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CAN NUCLEAR MATTER CONSIST OF α -PARTICLES?

A sufficient condition for the spatial collapse in an infinite system of interacting Bose particles is obtained on the basis of the variational principle with the use of trial functions with the Jastrow pair correlation factors. The instability of a hypothetical infinite system of α -particles with respect to the spatial collapse is shown under the assumption of the Ali–Bodmer interaction potentials between such Bose particles. Thus, it becomes clear why the hypothetical nuclear matter is naturally treated with the use of at least the nucleon degrees of freedom.

Keywords: Bose-system of α -particles, spatial collapse, nuclear matter.

1. Introduction

The ${}^4\text{He}$ nucleus, or an α -particle, is known to be a strongly bound system of two protons and two neutrons (with the binding energy of about 28.3 MeV) with zero spin and isospin. The structure of an α -particle is well-known (see, for example, [1]). Inside the nuclei, one can reveal the α -clusters. A number of light nuclei can be treated as a definite number of interacting α -particles (see [2] and references *ibid*) or a system of α -particles with additional nucleons [3–6]. There exist some approaches [7] treating the nuclei (up to the heaviest ones) with even and equal numbers of protons and neutrons as the systems of α -particles.

But what about the hypothetical nuclear matter: can one treat this system as consisting of α -particles? This question is analyzed in the present paper, and it is shown that the system of α -particles is unstable with respect to the spatial collapse if one ignores the Coulomb repulsion between the particles. To prove this assertion from the very first principles, we consider a Hamiltonian of N interacting Bose particles with pairwise interactions. We use the variational principle and derive a sufficient condition for the spatial collapse in this system. The obtained sufficient condition is a generalization of the one obtained in [8], and it is more refined. Using this criterion, we show that an infinite system of α -particles with the known Ali–Bodmer interactions [9]

between them is unstable with respect to the spatial collapse. As it is clear from the criterion formula given below, the same result is valid for any other possible realistic versions of α - α interactions of the same type with attraction at some distances of about a few fermi and any non-singular repulsion at short distances. It should be noted that, as usual in the nuclear matter theory, we do not consider the Coulomb repulsion between particles, otherwise the system will be destroyed by this long-range repulsion and will not be bound at all. Just due to the Coulomb repulsion, the nuclei heavier than those with $Z > \sim 10^2$ are not stable, nothing to say about $Z \rightarrow \infty$.

In the next section, we obtain a sufficient condition for the spatial collapse to take place in an infinite system of Bose particles. We start with the Hamiltonian containing the pairwise potentials, and then we use the Ritz variational principle to derive this condition. As compared to the criterion [8] obtained with the use of trivial trial function in the form of the product of one-particle functions, we now account for the Jastrow pair correlation factors and obtain a more delicate criterion.

We stress that, as it was shown in [8], the spatial collapse of an infinite system of interacting Bose particles can not be analyzed on the basis of the well-known Gross–Pitaevskii equation [10–12] because the presence or absence of the spatial collapse is not determined by the two-particle scattering length value.

2. Variational Estimation for the Energy with the Use of Trial Functions with the Jastrow Factors

We consider a system of N identical interacting Bose particles of mass m with the Hamiltonian

$$\hat{H} = \sum_{k=1}^N \frac{\hat{\mathbf{p}}_k^2}{2m} + \sum_{n>k=1}^N V(|\mathbf{r}_n - \mathbf{r}_k|), \quad (1)$$

where the short-range pairwise potential depends on the distance between particles. We assume that the profile of this potential has the general form with some (possible) attraction of finite range and some (possible) repulsion at short distances.

Now, we are going to find a sufficient condition for the spatial collapse of the system under consideration to take place at $N \rightarrow \infty$. In the previous paper [8], we proposed a simple sufficient condition of the spatial collapse in an infinite system of interacting Bose particles to be $\int V(r) d\mathbf{r} < 0$. But, for a system of α -particles, this simple criterion appears to be too mild in order to decide unambiguously whether one has the spatial collapse in an infinite system of α -particles (without the Coulomb repulsion). Really, let us consider the α - α interaction potential in the form proposed by Ali and Bodmer [9]. Then, only for two potentials denoted by a_0 and b_0 , one obtains $\int V(r) d\mathbf{r} < 0$, i.e., the collapse takes place without any doubts. For the rest versions of Ali–Bodmer potentials, this sufficient condition is not fulfilled, and the answer is undetermined. This only means that the used criterion is to be improved.

