forward direction to the contribution of f_c to the total cross section, see (33).

In the quasiclassical (short-wavelength, $k \to \infty$) limit, the vortex-thickness effects are prevailing, and f_c approximates fairly the whole scattering amplitude. The unitarity condition in this limit takes the form (34), and the optical theorem is given by (35). The scattering amplitude in the quasiclassical limit (36) consists of two parts: the one corresponding to the Fraunhofer diffraction on the vortex in the forward direction and the other one corresponding to the classical reflection from the vortex. The latter, apart from the phase factor, is flux-independent, whereas the former is essentially flux-dependent, being periodic in the value of the flux with the period equal to the London flux quantum. We conclude that the persistence of the Fraunhofer diffraction in the short-wavelength limit is crucial for maintaining the optical theorem in the quasiclassical limit, since the classical reflection vanishes in the forward direction.

Thus, the quantum-mechanical scattering theory in the quasiclassical limit yields an effect which is alien to classical mechanics: a particle wave penetrates in the forward direction behind an obstacle, and the Fraunhofer diffraction persisting in the short-wavelength limit is an essential ingredient of this effect. If the obstacle is an impenetrable magnetic vortex, then the effect of the wave penetration in the forward direction is modulated by cosine of the value of the vortex flux, see (46) and (43), and this is revealed by optical theorem (35). It should be emphasized that only the contribution of the diffraction peak is involved on the left-hand side of (35) in the nonvanishing order, whereas the right-hand side of (35) includes both the contributions of the diffraction peak and the classical reflection. Separate flux-dependent terms on the left-hand side of (35) compensate each other to yield the flux-independent right-hand side of (35), which is equal to the doubled diameter of the vortex, i.e. the doubled classical total cross section.

It should be noted that the experimental verification of the Aharonov–Bohm effect is based exclusively on the observation of a fringe shift in the interference pattern due to two coherent particle beams under the influence of an impenetrable magnetic vortex placed between the beams. In a somewhat different setup, one considers the scattering of a particle beam directly on an impenetrable magnetic vortex. Although this second setup is more elaborate from the theoretical point of view (see, e.g., [6–9, 16, 17] and the present paper), its experimental realization is hardly possible with the use of long-wavelength (slowly moving) particles.

On the contrary, a direct scattering experiment with the use of short-wavelength (fast-moving) particles is quite feasible. For the case of an impenetrable tube with no magnetic flux, the classical reflection is surely observed, whereas the forward peak of the Fraunhofer diffraction is elusive to experimental measurements: as was noted in [10], it seems more likely that the measurable quantity is the classical cross section, although the details of this phenomenon depend on the method of measurement. However, almost six decades have passed from the time, when this assertion was made in [10], and experimental facilities have improved enormously since then. In the present paper, we would like to draw attention to this long-standing experimental problem by pointing at the circumstances when the detection of the forward diffraction peak will be the detection of the Aharonov–Bohm effect persisting in the quasiclassical limit (see also [13]). The flux of an impenetrable magnetic vortex serves as a gate for the propagation of short-wavelength, almost classical, particles, and the validity of the optical theorem (35) is to be verified in a direct scattering experiment with such particles.

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S-МАТРИЦЯ ТА ЕФЕКТ ААРОНОВА–БОМА

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Р е з ю м е

Розглянуто S-матрицю у випадку розсіяння Ааронова–Бома і виведено оптичну теорему. Показано, що існування оптичної теореми в квазікласичній межі зумовлене незниканням дифракції Фраунгофера в короткохвильовій межі.