DIRAC ELECTRONS IN A PLANAR POTENTIAL WELL IN A MAGNETIC FIELD

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PACS 71.30.+h, 71.10.-w, 81.05.Uw

We study the Dirac equation in two spatial dimensions for quasiparticles in a potential well in graphene in a homogeneous magnetic field. It is shown that, at some critical value of the potential strength, the lowest empty energy level crosses a filled negative energy level leading to the instability of the system. The critical potential strength decreases with decrease of a quasiparticle gap and becomes zero in the gapless case. It is argued that the magnetically driven instability of a charged center can be considered as a quantum mechanical counterpart of the magnetic catalysis phenomenon in graphene.

1. Introduction

The supercritical Coulomb center is an old well-known problem in quantum electrodynamics (QED) [1, 2]. The solution of the Dirac equation in the Coulomb field of a point charge Ze is singular for $Ze^2/\hbar c > 1$ ($Z > 137$) due to a singular behavior of the Coulomb potential as $r \to 0$. Considering a finite size R of an atomic nucleus regularizes the potential and eliminates the singularity [3] so that discrete levels exist for $Z > 137$). When $Z \to Z_{\rm cr}$ (~ 173 for the $1S$ state), the lowest bound state enters continuum states and becomes a resonance. Therefore, for a supercritical charge, there occurs a spontaneous electron-positron pair creation from vacuum with the electrons shielding the supercritical charge of the nucleus to a subcritical value and with the positrons carrying an excessive positive charge to infinity.

Graphene provides an interesting (2+1)-dimensional analog of the problem of a supercritical Coulomb center in QED. The supercritical charge problem in graphene was thoroughly studied in the literature [4, 5], where it was found that the supercritical charge is defined by $Z_c\alpha = 1/2 + \pi^2/\log^2(c\Delta R/\hbar v_{\rm F})$ [6], where $c \approx 0.21$. Here, Δ is the quasiparticle gap, $\alpha = e^2/\hbar v_{\rm F}$ is the effective coupling constant, and $v_{\rm F} \approx 10^6$ m/s is the Fermi velocity of quasiparticles in graphene. Since the electrons and the holes strongly interact by means of the Coulomb interaction, one may expect [6, 7] the appearance of an excitonic instability in graphene with a subsequent phase transition to the phase with gapped quasiparticles that may turn graphene into an insulator. For this semimetal-insulator transition in graphene, the numerical simulations in the literature give the critical coupling constant $\alpha_c \approx 1.19$ [8, 9].

It is an interesting question how the presence of an homogeneous magnetic field $\mathbf{B}$ influences the supercritical Coulomb center problem and whether $Z_c\alpha$ decreases or increases, as the magnetic field increases? In QED in (3+1) dimensions, the Coulomb center problem in a magnetic field was studied in [10]. It was found that the magnetic field confines the transverse electronic motion, and the electron in a magnetic field is closer to the nucleus than that in a free atom. Thus, it feels a stronger Coulomb field. Therefore, $Z_c\alpha$ decreases with B. This result is consistent with the magnetic catalysis phenomenon [11], according to which the magnetic field catalyzes the gap generation and leads to the zero critical coupling constant for massless fermions in both (3+1)- and (2+1)-dimensional theories. Unfortunately, the Coulomb center problem in a magnetic field is not exactly solvable problem. In this paper, we will study the Dirac equation in two spatial dimensions for the electron in a radially symmetric potential well in an external homogeneous magnetic field, which admits a complete analytic solution.

2. Model

The electron quasiparticle states in a vicinity of the $K_\pm$ points of graphene in the field of a charged impurity and in a homogeneous magnetic field perpendicular to the plane of graphene are described by the Dirac Hamiltonian in 2+1 dimensions

$$H = \hbar v_{\rm F}\boldsymbol{\tau}\mathbf{p} + \xi\Delta\tau_3 + V(r), \tag{1}$$

where the canonical momentum $\mathbf{p} = -i\boldsymbol{\nabla} + e\mathbf{A}/c$ includes the vector potential $\mathbf{A}$ corresponding to the external magnetic field $\mathbf{B}$, and Δ is a quasiparticle gap. The two-component spinor $\Psi_{\xi s}$ carries the valley ($\xi = \pm$) and spin ($s = \pm$) indices. We will use the standard convention: $\Psi^T_{+s} = (\psi_A, \psi_B)_{K_+s}$, whereas $\Psi^T_{-s} = (\psi_B, \psi_A)_{K_-s}$, where A and B refer to two sublattices of the hexagonal graphene lattice, and the Pauli matrices τ_i act in the sublattice space. The potential well interaction $V(r) = -V_0\theta(r_0 - r)$ with $V_0 > 0$ does not depend on the spin. Therefore, we will omit the spin index s in what follows.