In the previous paper [8], we used the trial function in the simplest form of a product of one-particle Gaussian functions:

$$\begin{aligned} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) &= \prod_{k=1}^N \exp\left(-(\mathbf{r}_k/R)^2\right) \equiv \\ &\equiv \exp\left(-\frac{1}{R^2} \sum_{k=1}^N \mathbf{r}_k^2\right), \end{aligned} \quad (2)$$

where R is a parameter of the order of the size of the system. To improve the variational estimation, we now consider the wave function with the Jastrow pair correlation factors taken into account:

$$\Phi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \prod_{k>n=1}^N f(r_{kn}) \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N), \quad (3)$$

where function Ψ has the form (2), and the Jastrow correlation factors $f(r_{kn})$ are chosen in the form

$$f(r_{kn}) \equiv 1 - \exp(-r_{kn}^2/r_0^2), \quad (4)$$

where $r_{nk} \equiv |\mathbf{r}_n - \mathbf{r}_k|$, and r_0 is the radius of the order of the range of repulsion in the pair potential $V(r)$ of interaction between particles. Note that the choice of a specific form of the correlation factors can change slightly the form of the criterion to be obtained. But the chosen form (4) is sufficient for our purpose to study the spatial collapse in an infinite system of α -particles with the Ali–Bodmer interaction potentials.

To apply the Ritz variational principle (see, for instance, [13]) for the ground-state energy of the system,

$$E \leq \frac{\langle \Phi | \hat{H} | \Phi \rangle}{\langle \Phi | \Phi \rangle}, \quad (5)$$

one has to calculate the matrix elements of the kinetic and potential energies, as well as the matrix element $\langle \Phi | \Phi \rangle$ in the denominator necessary to normalize the trial function (3). To carry on calculations in an explicit form, we consider, for simplicity, the corresponding expressions in the limit $R \gg r_0$, where the radius R of the system is fixed, but much greater than the radius r_0 of correlations. We also assume that R is much greater than the radius of short-range forces being of the same order or compared with r_0 . Thus, the limit $R \gg r_0$ means also that R is much greater than the potential radius. Then we consider the matrix element of the potential energy in this limit:

$$\begin{aligned} \langle \Phi | \sum_{n>k=1}^N V(r_{nk}) | \Phi \rangle &= \frac{N(N-1)}{2} \langle \Phi | V(r_{12}) | \Phi \rangle \equiv \\ &\equiv \frac{N(N-1)}{2} \int d\mathbf{r}_1, \dots, d\mathbf{r}_N \Psi^2(\mathbf{r}_1, \dots, \mathbf{r}_N) \times \\ &\times V(r_{12}) \prod_{k>n=1}^N f^2(r_{kn}) \xrightarrow{R \gg r_0} \\ &\xrightarrow{R \gg r_0} \frac{N(N-1)}{2} C_{3,N} \int d\mathbf{r}_1 d\mathbf{r}_2 f^2(r_{12}) V(r_{12}) \times \\ &\times \exp\left(-\frac{2}{R^2}(r_1^2 + r_2^2)\right), \end{aligned} \quad (6)$$

where we used the notation

$$C_{K,N} \equiv \int d\mathbf{r}_K, \dots, d\mathbf{r}_N \prod_{k>n=K}^N f^2(r_{kn}) \times$$

$$\times \exp\left(-\frac{2}{R^2} \sum_{k=K}^N \mathbf{r}_k^2\right). \quad (7)$$

In particular, the normalization matrix element $\langle \Phi | \Phi \rangle \equiv C_{1,N}$ can be rewritten (at $R \gg r_0$) as

$$C_{1,N} \xrightarrow{R \gg r_0} \left(\frac{\pi}{2}\right)^3 R^6 C_{3,N}. \quad (8)$$

If one use the new variables $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ and $\boldsymbol{\rho} \equiv \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$, one obtains, instead of (6),

$$\begin{aligned} \langle \Phi | \sum_{n>k=1}^N V(r_{nk}) | \Phi \rangle &\xrightarrow{R \gg r_0} \\ &\xrightarrow{R \gg r_0} \frac{N(N-1)}{2} C_{3,N} \left(\frac{R}{2}\right)^3 \pi^{3/2} \int f^2(r) V(r) d\mathbf{r} \end{aligned} \quad (9)$$

in the limit $R \gg r_0$. Thus, the matrix element of the potential energy $\langle \Phi | \hat{V} | \Phi \rangle$ (9) divided by the normalization matrix element (8) becomes as follows:

$$\frac{\langle \Phi | \hat{V} | \Phi \rangle}{\langle \Phi | \Phi \rangle} \xrightarrow{R \gg r_0} \frac{N(N-1)}{2} R^{-3} \pi^{-\frac{3}{2}} \int f^2(r) V(r) d\mathbf{r}. \quad (10)$$

