EXACT MATHISSON–PAPAPETROU EQUATIONS IN THE SCHWARZSCHILD METRIC WITH INTEGRALS OF MOTION

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A new representation for exact Mathisson–Papapetrou equations under the Mathisson–Pirani condition in the Schwarzschild gravitational field, which does not contain third-order derivatives with respect to spinning-particle coordinates, has been obtained. For this purpose, the integrals of energy and angular momentum of a spinning particle, as well as a differential relation following from the Mathisson–Papapetrou equations for an arbitrary metric, are used. The form of the equations obtained is adapted for their computer integration and further researches dealing with the influence of the spin-curvature interaction on particle's behavior in the gravitational field imposing no restrictions on the particle's velocity and spin orientation.

1. Introduction

In the framework of general theory of relativity, the equations that describe the motion of a macroscopic particle with spin (the spinning particle) in a gravitational field look like [1, 2]

$$\frac{D}{ds}\left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{ds}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda_{\pi\rho\sigma}, \quad (1)$$

$$\frac{DS^{\mu\nu}}{ds} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{ds} = 0, \quad (2)$$

where $u^\lambda \equiv dx^\lambda/ds$ is the 4-velocity of the particle, $S^{\mu\nu}$ the tensor of its spin, m its mass, D/ds the covariant derivative, and $R^\lambda_{\pi\rho\sigma}$ the Riemann curvature tensor of the space-time. Hereafter, the system of units is used, in which numerical values of the gravitational constant and the velocity of light in vacuum are equal to 1; the Greek indices run from 1 to 4, and the Latin ones from 1 to 3. System (1), (2) is called the Mathisson–Papapetrou (MP) equations, although the other authors derived them using various methods after publications [1, 2]. In particular, Dixon [3] paid a lot of attention to those equations and their generalization, which took the particle's quadrupole moment into account; consequently, they are often referred to as the Mathisson–Papapetrou–Dixon equations.

The system of equations (1), (2) is incomplete. Consequently, fixed initial values for particle's coordinates, velocity, and spin do not allow one to obtain a unique solution. It is so, because Eqs. (1) and (2) do not specify a point, with respect to which the angular momentum of the particle (body) is calculated and the motion of which corresponds to the motion of the body, as a whole, in the space. Certainly, if the matter concerns the angular momentum that characterizes the particle spinning around its own axis, it is natural to select the particle's center of mass as such a representative point. However, it is known that, in the relativistic mechanics, the point, where the center of mass of a body revolving around its axis is located, depends on the reference frame [4]. Therefore, the relation

$$S^{\lambda\nu}u_\nu = 0, \quad (3)$$

the nonrelativistic analog of which identifies the position of the body's center of mass, gives a set of centers of mass rather than a unique point under relativistic conditions [5]. As a consequence, Eqs. (1)–(3), along with the solutions that describe straight-line motions, have solutions in the form of helical (in particular, circular) curves in the Minkowski space [6, 7]. However, superfluous solutions of this type are absent, if, instead of condition (3), the relation

$$S^{\lambda\nu}P_\nu = 0 \tag{4}$$

is used, where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{ds} \tag{5}$$

is the 4-momentum of the particle [3, 8]. (Relation (3) in the context of MP equations is traditionally referred to as the Pirani condition, bearing in mind work [9], although it was Mathisson [6] who used it for the first time.)

If the velocity of a particle with spin with respect to the gravitational field source is not very close to the light speed, and the field itself is not very strong, the solutions of MP equations obeying conditions (3) and (4) differ very little from each other [10, 11]. (The matter concerns those solutions under condition (3) that do not belong to the helical type, but describe the motion of the own center of mass [4]. The own reference frame is a frame, in which the rotation axis is motionless.) Under the same conditions, the deviation of world lines of a particle with spin from geodesic ones is very small.

If the velocity of a particle with spin is close enough to the light speed, the picture changes. First, there appear the situations where the solutions of Eqs. (1) and (2) under conditions (3) or (4) remain close to each other, but differ very much from the corresponding solutions of the equations for geodesic trajectories. These are world lines and trajectories of an ultrarelativistic particle with spin that starts its motion in a narrow spatial region about $r = 1.5r_g$ in the Schwarzschild field (r_g is the Schwarzschild radius of the event horizon) [12] or near $r = r_{\rm ph}^{(-)}$ in the Kerr field ($r_{\rm ph}^{(-)}$ is the radius of a geodesic orbit of a photon in the counter-rotation case) [13]. Second, for other initial conditions, the trajectories of an ultrarelativistic particle at their description by Eqs. (1) and (2) and either of relations (3) and (4) can differ substantially both from each other and from geodesic trajectories. In this case, the influence of terms nonlinear in the spin is important [14]. Then, a question arises: Is either of conditions (3) and (4) adequate, in general, for the description of ultrarelativistic motions of a particle with spin in a gravitational field or not? By the way, concerning a similar issue in the context of studying the world lines of particles with spin and zero mass on the basis of equations of the MP type, a conclusion was drawn that condition (3) may probably be a unique physically reasonable condition in this case [15]. Does it imply that not only a massless particle with spin, which moves at the speed of light, but also a particle with a nonzero mass and a velocity very close to the light one can be correctly described just provided condition (3) rather than (4)? A well-reasoned answer to that question can be obtained only after having analyzed those solutions of the exact equations (1) and (2) obtained under condition (3), which characterize the motions of the own center of mass (i.e. which are not oscillatory in the sense of works [6, 7]), and having compared those solutions with the corresponding solutions of the same equations obtained under condition (4).