The Dirac equation for the electron in a radially symmetric potential well in a homogeneous magnetic field perpendicular to the graphene plane in the symmetric gauge $(A_x, A_y) = (B/2)(-y, x)$ takes the following form in the cylindrical coordinates $x + iy = re^{i\phi}$:

$$f' - \frac{j+1/2}{r}f - \frac{r}{2l^2}f + \frac{E + \xi\Delta - V(r)}{\hbar v_{\rm F}}g = 0, \qquad (2)$$

$$g' + \frac{j-1/2}{r}g + \frac{r}{2l^2}g - \frac{E - \xi\Delta - V(r)}{\hbar v_{\rm F}}f = 0, \qquad (3)$$

where the wave function

$$\psi^T = (e^{i(j-1/2)\phi}f(r), ie^{i(j+1/2)\phi}g(r))/r, \quad l = \sqrt{\hbar c/|eB|}$$

is the magnetic length, and $\xi = \pm$ for quasiparticles near the K_+ and K_- points in graphene.

Equations (2) and (3) are easily solved in two regions $r < r_0$ and $r > r_0$ in terms of confluent hypergeometric functions. In the region $r < r_0$, eliminating the function $g(r)$, we obtain the second-order differential equation for the function $f(r)$:

$$f'' - \frac{1}{\rho}f' + \left[2p_V^2 - j - \frac{1}{2} - \frac{j^2 - j - 3/4}{\rho^2} - \frac{\rho^2}{4}\right]f = 0, \quad (4)$$

and, in the region $r > r_0$, we have the same equation but with $V_0 = 0$. Here, we introduced the following dimensionless quantities:

$$p_V^2 = (\epsilon + v_0)^2 - m^2, \quad p^2 = \epsilon^2 - m^2, \quad \epsilon = lE/(\sqrt{2}\hbar v_{\rm F}),$$

$$m = l\Delta/(\sqrt{2}\hbar v_{\rm F}), \quad v_0 = lV_0/(\sqrt{2}\hbar v_{\rm F}), \quad \rho = r/l.$$

The solution of (4) regular at $r = 0$ reads

$$f(\rho) = \rho^{j+\frac{1}{2}}e^{-\rho^2/4}\frac{C_1}{\Gamma(j+1/2)}\times$$

$$\times\Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{1}{2}; \frac{\rho^2}{2}\right), \qquad (5)$$

$$g(\rho) = (\epsilon + v_0 - \xi m)\rho^{j+\frac{3}{2}}e^{-\rho^2/4}\frac{C_1}{\Gamma(j+3/2)}\times$$

$$\times\Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{3}{2}; \frac{\rho^2}{2}\right). \qquad (6)$$

The solution of (4) that decreases at infinity is given by

$$f(\rho) = C_2\rho^{j+\frac{1}{2}}e^{-\rho^2/4}\Psi\left(j + \frac{1}{2} - p^2, j + \frac{1}{2}; \frac{\rho^2}{2}\right), \qquad (7)$$

$$g(\rho) = \frac{C_2}{\epsilon + \xi m}\rho^{j+\frac{3}{2}}e^{-\rho^2/4}\Psi\left(j + \frac{1}{2} - p^2, j + \frac{3}{2}; \frac{\rho^2}{2}\right). \quad (8)$$

Note that both these expressions are valid at all $j = \pm 1/2, \pm 3/2, \ldots$.