To calculate the average of the kinetic energy, it may be suitable to represent the matrix element of the kinetic energy operator in the form

$$\begin{aligned} \langle \Phi | \sum_{k=1}^N \frac{\hat{\mathbf{p}}_k^2}{2m} | \Phi \rangle &\equiv -\frac{\hbar^2}{2m} \sum_{k=1}^N \langle \Phi | \Delta_k | \Phi \rangle = \\ &= -\frac{\hbar^2}{2m} N \langle \Phi | \Delta_1 | \Phi \rangle = \frac{\hbar^2}{2m} N \langle \nabla_1 \Phi | \nabla_1 \Phi \rangle, \end{aligned} \quad (11)$$

where the gradient

$$\begin{aligned} \nabla_1 \Phi &= \prod_{n>k=2}^N f(r_{nk}) \exp\left(-\frac{1}{R^2} \sum_{s=2}^N r_s^2\right) \times \\ &\times \nabla_1 \left(\exp\left(-\frac{r_1^2}{R^2}\right) \prod_{j=2}^N f(r_{1j}) \right) \end{aligned} \quad (12)$$

can be found explicitly. Substituting the result of differentiation into (11) and integrating over $d\mathbf{r}_1$ and (for convenience) over $d\mathbf{r}_2$, in the limit $R \gg r_0$, one has

$$\langle \Phi | \sum_{k=1}^N \frac{\hat{\mathbf{p}}_k^2}{2m} | \Phi \rangle \xrightarrow{R \gg r_0}$$

$$\xrightarrow{R \gg r_0} \frac{3}{8} \frac{\hbar^2}{2m} \pi^3 R^4 C_{3,N} \cdot N \left(1 + \frac{N-1}{2\sqrt{2}} \frac{r_0}{R} + o\left(\frac{r_0}{R}\right)\right). \quad (13)$$

Dividing this expression by the normalization matrix element in the form (8) and adding (10), one ultimately has the variational estimation for the energy of the ground state in the limit $R \gg r_0$:

$$\frac{E}{N} \leq \frac{3}{2} \frac{\hbar^2}{mR^2} + \frac{N-1}{2R^3} \left(\frac{3}{2} \frac{\hbar^2}{m} \frac{r_0}{\sqrt{2}} + \frac{1}{\pi^{\frac{3}{2}}} \int f^2(r) V(r) d\mathbf{r} \right). \quad (14)$$

It is obvious from the obtained expression that, under the condition

$$A \equiv \frac{3}{2} \frac{\hbar^2}{m} \frac{r_0}{\sqrt{2}} + \frac{1}{\pi^{\frac{3}{2}}} \int f^2(r) V(r) d\mathbf{r} < 0, \quad (15)$$

one has the spatial collapse in the system of interacting Bose particles at $N \rightarrow \infty$. Really, in this case, the energy (per one particle!) goes to minus infinity, as it is clear from (14). At the same time, at a fixed parameter R , the system of particles has a finite volume $\sim R^3$, but the number of particles tends to infinity resulting in an infinite density of particles. Note that the obtained sufficient condition (15) generalizes our more simple criterion [8] $\int V(r) d\mathbf{r} < 0$, which follows from (15) at $r_0 \rightarrow 0$.

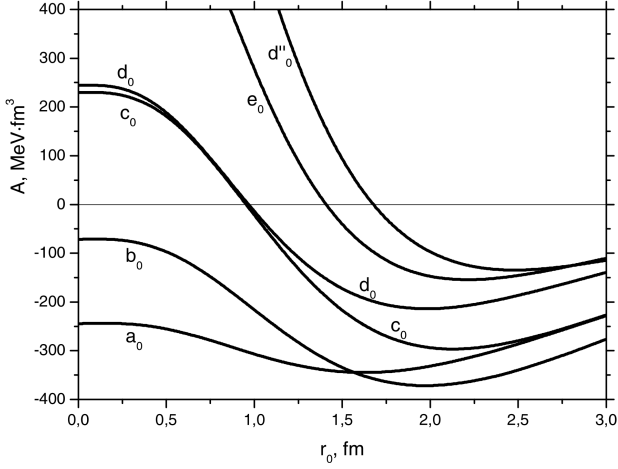
In the next section, we use the obtained criterion of the spatial collapse in a Bose system to analyze whether a hypothetical system of α -particles can form a nuclear matter. There, the negative answer will be obtained, since an infinite system of α -particles with typical α - α interaction potentials (without regard for the Coulomb repulsion) obeys condition (15) of spatial collapse.