The analysis of the solutions of Eqs. (1) and (2) at various supplementary relations for specific gravitational fields was initiated in work [6] for the Schwarzschild metric and continued in many publications for this and other metrics [5, 14, 17–27]. In [28, 29], the substantial attention was paid to elucidate physical effects governed by the interaction between a spin and a gravitational field in the framework of MP equations. In the last 10–12 years, there has been the growing interest in the study of physical consequences of MP equations [30–46]. In particular, the authors of work [31] continued the research of possibilities for the chaos to manifest itself in dynamic systems owing to the spin, which was started in work [25]. Works [38–40, 42, 45] were devoted to spin effects in circular orbits, including the spin precession phenomenon [39, 42] and the "clock effect" [33–35]. Attention in those works was mainly concentrated on the analysis of physical effects that follow from the equations concerned in rather weak gravitational fields and at relatively low (in comparison with the speed of light) velocities of a particle with spin with respect to the gravitational field source.

The rigorous MP equations (1) and (2) with condition (3) contain the third-order derivatives of particle coordinates. Those equations compose a complicated system of ordinary, essentially nonlinear differential equations even in rather a simple central-symmetric case of the Schwarzschild metric. For their solutions to be obtained, the numerical computer-assisted integration is required. This stage is preceded by a choice of a convenient representation for those equations by carrying out the corresponding analytical transformations. The availability of such a representation in the cases of the Schwarzschild and Kerr metrics is provided by the fact that, in those cases, the MP equations have the integrals of motion–the energy and the momentum [17, 20, 47].

As a certain analogy, let us point to the known approach in studying the solutions of equations for geodetic lines in the Schwarzschild and Kerr metrics, when their integrals of energy and momentum are effectively used for the analysis and the classification of possible types of spinless particle orbits in those metrics [48, 49], because a reduction of the whole problem to the consideration of differential equations with an order less by one than the order of the initial equations of geodetic lines becomes possible. In the case of Eqs. (1)–(3), a similar procedure was applied in work [14] to the motion of a spinning particle in the equatorial plane $\theta = \pi/2$ of the Schwarzschild metrics, when the spin is orthogonal to the plane. For such motions, the spin part of the MP equations (2) can be integrated irrespective of Eq. (1), the procedure being impossible for general motions, when the spin changes its orientation, and the orbit is not planar anymore. That is, to achieve the strategic target – to analyze and to classify possible orbits for a particle with spin in the Schwarzschild and Kerr fields – the results obtained in work [14] are not enough. Therefore, the prime task is their generalization to the cases of nonplanar motions with an arbitrary oriented spin both in the Schwarzschild and Kerr fields.

The fulfillment of this task is connected with complicated analytical calculations, especially for the Kerr metric. It is known that already the calculations for the equations of geodesic lines in the Kerr metric become much more involved in comparison with those in the Schwarzschild metric case. For the MP equations, this difference is even more substantial. Therefore, the subsequent researches should be divided into two stages. First of all, it is worth obtaining a presentation for the MP equations (1)–(3), which would be free of the third-order derivatives with respect to particle coordinates for arbitrary motions in the Schwarzschild field. Afterwards, we should repeat a similar procedure for the Kerr field. The equations obtained at the first stage, should be used to test the corresponding equations for the Kerr field, because, if the internal angular momentum of the field source is put equal to zero, the Kerr set of equations should transform into the Schwarzschild one. A monitoring over the performance of such a passage to the limit at every stage of analytical calculations would enhance their reliability.

This work aimed at executing the first indicated stage of researches, namely, to generalize the presentation of the exact MP equations (1) and (2) under condition (3) in terms of the integrals of energy and momentum, which was obtained in work [14] for planar motions of a spinning particle in the Schwarzschild field, to arbitrary particle motions in the same field and to illustrate some solutions of those equations obtained by computer-assisted integration. A generalization of those results to the case of the Kerr gravitational field will be considered in a separate work.

It is known that the MP equations (1) and (2) have two Killing vectors $\boldsymbol{\xi}$ in the Kerr field. In the Boyer–Lindquist coordinates, those vectors are written down as $\boldsymbol{\xi}^t = \partial/\partial t$ and $\boldsymbol{\xi}^\varphi = \partial/\partial\varphi$ [31]. They correspond to two integrals of motion: the energy E and the z-components of momentum, J_z [17, 20, 31, 47],

$$E = P_t - \frac{1}{2} g_{t\mu,\nu} S^{\mu\nu}, \quad J_z = -P_\varphi + \frac{1}{2} g_{\varphi\mu,\nu} S^{\mu\nu}.$$

In the case of Schwarzschild field, these expressions, taking relation (5) into account, look like

$$E = m u_4 + g_{44} u_\mu \frac{DS^{4\mu}}{ds} + \frac{1}{2} g_{44,\mu} S^{\mu 4}, \tag{6}$$

$$J_z = -m u_3 - g_{33} u_\mu \frac{DS^{3\mu}}{ds} - \frac{1}{2} g_{33,\mu} S^{\mu 3} \tag{7}$$

(hereafter, we use the metric in the standard Schwarzschild coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and $x^4 = t$ with the signature $(-,-,-,+)$).

In contrast to the axisymmetric Kerr field, the centrally symmetric Schwarzschild field is characterized by two additional Killing vectors, which provide the conservation for the components J_x and J_y of the momentum, the explicit expressions for which are given, e.g., in work [25].

