1. Introduction

Anomalous diffusion is a process in which a mean square displacement, \( \langle x^2(t) \rangle \), satisfies the relation [1]

\[
\langle x^2(t) \rangle \sim t^\alpha,
\]

where \( \alpha \) is an anomalous diffusion parameter. For \( 0 < \alpha < 1 \), we are dealing with subdiffusion; for \( \alpha = 1 \), we have a situation of normal diffusion, and, for \( \alpha > 1 \), we encounter superdiffusion. To describe the anomalous diffusion, one usually uses differential equations with derivatives of fractional order [1, 2].

In our paper, we briefly discuss the subdiffusion in a system with a thin membrane, which has been intensively studied experimentally [3–6] (see also [7] and references therein). Subdiffusion is a process which is qualitatively different from normal diffusion. Due to the complex structure of the various forms of media, a subdiffusive particle waits an anomalously long time to make its step, and the mean value of this time is infinite. The exact formula for the mean square displacement in one-dimensional system reads [1]

\[
\langle x^2(t) \rangle = \frac{2D_\alpha}{\Gamma(1 + \alpha)} t^\alpha,
\]

for \( 0 < \alpha < 1 \); \( D_\alpha \) is the subdiffusion coefficient measured in the units \( m^2/s^\alpha \), and \( \Gamma(z) \) denotes the Gamma function. In what follows, we assume that the anomalous diffusion parameters are independent of the time and the space variable.

There is a problem with distinguishing between subdiffusion and normal diffusion with a very small diffusion coefficient. We present a method which allows us to extract the subdiffusion parameters from experimental data. The method utilizes a membrane system where a thin membrane separates a homogeneous solute from a pure solvent at the initial time moment. Since the concentration profiles of the diffusing substance have been experimentally investigated, the subdiffusion parameters \( \alpha \) and \( D_\alpha \) are treated as the fitting parameters of theoretical functions in relation to the experimental data. However, the concentration profiles are usually measured with relatively large errors, so the obtained values of the parameters are not very accurate. To estimate the parameters with greater precision, the time evolution of the thickness of the so-called near membrane layers is used according to this method [3, 4].

2. Anomalous Diffusion Equations

Let us consider a one-dimensional random walk of a particle (see Fig. 1). By \( \omega(t) \), we denote the probability density of waiting time between two successive jumps, and \( \lambda(x) \) is the probability density of the jump length.

Fig. 1. Schematic representation of a one-dimensional random walk.

Within the Continuous Time Random Walk approach, the type of transport process is determined by the functions \( \omega(t) \) and \( \lambda(x) \) (strictly speaking, by the moments of these functions). There are three main possibilities [1].

1. When \( \omega(t) \sim e^{-t/\tau} \) and \( \lambda(x) \sim e^{-x^2/2\sigma} \), which ensures that \( \langle \omega(t) \rangle \) and \( \langle \lambda^2(x) \rangle \) are finite, the particle transport constitutes normal diffusion which is described by the equation

\[
\frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2},
\]
where $D$ is the normal diffusion coefficient. The parameters $\tau$ and $\sigma$ satisfy the relation $D = \tau / \sigma$.

3. Subdiffusion in a Membrane System

Let us consider the system where a thin membrane separates a homogeneous solution from a pure solvent (see Fig. 2) (we should mention that such a system has often been used in experimental studies [3–6,9]). Assuming that the length of the system is very long as compared with the regions in which the concentration profiles change noticeably with respect to their initial values, we can treat the system as unbounded.

Thus, we choose the initial condition as

$$C(x, 0) = \begin{cases} C_0, & x < 0, \\ 0, & x > 0. \end{cases}$$

We denote the concentration and the flux in the region $x < 0$ as $C_1(x, t)$ and $J_1(x, t)$ and in the region $x > 0$ as $C_2(x, t)$ and $J_2(x, t)$, respectively. The subdiffusion parameters in both parts of the system are $\alpha$ and $D_\alpha$.

To solve the subdiffusion equation, we need four boundary conditions, two of them should be set at the thin membrane. The first boundary condition requires the continuity of the flux at the membrane

$$J_1(0^-, t) = J_2(0^+, t) (\equiv J(0, t)),$$

where the subdiffusive flux $J(x, t)$ is given by generalized Fick’s law

$$J(x, t) = -D_\alpha \frac{\partial^{1-\alpha} C(x, t)}{\partial x^{\alpha}}.$$

There is no obvious predetermined choice for the second boundary condition. We will assume that this boundary condition is provided by a linear combination of concentrations and a flux at the membrane

$$b_1 C_1(0^-, t) + b_2 C_2(0^+, t) + b_3 J(0, t) = 0,$$

where $b_1b_3 \leq 0$ and $b_2b_3 \geq 0$. Equations (5) and (6) provide the commonly used membrane boundary conditions at a thin membrane [10, 11]. The physical interpretation has the quotients of the coefficients $b_1, b_2$ and $b_3$.