3. The Spatial Collapse of a Hypothetical Infinite System of α -Particles without the Coulomb Repulsion

Now, consider typical interaction potentials between α -particles in the form [9]

$$V(r) = V_r \exp(-\mu_r^2 r^2) - V_a \exp(-\mu_a^2 r^2) \quad (16)$$

with a few sets of parameters for attraction and repulsion given in the Table. It is worth to note that the α - α interaction [9] depends on the the angular momentum. But, in the states with $l \neq 0$, the interaction potentials are more attractive than in the state



Dependences of the left hand side A of the criterion (15) on the radius of correlations r_0 for different Ali-Bodmer α - α potentials [9] (see also the Table)

with $l = 0$. We simplify the problem and assume the potentials to be the same in all the states. In this case, the interaction generally becomes a little bit less attractive. If we shall demonstrate that, even with such a simplified interaction, the hypothetical α -particle matter does collapse, then this effect should be observed for original versions of α - α interaction [9] even more so.

Criterion (15) contains the radius of correlations r_0 which can be chosen in such a way that to make the contribution of repulsion of the α - α potential into the integral in (15) sufficiently small. As a result, the integral of the potential with the correlation factor squared becomes negative. Due to the rather small contribution of the first term originating from the kinetic energy matrix element, the correlation factors at a definite r_0 turn the left-hand side of (15) to negative

Parameters of some α - α potentials from [9]. The first column contains a notation [9] of the corresponding version of a potential

Potential	μ_α (fm $^{-1}$)	V_α (MeV)	μ_r (fm $^{-1}$)	V_r (MeV)
a_0	0.35	30	0.65	125
b_0	0.42	150	0.55	325
c_0	0.45	190	0.6	500
d_0	0.475	130	0.7	500
d_0''	0.475	130	0.8	1300
e_0	0.5	150	0.8	1050

values at almost any short-range repulsion (except the singular repulsion, in particular like “hard core”).

In Figure, we show how the left-hand side A of expression (15) depends on r_0 . It is seen that, for potentials a_0 and b_0 from the Table, the criterion (15) is fulfilled already at $r_0 = 0$. But, for the rest Ali-Bodmer potentials, the inequality (15) is valid at nonzero r_0 . We do not depict the dependences at $r_0 \rightarrow \infty$ for two reasons. First, the term originating from the kinetic energy in (15) is proportional to r_0 , while the potential energy, due to the increase of the radius of correlations, vanishes at $r_0 \rightarrow \infty$, and thus A becomes positive at a definite r_0 (depending on the version of potential). Second, obtaining expression (15), we assumed that $r_0 \ll R$, where R is fixed (although may be rather large as compared to the radius of forces). Therefore, it is not correct to tend the r_0 to infinity in expression (15).

As it is clear, all the other possible versions of “realistic” local α - α potentials should also give $A < 0$ at a definite radius of correlations r_0 . Thus the hypothetical system of α -particles is unstable with respect to the spatial collapse.

4. Conclusions

To summarize, we note that a variational estimation with the account for Jastrow pair correlation factors enabled us to demonstrate that an infinite system of α -particles can not form a hypothetical nuclear matter due to its instability with respect to the spatial collapse (if the Hamiltonian contains the Ali-Bodmer or similar α - α short-range interaction potentials and does not contain the Coulomb long-range repulsion). We stress once more that, as shown in [8], the sign and value of the scattering length of the pair potential can not be used as a criterion of the effect of spatial collapse in Bose systems of interacting particles.

The obtained criterion of the spatial collapse of an infinite system of Bose particles can be used for studying a possible spatial collapse in other physical systems including imperfect Bose gases.

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ЧИ МОЖЕ ЯДЕРНА
МАТЕРІЯ СКЛАДАТИСЯ З α -ЧАСТИНОК?

На основі варіаційного принципу з використанням пробних функцій з парними кореляційними факторами Ястрова отримано достатню умову просторового колапсу в нескінченній системі взаємодіючих бозе-частинок. Показано, що гіпотетична нескінченна система α -частинок нестабільна по відношенню до просторового колапсу у припущенні потенціалів взаємодії Алі–Бодмера між такими бозе-частинками. В результаті стає зрозумілим, чому гіпотетичну ядерну матерію природно розглядати принаймні з використанням нуклонних ступенів вільності.

Ключові слова: бозе-система з α -частинок, просторовий колапс, ядерна матерія.