As was indicated above, the presentation of the MP equations (1)–(3) in terms of integrals of motion in the Schwarzschild field is the first stage in the researches aimed at the solution of a more complicated problem, namely, in obtaining a similar presentation for arbitrary motions of a spinning particle in the Kerr field. In this connection, the attention should be paid to two circumstances: 1) the presentation of MP equations in the Kerr field in terms of integrals of motion includes only E and J_z, because J_x and J_y are not constants in this field; 2) as a result of computation difficulties in the case of Kerr field, it is important to check the performance of the passage to the Schwarzschild limit at every calculation stage. Therefore, it is important to obtain such a presentation of the MP equations (1)–(3) for the Schwarzschild field, which would include, as it was in the case of Kerr field, only the integrals E and J_z, without engaging J_x and J_y.

We emphasize that a generalization of the procedure aimed at excluding the third-order derivatives with respect to particle coordinates from the MP equations under condition (3), which was applied in work [14] for planar motions in the Schwarzschild field, to arbitrary motions in the Kerr field is nontrivial *per se*. It turns out that, for such a generalization to be made, not only the integrals of motion are to be used, but a differential relation, which follows from Eqs. (1)–(3) and becomes a trivial identity in the case of planar motions, should also be taken into account. This relation is important, because, unlike the MP equations, it does not contain any third-order derivative with respect to coordinates. Therefore, we now derive this relation. Moreover, we will do it in the general form for an arbitrary metric. This means that this relation will have an independent importance, and it can be used for the analysis of solutions of Eqs. (1)–(3) in the fields different from the Schwarzschild or Kerr one.

The interest in the researches of physical consequences that follow from the MP equations is enhanced by the fact that those equations are, to some extent, a classical approximation of the generally covariant Dirac equation [50] and can be used to study regularities in the behavior of high-energy spin particles in the content of cosmic rays in the vicinity of compact astrophysical objects (black holes, quasars, and so forth).

2. A Relation That Follows from Eqs. (1)–(3)

Consider the first three equations of subsystem (1) with the superscript $\lambda = 1, 2, 3$. Let us multiply them by S^{23}, S^{31}, and S^{12}, respectively, and sum the products. We obtain

$$m\varepsilon_{ikl}S^{ik}\frac{Du^l}{ds}+\frac{D}{ds}\left(\varepsilon_{ikl}S^{ik}u_\mu\frac{DS^{l\mu}}{ds}\right)-$$

$$-\varepsilon_{ikl}\frac{DS^{ik}}{ds}u_\mu\frac{DS^{l\mu}}{ds}=-\frac{1}{2}\varepsilon_{ikl}S^{ik}u^\pi S^{\rho\sigma}R^l{}_{\pi\rho\sigma}. \quad (8)$$

Notice that, under conditions (3), the quantity m in Eq. (1) is a constant. We intend to demonstrate that each of two groups of terms of the form

$$\varepsilon_{ikl}S^{ik}u_\mu\frac{DS^{l\mu}}{ds}, \quad (9)$$

and

$$\varepsilon_{ikl}\frac{DS^{ik}}{ds}u_\mu\frac{DS^{l\mu}}{ds} \quad (10)$$

in expression (8) identically equals to zero under condition (3). Really, a consequence of the covariant differentiation of condition (3) is the relation

$$u_\mu\frac{DS^{\lambda\mu}}{ds}+S^{\lambda\mu}\frac{Du_\mu}{ds}=0. \quad (11)$$

In view of Eq. (11), expression (9) can be written down as follows:

$$\varepsilon_{ikl}S^{ik}S^{l4}\frac{Du_4}{ds}. \quad (12)$$

Here, we took into account that the multipliers at Du_i/ds, where $i = 1, 2, 3$, are identically equal to zero, because the spin tensor is antisymmetric. To evaluate the parenthesized expression on the right-hand side of Eq. (12), we take into account that condition (3) immediately results in the relations

$$S^{i4}=\frac{1}{u_4}S^{li}u_l. \quad (13)$$

Therefore, we have

$$\varepsilon_{ikl}S^{ik}S^{l4}=0, \quad (14)$$

which is also a consequence of the spin tensor antisymmetry. Hence, expression (9) is identically equal to zero.

To evaluate expression (10), we use relations that immediately follow from a subsystem of Eqs. (2),

$$\varepsilon_{ikl}\frac{DS^{kl}}{ds}=\varepsilon_{ikl}u_\mu\frac{DS^{k\mu}}{ds}u^l-\varepsilon_{ikl}u_\mu\frac{DS^{l\mu}}{ds}u^k. \quad (15)$$

Substituting Eq. (15) into Eq. (10), we obtain

$$\varepsilon_{ikl}\left(u_\nu\frac{DS^{i\nu}}{ds}u^k-u_\nu\frac{DS^{k\nu}}{ds}u^i\right)u_\mu\frac{DS^{l\mu}}{ds}=$$

$$=\varepsilon_{ikl}u^iu_\mu u_\nu\left(\frac{DS^{k\mu}}{ds}\frac{DS^{l\nu}}{ds}-\frac{DS^{l\mu}}{ds}\frac{DS^{k\nu}}{ds}\right)=0, \quad (16)$$

because every term contains a convolution of the symmetric tensor $u_\mu u_\nu$ with an expression in the parentheses, which is antisymmetric with respect to superscripts μ and ν.