3. Energy Spectrum

Sewing the above solutions at $r = r_0$, we obtain the following equation for the energies of solutions with the total angular momentum j:

$$\frac{(j+\frac{1}{2})\Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{1}{2}; \frac{\rho_0^2}{2}\right)}{(\epsilon + v_0 - \xi m)\Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{3}{2}; \frac{\rho_0^2}{2}\right)} =$$

$$= (\epsilon + \xi m)\frac{\Psi\left(j + \frac{1}{2} - p^2, j + \frac{1}{2}; \frac{\rho_0^2}{2}\right)}{\Psi\left(j + \frac{1}{2} - p^2, j + \frac{3}{2}; \frac{\rho_0^2}{2}\right)}. \qquad (9)$$

Before analyzing Eq. (9) and finding the energy spectrum of the problem under consideration, we recall the Landau energy levels for the electron states in graphene in a homogeneous magnetic field. If the interaction vanishes ($V_0 = 0, r_0 \to 0$), Eq. (9) gives the following well-known spectrum of infinitely degenerate Landau levels:

$$E = -\xi\Delta, \quad j \le -\frac{1}{2}, \qquad (10)$$

$$E = \pm\sqrt{\Delta^2 + 2n\left(\frac{\hbar v_{\rm F}}{l}\right)^2}, \; n = 1, 2, \ldots, \; j + \frac{1}{2} \le n. \qquad (11)$$

Note that the level $E = \Delta$ ($E = -\Delta$) is present only at the K_- (K_+) point.

For nonzero V_0, the Landau energy levels are no longer degenerate. Using the sewing equation (9), we can determine the evolution of degenerate solutions with V_0.

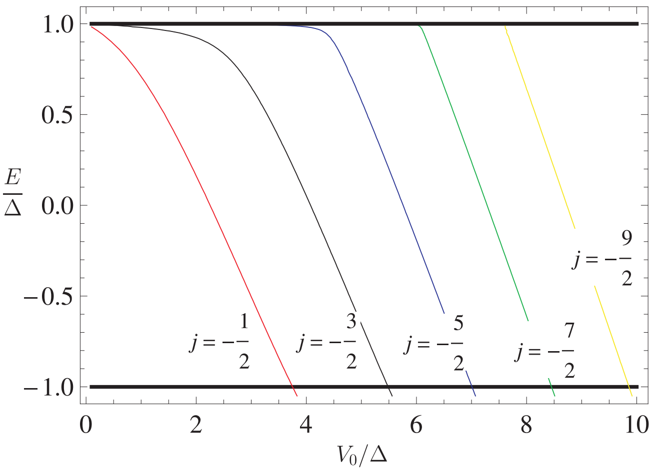

Fig. 1. Evolution of degenerate solutions of the lowest Landau level at the K_- point as a function of the dimensionless ratio V_0/Δ

For solutions of the Landau level $E=\Delta$ with different j, their energies as functions of V_0/Δ (at a fixed magnetic field B) are plotted in Fig. 1 for $m=0.1$ and $\rho_0=r_0/l=0.02$.

We see that, as V_0 increases more and more, the solutions with different j cross the energy level $\epsilon=-m$. Clearly, as soon as a vacant state appears, the system will try to fill it [1, 2] emitting a positively charged hole going to infinity. The critical potential is defined as V_0 for which the first crossing occurs. Figure 1 implies that the first crossing takes place for the state with the lowest centrifugal barrier $j=-1/2$. Therefore, we should pay a special attention to the evolution of this state with V_0.

4. Instability

For the states with negative angular momenta $j=-1/2-k$, $k=0,1,\ldots,$ Eq. (9) becomes

$$(\epsilon+v_0+\xi m)\frac{\Phi\left(1-p_V^2,k+2;\frac{\rho_0^2}{2}\right)}{\Phi\left(-p_V^2,k+1;\frac{\rho_0^2}{2}\right)}=$$

$$=-(\epsilon+\xi m)\frac{\Psi\left(1-p^2,k+2;\frac{\rho_0^2}{2}\right)}{\Psi\left(-p^2,k+1;\frac{\rho_0^2}{2}\right)}, \qquad (12)$$

where we used the formulae

$$\lim_{c\to -k}\frac{\Phi(a,c;x)}{\Gamma(c)}=\frac{\Gamma(a+k+1)}{\Gamma(a)(k+1)!}x^{k+1}\times$$