When $b_3 \neq 0$, Eq. (6) yields

$$J = \lambda_1 C(0^-, t) - \lambda_2 C(0^+, t),$$

with $\lambda_1 = -b_1/b_3$ and $\lambda_2 = -b_2/b_3$ being membrane permeability coefficients. For the symmetric membrane ($\lambda_1 = \lambda_2 \equiv \lambda$), we obtain the often used boundary conditions where the flux flowing through the membrane is proportional to the difference of concentrations at the membrane. When $b_3 = 0$, for $b_1 \neq 0$ and $b_2 \neq 0$, we obtain

$$C(0^-, t) = \sigma C(0^+, t),$$

with $\sigma = -b_2/b_1$ interpreted as the membrane reflection coefficient; this membrane boundary condition can be obtained from the generalized method of images [10]. We note that the cases $b_3 = b_1 = 0$, $b_2 \neq 0$ and $b_3 = b_2 = 0$, $b_1 \neq 0$ correspond to the fully absorbing membrane, whereas $b_1 = 0$, $b_2 \neq 0$, $b_3 \neq 0$ and $b_2 = 0$, $b_1 \neq 0$,
Since Eq. (6) is a generalized form of the linear boundary conditions at the membrane, we will use $b_1$, $b_2$, and $b_3$ as the coefficients for the description of the permeability of the membrane. Therefore, the final solutions contain these parameters.

Additional boundary conditions are chosen as

$$C_1(-\infty, t) = C_0, \quad C_2(\infty, t) = 0. \quad (7)$$

The solutions to Eq. (2) with boundary conditions (4), (6), (7) and the initial one (3) are as follows (the procedure we used to solve the equation is based on the Laplace transformation and is presented in [12]).

For $b_3 \neq 0$,

$$C_1(x, t) = C_0 \left[ 1 - \frac{2\eta}{\alpha(b_1 - b_2)} \times \right. \left. \times H_1^{0,0} \left( \left( \frac{-x}{\sqrt{D_\alpha t^\alpha}} \right)^{2/\alpha} \right) \right],$$

$$C_2(x, t) = \frac{2C_0 b_1}{\alpha(b_1 - b_2)} H_1^{0,1} \left( \left( \frac{x}{\sqrt{D_\alpha t^\alpha}} \right)^{2/\alpha} \right),$$

where $H$ is the Fox function

$$H_1^{10} \left( \frac{a^{1/\beta}}{t} \right) \left| \begin{array}{c} 1 \\ 1/\beta \\ 1 \\ 1/\beta \\ 1/\beta \\ \end{array} \right\} = \beta \left( \frac{(1 + \nu)/\beta}{t^{1+\nu}} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(-k/\beta - \nu)} \left( \frac{a}{t^\beta} \right)^k \right).$$

For $b_3 \neq 0$,

$$C_1(x, t) =$$

$$= C_0 \left[ 1 - \frac{2\eta}{\alpha D_\alpha} \sum_{n=0}^{\infty} \gamma^n \left( \frac{\sqrt{D_\alpha}}{-x} \right)^{2/n} \times \times H_1^{0,0} \left( \left( \frac{x}{\sqrt{D_\alpha t^\alpha}} \right)^{2/\alpha} \right) \right],$$

$$C_2(x, t) =$$

$$= \frac{2C_0 \eta}{\alpha D_\alpha} \sum_{n=0}^{\infty} \gamma^n \left( \frac{\sqrt{D_\alpha}}{-x} \right)^{2/n} \times \times H_1^{0,0} \left( \left( \frac{-x}{\sqrt{D_\alpha t^\alpha}} \right)^{2/\alpha} \right),$$

where

$$\eta = -\frac{b_1}{b_3}, \quad \gamma = \frac{b_1 - b_2}{b_3 \sqrt{D_\alpha}}.$$
4. Conclusions

The method presented above was used to identify the kind of glucose and sucrose transport in gel (1.5% water solution of agarose), which proved to constitute the subdiffusion with $\alpha = 0.90 \pm 0.01$ and to determine the values of subdiffusion coefficients: $D_{0.90} = (9.8 \pm 1.0) \times 10^{-4} \text{ mm}^2/s^{0.90}$ for glucose and $D_{0.90} = (6.3 \pm 0.9) \times 10^{-4} \text{ mm}^2/s^{0.90}$ for sucrose [3, 4].

We note that the power functions, similar to (1), characterize the subdiffusion not only in a membrane system but, among others, the subdiffusive transport in a system with chemical reactions. In this system, the power function describes the time evolution of the reaction front [9, 13].

This research was presented at the 5th International Conference “Physics of Liquid Matter: Modern Problems” (Kyiv, Ukraine, 2010), http://www.plmmp.org.ua.


Received 21.01.11