Hence, as a result of Eqs. (12), (14), and (16), relation (8) reads

$$m\varepsilon_{ikl}S^{ik}\frac{Du^l}{ds}=-\frac{1}{2}\varepsilon_{ikl}S^{ik}u^\pi S^{\rho\sigma}R^l{}_{\pi\rho\sigma}. \quad (17)$$

Instead of the spatial components of the spin tensor, it is convenient to use a three-component quantity S_i, for which [14, 40]

$$S_i=\frac{1}{2}\sqrt{-g}\varepsilon_{ikl}S^{kl},\quad S^{kl}=\frac{1}{\sqrt{-g}}\varepsilon^{kli}S_i. \quad (18)$$

Here, g is the determinant of the metric tensor, and ε^{klm} is the Levi–Civita symbol. Note that, when considering the MP equations, the spin 4-vector s_λ is widely used in the literature along with the spin tensor $S^{\lambda\mu}$. There exists a simple relation between S_i and s_λ: $S_i = u_i s_4 - u_4 s_i$ [14]. Whence, it follows that, in the own coordinate system of the particle, in which $u^i = 0$ and $s_4 = 0$, the components S_i are proportional to the spatial components of the spin 4-vector s_i. Below, we use the quantity S_i, because it provides a more compact form for the relevant equations. Taking Eq. (18) into account, relation (17) acquires the form

$$mS_i \frac{Du^i}{ds} = -\frac{1}{2} u^\pi S^{\rho\sigma} S_j R^j_{\pi\rho\sigma}. \tag{19}$$

We emphasize that, unlike the equations of subsystem (1), each of which, under condition (3), includes the third-order derivatives with respect to coordinates, relation (19) includes derivatives not higher than the second-order ones.

3. System of Exact MP Equations of the Second Order for Coordinates in the Schwarzschild Metric

First of all, we specify relation (19) in the case of Schwarzschild metric. In the standard coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and $x^4 = t$, the following components of the metric and Riemann tensors are different from zero [48]:

$$g_{11} = -\left(1 - \frac{2M}{r}\right)^{-1}, \quad g_{22} = -r^2,$$

$$g_{33} = -r^2 \sin^2\theta, \quad g_{44} = 1 - \frac{2M}{r}, \tag{20}$$

$$R_{1212} = \frac{M}{r-2M}, \quad R_{1313} = \frac{M}{r-2M}\sin^2\theta,$$

$$R_{2323} = -2Mr\sin^2\theta, \quad R_{4141} = \frac{2M}{r^3},$$

$$R_{4242} = \frac{M(2M-r)}{r^2}, \quad R_{4343} = \frac{M(2M-r)}{r^2}\sin^2\theta. \tag{21}$$

Taking Eqs. (18), (20), and (21) into account, relation (19) can be written down as follows:

$$m\sqrt{-g} S_i \frac{Du^i}{ds} = 6S_1(R^1_{313} S_2 u^3 - R^1_{212} S_3 u^2). \tag{22}$$

Note that, in the partial case of particle motions in the equatorial plane $\theta = \pi/2$ of the Schwarzschild field, for which the spin components S_1 and S_3 vanish, the left- and right-hand sides of relation (22) are identically equal to zero. However, for general motions, relation (22) is not trivial, and, as we show below, it plays an important role in obtaining a closed system of differential equations of the second order in particle coordinates, which contain the integrals of motion E and J_z as parameters.

The further transformations of relation (22) are carried out with the use of the integrals of motion (6) and (7). In particular, from a relation for E, which follows from relation (6) after taking Eq. (20) into account, we express the derivative Du^3/ds in terms of Du^1/ds and Du^2/ds, as

$$\frac{Du^3}{ds} = (S_2u_1 - S_1u_2)^{-1} g^{33} \Big[g_{11}(S_2u_3 - S_3u_2)\frac{Du^1}{ds} + $$

$$+ g_{22}(S_3u_1 - S_1u_3)\frac{Du^2}{ds} + mu_4u^4\sqrt{-g} - $$

$$- Eu^4\sqrt{-g} + \frac{g_{44,1}}{2g_{44}}(S_2u_3 - S_3u_2)\Big]. \tag{23}$$

Similarly, from an expression for J_z, which follows from expression (7), we also express Du^4/ds in terms of Du^1/ds and Du^2/ds:

$$\frac{Du^4}{ds} = (S_2u_1 - S_1u_2)^{-1} u^4 \Big(g_{11}S_2\frac{Du^1}{ds} - g_{22}S_1\frac{Du^2}{ds} - $$

$$- mu^3\sqrt{-g} - Jg^{33}\sqrt{-g} + \frac{g_{33,1}}{2g_{33}}S_2 - \frac{g_{33,2}}{2g_{33}}S_1\Big). \tag{24}$$

Substituting expressions (23) and (24) into Eq. (22), we obtain

$$[S_1g_{33}(S_2u_1 - S_1u_2) + S_3g_{11}(S_2u_3 - S_3u_2)]\frac{Du^1}{ds} + $$

$$+ [S_2g_{33}(S_2u_1 - S_1u_2) + S_3g_{22}(S_3u_1 - S_1u_3)]\frac{Du^2}{ds} + $$

$$+ S_3[mu_4u^4\sqrt{-g} - Eu^4\sqrt{-g} + \frac{g_{44,1}}{2g_{44}}(S_2u_3 - S_3u_2)] + $$

$$+ \frac{6S_1g_{33}}{m\sqrt{-g}}(S_2u_1 - S_1u_2)(R^1_{212}S_3u^2 - R^1_{313}S_2u^3) = 0. \tag{25}$$

We recall that, in the general case of an arbitrary metric, the MP equations have the integral of motion

$$u_\mu u^\mu = \text{const} \tag{26}$$

(choosing a corresponding time scale, this equality is usually written down in the form $u_\mu u^\mu = 1$). By the way, this integral of motion is also characteristic of the equation for geodesic lines. The covariant differentiation of relation (26) gives rise to the expression

$$u_1 \frac{Du^1}{ds} + u_2 \frac{Du^2}{ds} + u_3 \frac{Du^3}{ds} + u_4 \frac{Du^4}{ds} = 0. \tag{27}$$