$$\times\Phi(a+k+1,k+2;x),\ k=0,1,\ldots, \qquad (13)$$

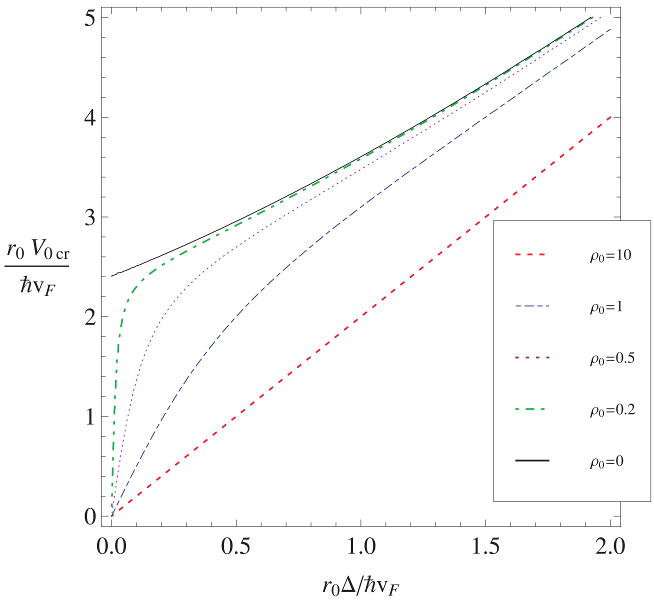

Fig. 2. Critical potential $V_{0\mathrm{cr}}$ as a function of the gap for different values of ρ_0. The case of zero magnetic field corresponds to $\rho_0=0$

$$\Psi(a,c;x)=x^{1-c}\Psi(a-c+1,2-c;x). \qquad (14)$$

For the weak coupling, $V_0\to 0$, we have $p_V^2\to p^2\to 0$. Then Eq. (12) simplifies:

$$\epsilon=-\xi m-v_0\frac{\Phi(1,k+2;\frac{\rho_0^2}{2})}{\Phi(1,k+2;\frac{\rho_0^2}{2})+\Psi(1,k+2;\frac{\rho_0^2}{2})}. \qquad (15)$$

Here, we took into account that $\Phi(0,c;x)=\Psi(0,c;x)=1$. This equation yields the following bound state with $j=-1/2$ ($k=0$) at the K_- point:

$$\epsilon=m-v_0\left(1-e^{-\rho_0^2/2}\right), \qquad (16)$$

where we made use of the particular values of hypergeometric functions, $\Phi(1,2;x)=(e^x-1)/x, \Psi(1,2;x)=1/x$. At large angular momenta, $k\to\infty$, using the corresponding asymptotics of hypergeometric functions, we find that the bound states accumulate near the energy $\pm m$:

$$\epsilon=-\xi m-\frac{v_0e^{-\rho_0^2/2}}{\Gamma(k+1)}\left(\frac{\rho_0^2}{2}\right)^{k+1}. \qquad (17)$$

As the coupling v_0 grows, the energy of the bound state with $j=-1/2$ decreases and finally crosses the level $\epsilon=-m$ at some critical value v_{cr}.

For $\epsilon=-m$, we have $p^2=0$, $p_V^2=v_0^2-2mv_0$. Then, for the state with $\xi=-$, $j=-1/2$, we find that Eq. (12)

defines the following equation for the critical interaction strength $V_{0\mathrm{cr}}$:

$$V_{0\mathrm{cr}} = 2\Delta\left[1 + \frac{2\Phi\left(-\kappa, 1, \frac{\rho_0^2}{2}\right)}{\rho_0^2\Phi\left(1-\kappa, 2, \frac{\rho_0^2}{2}\right)}\right], \tag{18}$$

where $\kappa = v_{\mathrm{cr}}(v_{\mathrm{cr}} - 2m)$. The critical potential strength $V_{0\mathrm{cr}}$ as a function of Δ is plotted in Fig. 2 for different values of the parameter ρ_0 which defines the ratio of the potential well width to the magnetic length. Analytically, it is not difficult to find that, for $\rho_0 \ll 1$, Eq. (18) yields

$$V_{0\mathrm{cr}} = 2\Delta(1 + 2l^2/r_0^2)\,. \tag{19}$$

It is clearly seen from Eq. (19) that the critical potential strength $V_{0\mathrm{cr}}$ decreases with the growth of a magnetic field (or, with the decrease of l) at fixed r_0 and Δ. The physical reason for that is that the magnetic field forces electron orbits to become closer to the charge center, thus making attraction stronger.

What is really surprising here is that $V_{0\mathrm{cr}}$ tends to zero as $\Delta \to 0$. If the magnetic field is absent, then it is not difficult to check that, for the Dirac equation with a potential well, the critical strength is given by the expression

$$V_{0\mathrm{cr}} = \Delta\left[1 + \sqrt{1 + \left(\frac{\hbar v_{\mathrm{F}}}{\Delta r_0}\right)^2 j_{0,1}^2}\right], \tag{20}$$

where $j_{0,1} \approx 2.41$ is the first zero of the Bessel function $J_0(x)$. Therefore, $V_{0\mathrm{cr}}$ tends to a finite value, $V_{0\mathrm{cr}} = (\hbar v_{\mathrm{F}}/r_0)j_{0,1}$, in the gapless limit $\Delta \to 0$. In this case, there is only the continuum spectrum for $V_0 < V_{0\mathrm{cr}}$. Whereas, for $V_0 > V_{0\mathrm{cr}}$, resonances with complex energies $\mathrm{Re}E < 0$, $\mathrm{Im}E < 0$ appear, by signalizing the instability of system. The presence of a magnetic field changes the situation dramatically. It leads to the instability of the potential well problem in the second quantized theory for *any* value of potential strength V_0.

5. The Local Density of States

It is interesting to see how a magnetic field and the charged center affect the local density of states (LDOS) of quasiparticles in graphene which can be directly measured in scanning tunneling microscope (STM) experiments. The crucial difference of the case of gapless quasiparticles from that of gapped ones in a magnetic field is that the critical charge is zero for gapless quasiparticles. Therefore, the energies of all previously degenerate states of the lowest Landau level become negative.

The LDOS at a distance $\mathbf{r}$ from the impurity is given by

$$\rho(E;\mathbf{r}) = -\frac{1}{\pi}\mathrm{tr}\mathrm{Im}G(\mathbf{r},\mathbf{r};E+i\eta), \quad \eta \to 0, \tag{21}$$

where the trace includes the summation over the valley, sublattice, and spin degrees of freedom, and the retarded Green's function $G(\mathbf{r},\mathbf{r}';E+i\eta)$ in a constant magnetic field has the form

$$G(\mathbf{r},\mathbf{r}';E) = e^{i\Phi(\mathbf{r},\mathbf{r}')}\tilde{G}(\mathbf{r},\mathbf{r}';E), \tag{22}$$

$$\Phi(\mathbf{r},\mathbf{r}') = \frac{e}{\hbar c}\int\limits_{\mathbf{r}'}^{\mathbf{r}} A_i^{ext}(z)dz^i, \tag{23}$$

where $\Phi(\mathbf{r},\mathbf{r}')$ is the phase, and $\tilde{G}(\mathbf{r},\mathbf{r}';E)$ is the gauge invariant part of the Green's function. The last one satisfies the Lippmann–Schwinger equation

$$\tilde{G}(\mathbf{r},\mathbf{r}';E) = \tilde{G}_0(\mathbf{r}-\mathbf{r}';E) + \int d\mathbf{r}''\tilde{G}_0(\mathbf{r}-\mathbf{r}'';E)\times$$

$$\times V(\mathbf{r}'')\tilde{G}(\mathbf{r}'',\mathbf{r}';E)e^{i[\Phi(\mathbf{r},\mathbf{r}'')+\Phi(\mathbf{r}'',\mathbf{r}')+\Phi(\mathbf{r}',\mathbf{r})]}. \tag{24}$$

[Note that the Green function $\tilde{G}(\mathbf{r},\mathbf{r}';E)$ is not translation-invariant in the presence of an impurity unlike the noninteracting function $\tilde{G}_0(\mathbf{r}-\mathbf{r}';E)$.] For the weak interaction, we can calculate the LDOS in the first order of perturbation theory,

$$\rho(E;\mathbf{r}) = \rho_0(E;\mathbf{r}) + \delta\rho(E;\mathbf{r}), \tag{25}$$

where $\rho_0(E;\mathbf{r})$ is the LDOS for free quasiparticles in a magnetic field, and

$$\delta\rho(E;\mathbf{r}) = -\frac{1}{\pi}\mathrm{Im}\int d\mathbf{r}'\mathrm{tr}\Big[\tilde{G}_0(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\tilde{G}_0(\mathbf{r}'-\mathbf{r})\Big]\,. \tag{26}$$