Substituting expressions (23) and (24) for the covariant derivatives Du^3/ds and Du^4/ds into expression (27), we obtain the following relation, which includes only the derivatives Du^1/ds and Du^2/ds:

$$g_{11}(S_2 - u_2 S_i u^i) \frac{Du^1}{ds} - g_{22}(S_1 - u_1 S_i u^i) \frac{Du^2}{ds} -$$

$$-Eu^3 u^4 \sqrt{-g} + \frac{g_{44,1}}{2g_44} u^3 (S_2 u_3 - S_3 u_2) +$$

$$u_4 u^4 g^{33}(-J\sqrt{-g} + \frac{1}{2} g_{33,1} S_2 - \frac{1}{2} g_{33,2} S_1) = 0. \tag{28}$$

Substituting Eqs. (23) and (24) directly into four equations of the MP subsystem (1) and taking Eqs. (25) and (28) into account, it is easy to verify that all those equations are satisfied identically.

Hence, expressions (25) and (28) compose a system of two linear algebraic equations for Du^1/ds and Du^2/ds. Leaving aside the fact that the corresponding calculations are a little cumbersome, it is easy to find expressions for Du^1/ds and Du^2/ds in terms of the particle coordinates, velocity components, and spin, and, therefore, expressions for ordinary derivatives du^1/ds and du^2/ds after taking the explicit form for Christoffel symbols in metrics (20) into account. Ultimately, from Eqs. (23) and (24), using the expressions obtained for du^1/ds and du^2/ds, we obtain the corresponding expressions for du^3/ds and du^4/ds. Thus, we arrive at a system of four differential equations

$$\frac{du^\lambda}{ds} = f_\lambda(x^\mu, u^\nu, S_i), \quad \lambda = 1, 2, 3, 4, \tag{29}$$

where f_λ are the corresponding functions. For the sake of compactness, the explicit forms for these functions are not presented here. It is the more so, because, instead of a system of four equations of the second order with respect to the coordinates x^λ, and so is system (29), it is convenient (in particular, in computer-assisted calculations) to consider a system of eight equations of the first order for eight unknown functions y_i related to the particle coordinates and the velocity in such a way that they should correspond to dimensionless quantities, namely, by definition,

$$y_1 = \frac{r}{M}, \quad y_2 = \theta, \quad y_3 = \varphi, \quad y_4 = \frac{t}{M},$$

$$y_5 = u^1, \quad y_6 = Mu^2, \quad y_7 = Mu^3, \quad y_8 = u^4. \tag{30}$$

In addition, we introduce the additional dimensionless quantities connected with the components S_i,

$$y_9 = \frac{S_1}{mM}, \quad y_{10} = \frac{S_2}{mM^2}, \quad y_{11} = \frac{S_3}{mM^2}, \tag{31}$$

as well as with the own particle time, s, the absolute value of spin, S, and the integrals of energy, E, and momentum, J_z,

$$x = \frac{s}{M}, \quad \varepsilon = \frac{S}{Mm}, \quad \mu = \frac{ME}{S}, \quad \nu = \frac{J_z}{S}. \tag{32}$$

(The MP equations are known to have the integral of motion $S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}$ in the general case of arbitrary metric.) Then, the eight mentioned differential equations for eight functions y_i, in accordance with definitions (29), read

$$\dot{y}_1 = y_5, \quad \dot{y}_2 = y_6, \quad \dot{y}_3 = y_7, \quad \dot{y}_4 = y_8,$$

$$\dot{y}_5 = A_1, \quad \dot{y}_6 = A_2, \quad \dot{y}_7 = A_3, \quad \dot{y}_8 = A_4, \tag{33}$$

where the dot means the ordinary differentiation with respect to x, and

$$A_1 = A + \left(y_1 y_6^2 + y_1 y_7^2 \sin^2 y_2 - \frac{y_8^2}{y_1^2} \right) q + \frac{y_5^2}{y_1^2} q^{-1},$$

$$A_2 = B - \frac{2}{y_1} y_5 y_6 + y_7^2 \sin y_2 \cos y_2,$$

$$A_3 = \left(-q^{-1} y_5 y_{10} + y_1^2 y_6 y_9 \right)^{-1} \times [q^{-1}(-y_7 y_{10} \sin^2 y_2 +$$

$$+ y_6 y_{11}) A + \left(-q^{-1} y_5 y_{11} + y_1^2 y_7 y_9 \sin^2 y_2 \right) B -$$

$$- y_8^2 q \sin y_2 + \mu y_8 \sin y_2 + \frac{1}{y_1^2} q^{-1} (y_7 y_{10} \sin^2 y_2 -$$

$$-y_6y_{11})]\sin^{-2}y_2-\frac{2}{y_1}y_5y_7-2y_6y_7\cot y_2,$$

$$A_4=y_8\left(-q^{-1}y_5y_{10}+y_1^2y_6y_9\right)^{-1}\times$$

$$\times[-q^{-1}y_{10}A+y_1^2y_9B-y_1^2y_7\sin y_2+$$

$$+\frac{\nu}{\sin y_2}+\frac{y_{10}}{y_1}-y_9\cot y_2]-\frac{2}{y_1^2}y_5y_8q^{-1},$$