The Green's function of free quasiparticles in a magnetic field is well known (see, e.g., [11,13]), and it has the form of a series over the Landau levels in the configuration space (we consider the zero-gap case),

$$\tilde{G}_0(\mathbf{r};E) = \frac{1}{2\pi l^2}\,e^{-\frac{\mathbf{r}^2}{4l^2}}\sum_{n=0}^{\infty}\frac{1}{(E+i\eta)^2 - M_n^2}\times$$

$$\times\left[E\left[P_-L_n\left(\frac{\mathbf{r}^2}{2l^2}\right)+P_+L_{n-1}\left(\frac{\mathbf{r}^2}{2l^2}\right)\right]+\right.$$

$$\left.+\,i\hbar v_{\mathrm{F}}\frac{\boldsymbol{\tau}\mathbf{r}}{l^2}L^1_{n-1}\left(\frac{\mathbf{r}^2}{2l^2}\right)\right], \qquad (27)$$

where $M_n = (\hbar v_{\mathrm{F}}/l)\sqrt{2n}$ are the energies of Landau levels, $P_\pm = (1\pm\tau_3)/2$ being the projectors, $L_n^\alpha(z)$ the generalized Laguerre polynomials (by definition, $L_n(z)\equiv L_n^0(z)$ and $L^\alpha_{-1}(z)\equiv 0$).

The sum over the Landau levels can be explicitly performed by means of the formula

$$\sum_{n=0}^{\infty}\frac{L_n^\alpha(x)}{n+b}=\Gamma(b)\Psi(b;1+\alpha;x) \qquad (28)$$

(see, Eq.(6.12.3) in [15]), leading to a closed expression for the free Green's function (see the recent papers [16, 17]),

$$\tilde{G}_0(\mathbf{r};E)=-\frac{e^{-\frac{\mathbf{r}^2}{4l^2}}}{4\pi\hbar^2v_{\mathrm{F}}^2}\left\{E\left[P_-\Gamma(-\lambda)\Psi\left(-\lambda;1;\frac{\mathbf{r}^2}{2l^2}\right)+\right.\right.$$

$$\left.+\,P_+\Gamma(1-\lambda)\Psi\left(1-\lambda;1;\frac{\mathbf{r}^2}{2l^2}\right)\right]+$$

$$\left.+\,i\hbar v_{\mathrm{F}}\frac{\boldsymbol{\tau}\mathbf{r}}{l^2}\Gamma(1-\lambda)\Psi\left(1-\lambda;2;\frac{\mathbf{r}^2}{2l^2}\right)\right\}. \qquad (29)$$

Here, $\Gamma(x)$ is the Euler gamma function, and $\lambda=(E+i\eta)^2l^2/(2\hbar^2v_{\mathrm{F}}^2)$.

The LDOS of free quasiparticles in a magnetic field does not depend on $\mathbf{r}$ and is given by

$$\rho_0(E)=-\frac{1}{\pi}\lim_{\mathbf{r}\to0}\mathrm{Imtr}[\tilde{G}_0(\mathbf{r};E+i\eta)]=\frac{1}{\pi^2\hbar^2v_{\mathrm{F}}^2}\times$$

$$\times\lim_{\mathbf{r}\to0}\mathrm{Im}\left\{(E+i\eta)\left[\Gamma(-\lambda)\Psi\left(-\lambda;1;\frac{\mathbf{r}^2}{2l^2}\right)+\right.\right.$$

$$\left.\left.+\,\Gamma(1-\lambda)\Psi\left(1-\lambda;1;\frac{\mathbf{r}^2}{2l^2}\right)\right]\right\}. \qquad (30)$$

The hypergeometric function $\Psi(a;c;x)$ at small x behaves as

$$\Psi(a;1;x)\simeq-\frac{1}{\Gamma(a)}[\ln x+\psi(a)+2\gamma]+O(x\ln x),$$

$$\Psi(a;2;x)\simeq\frac{1}{\Gamma(a)x}+\frac{1}{\Gamma(a-1)}[\ln x+\psi(a)+$$

$$+2\gamma-1]+O(x\ln x), \qquad (31)$$

where $\psi(z)$ is the digamma function. Therefore,

$$\rho_0(E)=-\frac{1}{(\pi\hbar v_{\mathrm{F}})^2}\mathrm{Im}\left[(E+i\delta)\left(\psi(-\lambda)+\psi(1-\lambda)\right)\right], \qquad (32)$$

and the LDOS of free quasiparticles in a magnetic field is found finally to be

$$\rho_0(E)=\frac{2}{\pi l^2}\left[\delta(E)+\sum_{n=1}^{\infty}[\delta(E-M_n)+\delta(E+M_n)]\right], \qquad (33)$$

(compare with Eq. (4.2) in [18]).