$$A=\varepsilon^{-2}q^{-1}\left(q^{-1}y_5y_{10}-y_1^2y_6y_9\right)^{-1}\times$$

$$\times\Big[[y_7(-y_5y_{10}^2y_1^{-2}+qy_9(y_6y_{10}+y_7y_{11}))+$$

$$+\frac{y_{11}}{y_1^2\sin^2y_2}(qy_9+y_5(y_5y_9+y_6y_{10}))]\times$$

$$\times[y_1^2q^{-1}\frac{\mu}{y_8}\sin y_2+q^{-2}\frac{1}{y_8^2}(y_7y_{10}\sin^2y_2-y_6y_{11})]+$$

$$+\frac{1}{y_1^2}[y_5y_{10}^2+\frac{y_5y_{11}^2}{\sin^2y_2}-qy_1^2y_9(y_6y_{10}+y_7y_{11})]\times$$

$$\times\left(\frac{\nu}{\sin y_2}+\frac{y_{10}}{y_1}-y_9\cot y_2\right)\Big]+$$

$$+\frac{1}{\varepsilon^2\sin y_2}[y_9+q^{-1}y_5(y_5y_9+y_6y_{10}+y_7y_{11})]\times$$

$$\times\Big[\frac{6y_9}{y_1^3y_8^2}q^{-1}(-y_7y_{10}\sin^2y_2+y_6y_{11})-$$

$$-y_{11}(q^{-1}y_5y_{10}-y_1^2y_6y_9)^{-1}\Big],$$

$$B=y_1^{-2}\left[y_9+q^{-1}y_5(y_5y_9+y_6y_{10}+y_7y_{11})\right]^{-1}\times$$

$$\times\big[q^{-1}(y_{10}+y_1^2y_6(y_5y_9+y_6y_{10}+y_7y_{11}))A+$$

$$+\mu y_1^2y_7y_8\sin y_2+q^{-1}y_7(y_7y_{10}\sin^2y_2-y_6y_{11})-$$

$$-qy_8^2(\nu\sin^{-1}y_2+y_{10}y_1^{-1}-y_9\cot y_2)\big]. \quad (34)$$

In expressions (34), the notation $q=1-\frac{2}{y_1}$ was used.

For three functions y_9, y_{10}, and y_{11}, there are three equations which follow from the spin part of the MP equations [12],

$$\dot y_9=A_5,\quad \dot y_{10}=A_6,\quad \dot y_{11}=A_7, \quad (35)$$

where

$$A_5=\frac{2y_5y_9}{y_1^2}q^{-1}+\frac{y_6y_{10}+y_7y_{11}}{y_1}-(y_5y_9+y_6y_{10}+$$

$$+y_7y_{11})\times\left[\left(A_1-\frac{y_5A_4}{y_8}-\frac{y_5^2}{y_1^2}q^{-1}\right)q^{-1}-y_1y_6^2-\right.$$

$$\left.-y_1y_7^2\sin^2y_2+\frac{y_8^2}{y_1^2}\right]+\frac{y_9A_4}{y_8},$$

$$A_6=-y_1y_6y_9\left(1-\frac{3}{y_1}\right)+\frac{y_5y_{10}}{y_1^2}\left(2q^{-1}+y_1\right)+$$

$$+y_7y_{11}\cot y_2+\frac{y_{10}A_4}{y_8}-(y_5y_9+y_6y_{10}+y_7y_{11})\times$$

$$\times\Big[y_1^2A_2-\frac{y_1^2y_6}{y_8}A_4+2y_5y_6\left(y_1-q^{-1}\right)-$$

$$-y_1^2y_7^2\cos y_2\sin y_2\Big],$$

$$A_7=\frac{y_5y_{11}}{y_1^2}(2q^{-1}+y_1)-y_1y_7y_9\left(1-\frac{3}{y_1}\right)\sin^2y_2+$$

$$+y_6y_{11}\cot y_2-y_7y_{10}\cos y_2\sin y_2+\frac{y_{11}A_4}{y_8}-$$

$$-(y_5y_9+y_6y_{10}+y_7y_{11})\times\Big[y_1^2A_3\sin^2y_2-\frac{y_1^2y_7}{y_8}A_4\times$$

$$\times\sin^2y_2+2y_5y_7(y_1-q^{-1})\sin^2y_2+$$

$$+2y_1^2y_6y_7\cos y_2\sin y_2\Big]. \quad (36)$$

Hence, Eqs. (33) and (35) comprise a complete system of exact MP equations, which describes the most general motions of a particle with spin in a gravitational field, without any restrictions imposed on the particle velocity and the spin orientation. The expressions for A_3, A_4, A, and B contain the quantities μ and ν which are proportional, according to definition (32), to the integrals E and J_z, respectively. This means that, while solving the Cauchy problem for Eqs. (33) and (35), fixed initial values for all functions y_i $(i = 1, ..., 11)$ do not provide a unique solution for those equations, as it has to be for exact MP equations under condition (3). Varying the values of parameters μ and ν at fixed initial values for y_i enables us to describe the motions of various centers of mass of a particle with spin. Among the set of (μ, ν)-pairs, there exists a unique one which describes the motions of the own center of mass. How to find this (μ, ν)-pair is a separate problem. One of the approaches aimed at solving it was formulated in work [51], where the method for distinguishing the non-oscillatory solutions of the exact MP equations under condition (3) was proposed.