The first-order correction to the LDOS due to the interaction is given by Eq. (26). To find the asymptotics in the case of the radial well at distances $r\gg r_0$, where r_0 is the range of the potential, we can put $r'=0$ in the arguments of the free Green's functions in Eq. (26) and get the behavior

$$\delta\rho(\mathbf{r},\mathbf{r};E)=V_0r_0^2\mathrm{Imtr}[\tilde{G}_0(\mathbf{r};E)\tilde{G}_0(-\mathbf{r};E)]\simeq$$

$$\simeq\frac{2V_0r_0^2}{(\pi\hbar v_{\mathrm{F}}l)^2}\mathrm{Im}[\lambda\psi\left(-\lambda\right)]\ln\frac{\mathbf{r}^2}{2l^2}, \qquad (34)$$

$$\frac{V_0r_0^2}{2(\pi\hbar v_{\mathrm{F}}l)^2}\mathrm{Im}\left[\lambda\Gamma^2(-\lambda)\right]e^{-\mathbf{r}^2/2l^2}\left(\frac{\mathbf{r}^2}{2l^2}\right)^{2|\lambda|} \qquad (35)$$

in the regions $l\gg r\gg r_0$ and $r\gg\max(l,r_0)$, respectively. As is seen, the correction to the free LDOS is an odd function of the energy and decreases exponentially at large distances.

6. Conclusion

Analyzing the Dirac equation with a radially symmetric potential well for quasiparticles with a gap in graphene in a homogeneous magnetic field, we found a critical value of the potential strength, when the lowest empty level crosses a filled negative energy level leading to the instability of the system. We showed that this critical potential strength tends to zero as the quasiparticle gap goes to zero, $\Delta\to0$. Consequently, the presence of a magnetic field dramatically affects the potential well problem in graphene making an arbitrarily shallow potential supercritical in the gapless theory. The crucial ingredient for the instability is the existence of the zero-energy level for gapless Dirac fermions in a magnetic field which is infinitely degenerate. In this case, any weak attraction leads to the appearance of empty states in the Dirac sea of negative-energy states and to the instability of the system.

One should stress a qualitative difference in the phenomenon of instability between gapped and gapless quasiparticles. In the case of gapped quasiparticles, there is a *finite* critical value for the strength of interaction, when the lowest unfilled level crosses the first filled one, forming a hole in the sea of filled states. As the coupling grows, more and more levels cross that level. Clearly, the system tries to rearrange itself, by filling in empty states, whose presence is a signal of instability.

This result suggests that the Coulomb center in gapless graphene in a magnetic field may be also unstable for any value Ze. In turn, since the electrons and the holes in graphene interact by means of the Coulomb interaction, this implies that the magnetically driven instability of the supercritical Coulomb center can be considered as a quantum mechanical counterpart of the magnetic catalysis phenomenon in graphene.

This work is supported partially by the SCOPES grant No. IZ73Z0_128026 of Swiss NSF, by the grant SIMTECH No. 246937 of the European FP7 program, by the joint Ukrainian-Russian SFFR-RFFR project F40.2/108 “Application of string theory and field theory methods to nonlinear phenomena in low dimensional systems”, and by the Program of Fundamental Research of the Physics and Astronomy Division of the NAS of Ukraine.

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ДIРАКIВСЬКI ЕЛЕКТРОНИ В ПЛАНАРНIЙ ПОТЕНЦIЙНIЙ ЯМI У ПРИСУТНОСТI МАГНIТНОГО ПОЛЯ

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Р е з ю м е

Дослiджено рiвняння Дiрака у двовимiрному просторi для квазiчастинок в потенцiйнiй ямi у постiйному магнiтному полi в графенi. Показано, що для деякого критичного значення потенцiалу ями найнижчий незаповнений енергетичний рiвень перетинає межу заповнених електронних станiв iз вiд'ємною енергiєю, що приводить до нестабiльностi системи. Критичний потенцiал зменшується зi зменшенням щiлини квазiчастинок i дорiвнює нулю у безщiлинному випадку. Аргументовано, що магнiтно iндукована нестабiльнiсть зарядженого центра може розглядатися як квантово-механiчний аналог явища магнiтного каталiзу в графенi.