The analysis of general motions, which are described by Eqs. (33) and (35) demands that detailed computer-assisted calculations should be carried out, so that this issue will be considered in a separate work. Below, as an example, we consider the solutions of those equations, for which the initial value of radial coordinate equals $3M = 1.5r_g$, and the initial values of velocity and spin vector components are close to the corresponding values of the known exact solution of the MP equations under condition (3) in the Schwarzschild field which describes an ultrarelativistic circular orbit with $r = 1.5r_g$. To be more specific, similarly to what was done in works [12, 13], we will analyze the variation character of this orbit, if the initial spin is not orthogonal to the plane $\theta = \pi/2$, and the radial spin component is a little different from zero. The difference is that, in works [12, 13], the MP equations were considered only in the linear-in-spin approximation, whereas now we use the exact MP equations (33) and (35).

4. Example

The ultrarelativistic circular orbit with $r = 3M$ is a common solution for both exact MP equations and their linear-in-spin approximation [12, 14]. Within a short time interval reckoned from the moment of a particle descend from this orbit owing to a certain variation in its velocity or spin orientation, the influence of linear terms considerably prevails over the contribution by the terms nonlinear in spin [12, 13]. While studying the particle behavior within longer time intervals, one may not confine the consideration to linear terms. It is necessary to solve the system of exact MP equations (33) and (35). Since the matter concerns the extension of already known solutions over longer time intervals, the issue on the selection of such values for the parameters μ and ν, which would correspond to the solution for the own center of mass, can be resolved in a simple way; namely, being the integrals of motion, they remain invariable since the motion starts. Therefore, while numerically integrating Eqs. (33) and (35), the values for parameters μ and ν which enter the expressions for A_3, A_4, A, and B (see Eq. (34)), were selected equal to those which were determined at the initial moment in the linear-in-spin approximation.

Note that a direct consequence of expression $S^2 = \frac{1}{2}S_{\mu\nu}S^{\mu\nu}$ considering Eqs. (3) and (18), as well as notations (30)–(32), is the relation

$$\varepsilon^2 = q^{-2}y_8^{-2}[(y_5y_9 + y_6y_{10} + y_7y_{11})^2+$$

$$+qy_9^2 + y_{10}^2y_1^{-2} + y_{11}y_1^{-2}\sin^{-2}y_2]. \quad (37)$$

In this case, as a result of the condition imposed on a test particle [18], the absolute value of parameter ε defined by formulas (32) has to be much smaller than 1.

As was done in work [12], relation (37) is taken into account, when the initial slope of the spin with respect to the equatorial plane $\theta = \pi/2$ changes, provided that the spin absolute value is fixed. Moreover, since the expression on the right-hand side of Eq. (37) is an integral of motion, we use it to check the accuracy of computer calculations.

The circular orbit with $r = 3M$ is associated with the following initial values for quantities y_i [12]:

$$y_1(0) = 3, \quad y_2(0) = \frac{\pi}{2}, \quad y_3(0) = 0, \quad y_4(0) = 0,$$

$$y_5(0) = 0, \quad y_6(0) = 0, \quad y_7(0) = u, \quad y_8(0) = v,$$

$$y_9(0) = 0, \quad y_{10}(0) = w, \quad y_{11}(0) = 0, \quad (38)$$

where

$$u = -\frac{1}{3\sqrt{2}}\sqrt{-1 + \frac{\sqrt{\varepsilon^2 + 12}}{\varepsilon}},$$

$$v = \sqrt{3}\sqrt{1 + 9u^2}, \quad w = \varepsilon v. \quad (39)$$

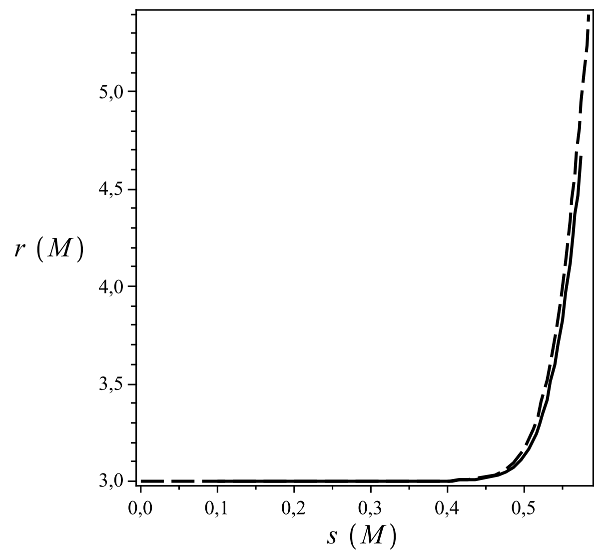

Fig. 1. Dependences $r(s)$ in terms of M-units for the exact MP equations (solid curve) and their linear-in-spin approximations (dashed curve)

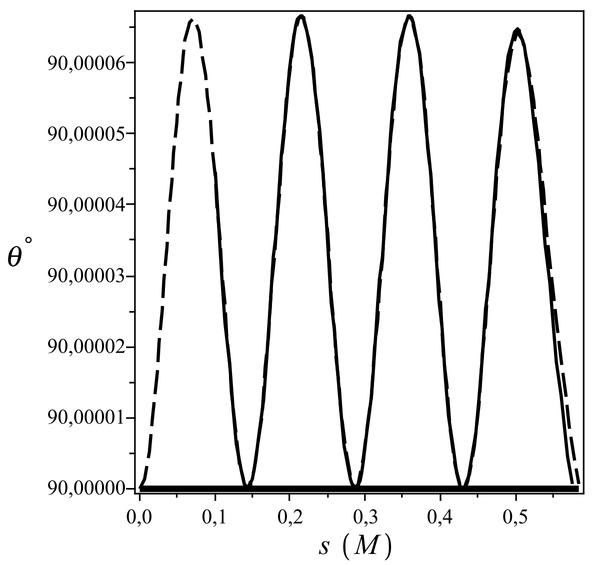

Fig. 2. Dependences $\theta(s)$ for the exact MP equations (solid curve) and their linear-in-spin approximations (dashed curve)

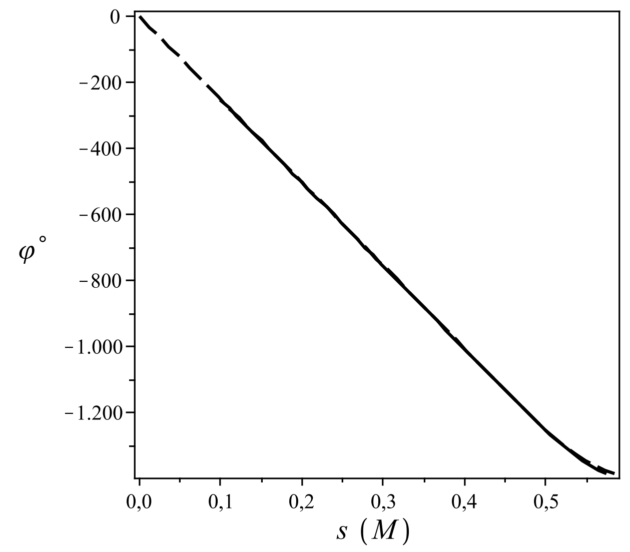

Fig. 3. Dependences $\varphi(s)$ for the exact MP equations (solid curve) and their linear-in-spin approximations (dashed curve)

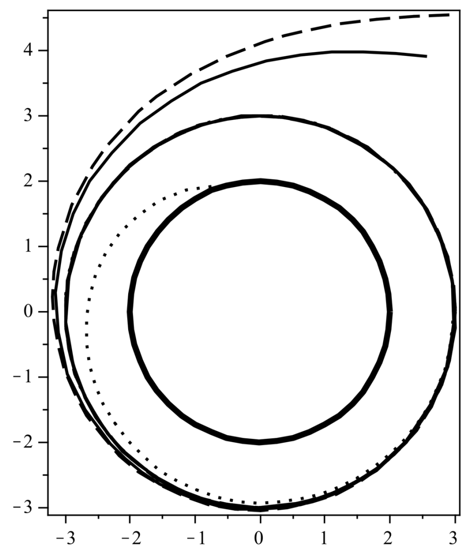

Fig. 4. Geodesic path (dotted curve) and the paths of a spinning particle for the exact MP equations (solid curve) and their linear-in-spin approximations (dashed curve). The circle with radius 2 corresponds to the horizon surface

Figures 1 to 4 illustrate the solutions of Eqs. (33)–(35) with the initial conditions for the majority of y_i-quantities equal to those given in formulas (38), but the quantities y_5, y_9, and y_{10}. The latter, according to notations (30), correspond to the radial velocity and two components of spin 3-vector. We gave y_5 a small initial value of 3.9×10^{-7}, and y_9 and y_{10} such values, at which the initial slope angle of the spin with respect to the equatorial plane was equal to 1°. We also put $\varepsilon = 10^{-4}$. These specific values were selected as, in a sense, typical and illustrative ones.

According to Fig. 1, the dependences $r(s)$ within the time interval from 0 to $0.5M$ are almost identical for the exact MP equations and their linear-in-spin approximations. The difference between the corresponding curves is small until the displacement along the radial coordinate reaches a value of about $5M$. According to Figs. 2 and 3, a similar behavior is inherent to the dependences $\theta(s)$ and $\varphi(s)$. Figure 4 demonstrates that more illustrative is the difference between particle paths. The dotted curve in the figure represents a path for a spinless particle that moves with the same initial values of coordinates and velocities as they are for a spinning particle. In this case, the spinless particle falls on the horizon surface at the time moment $s \approx 0.228M$.

5. Conclusions

Hence, the application of the integrals of motion for the exact MP equations (1) and (2) under condition (3) in the Schwarzschild field, i.e. the energy E and the momentum J_z, together with the use of relation (22), enabled us to obtain a complete system of equations (33) and (35) for the description of the most general motions of a spinning particle in this field. Since relation (22) is a particular case of general relation (19), this approach can be applied to other gravitational fields, in particular, the Kerr field, for which the corresponding integrals of motion are also available.

Figures 1 to 4 show that both the approximate (linear in spin) and exact MP equations allow, in the Schwarzschild field, the effects of a considerable counteraction of the interaction between a spin and the space-time curvature to the ordinary attraction inherent to the influence of the gravitation on a spinless particle.

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ТОЧНІ РІВНЯННЯ
МАТІСОНА–ПАПАПЕТРУ
ДЛЯ МЕТРИКИ ШВАРЦШИЛЬДА
З ВИКОРИСТАННЯМ ІНТЕГРАЛІВ РУХУ

Р.М. Пляцко, О.Б. Стефанишин

Р е з ю м е

Отримано нове представлення точних рівнянь Матісона–Папапетру за умови Матісона–Пірані у гравітаційному полі Шварцшильда, що не містить третіх похідних від координат частинки зі спіном. Для цього використано інтеграли енергії та моменту кількості руху частинки, а також одне диференціальне співвідношення, яке випливає з рівнянь Матісона–Папапетру для довільної метрики. Запис рівнянь адаптовано для їх комп'ютерного інтегрування з метою подальших досліджень впливу взаємодії спіну частинки з кривизною простору-часу на її поведінку в гравітаційному полі без обмежень на швидкість і орієнтацію спіну.