DIFFUSION IN A FROZEN RANDOM VELOCITY FIELD

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PACS 05.20.-Dd, 02.70.-c

Particle diffusion in a frozen isotropic 2D random velocity field is studied by simulation, and the results are compared with the prediction of a simple model. The model accounts for the effects of particle trapping and infinite correlation time.

1. Introduction

We are interested in the statistical description of particles whose behavior is determined by a incompressible homogeneous isotropic velocity field with statistical properties of a Gaussian process. A similar problem was studied by Kraichnan in the 1970s [1]. In addition, we assume that the field is two-dimensional and frozen, i.e. it does not depend on time. The latter two features cause the particle trapping, infinite correlation time, and growing deviation of the distribution function of particles in the coordinate space from the Gaussian form. Dispersion manifests a subdidusive regime, and the diffusion coefficient tends asymptotically to zero. It is follows from the notion that particle orbits along contour lines of the stream function are closed and the particle spread would be terminated somewhere. Such subdiffusion is strictly different from the normal diffusion. In his paper, Kraichnan has noted that "the static diffusion is a simple and severe test for statistical theories" [1]. He made conclusion that the direct interaction approximation (DIA) fails to represent trapping effects which are inherited from frozen fields.

Later, the diffusion in a random 2D field was considered as a percolation problem [2]. The theory gives an asymptotic value of diffusion coefficient. For the field varying in time, a scaling of the diffusion coefficient on the Kubo number was found. Then this scaling was supported by the simulation in some interval of Kubo numbers [3]. However, for a frozen field with infinitely large Kubo number, the diffusion coefficient is zero.

In the last decade, the problem was studied by the decorrelated trajectory (DCT) method [4–7]. This in part phenomenological method assumes the partition of the ensemble of particles into a set of subensembles according to the initial values of velocity and potential in a starting point of the trajectory. Different groups of particles are governed by different statistics, and the whole picture is found as a superposition of subensembles. In paper [4], this method was applied to the calculation of a temporal evolution of the diffusion coefficient in a frozen random velocity field. Qualitatively, it gives expected curves for the diffusion coefficient which comes through the maximum and tends asymptotically to zero. However, no direct comparison with the result obtained by a numerical simulation was performed there.

For the prediction of the results of a simulation, we proposed the analytical model that explicitly corresponds to the microscopic equations used in the simulation. In the isotropic homogeneous field, an average particle displacement is zero due to symmetry, and the first non-vanishing moment is a dispersion. The analytical model is formulated in terms of a dispersion. The model accounts for the effects of particle trapping and infinite correlation time.

2. Model for Numerical Simulation

We consider a motion of test particles in a frozen isotropic random velocity field. An incompressible 2D velocity field is given through a potential (stream function) $\varphi(\mathbf{r})$ as

$$v_x = -\frac{\partial}{\partial y}\varphi(\mathbf{r}), \qquad v_y = \frac{\partial}{\partial x}\varphi(\mathbf{r}), \tag{1}$$

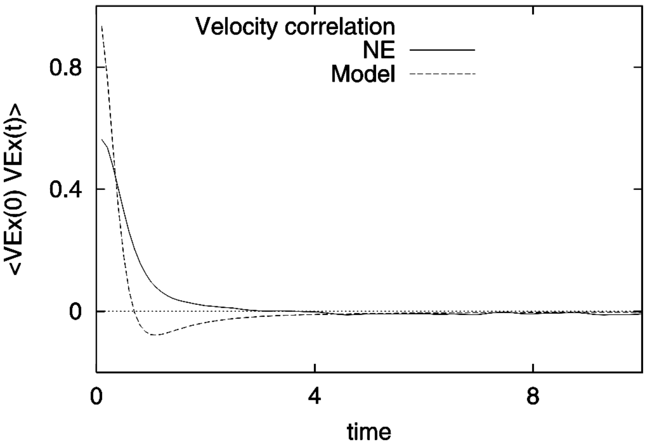

Fig. 1. Velocity correlation function. NE – simulation, Model – solution of Eqs. (4)–(7) in the analytical model

where $\mathbf{r} = (x, y)$ in Cartesian coordinates or (r, θ) in polar coordinates. The potential of the isotropic field is taken as a superposition of $5N_k \times N_\phi$ waves

$$\varphi(x,t) = \sum_{i=1}^{5N_k} \sum_{j=1}^{N_\phi} \varphi_i \cos\left(k_i r \cos\left(\phi_j - \theta\right) + \alpha_i + \alpha_j\right). \quad (2)$$

The field intensity is distributed between the partial harmonics according to the Gaussian,

$$\varphi_i{}^2 = \frac{8}{\sqrt{\pi}} \frac{v_0^2}{\Delta k^2} \frac{1}{N_k N_\phi} \exp\left(-\frac{k_i^2}{\Delta k^2}\right), \quad (3)$$

where $k_i = i \cdot \Delta k / N_k$ $(i = 1, 5N_k)$, $\phi_j = j \cdot 2\pi/N_\phi$ $(j = 1, N_\phi)$, v_0 is the velocity amplitude, and Δk is the spectrum width. Randomness of the field is generated by random phases α_i and α_j. The spectrum includes small wave numbers, and the effect of long correlation would be noticeable.

Equations (1), (2) describe, in particular, the $\mathbf{E} \times \mathbf{B}$ drift motion of test particles in plasma with constant magnetic field directed along the z-axis and with electrostatic random field.

The equations of motion (1) were solved numerically, and the particle dispersion and the Lagrangian correlation function were found by averaging over realizations. In the absence of a field, particles do not move. Thus, the zero approximation does not exist, and the particle motion is purely diffusive. This distinguishes the problem of particle diffusion in a velocity random field from the diffusion in a Langmuir type random field [8, 9]. For the former, more runs are needed to obtain a smooth curve.

3. Analytical Model

The equations of our analytical model are the following. First, we use the Taylor relation between the time-dependent diffusion coefficient $D(t)$ and the Lagrangian correlation function $V_L(t)$,

$$D(t) = \int_0^t V_L(t) dt. \quad (4)$$

The particle dispersion $\langle r^2 \rangle \equiv \langle x^2 \rangle + \langle y^2 \rangle$ is related to the diffusion coefficient by

$$\langle r^2 \rangle = 4 \int_0^t D(t) dt. \quad (5)$$

The Eulerian velocity correlation function for the field (1) is of the form

$$V_E(r^2) \equiv \langle V_x(\mathbf{0}) V_x(\mathbf{r}) \rangle = \\ = v_0^2 \exp(-\alpha)\{(1 - 2\alpha) I_0(\alpha) + 2\alpha I_1(\alpha)\}, \quad (6)$$

where $\alpha = (1/8)\Delta k^2 r^2$. The Eulerian velocity correlation function depends on the coordinate through r^2.

We close the system of equations, by assuming

$$V_L(t) = V_E(\langle r^2(t) \rangle). \quad (7)$$

Note that the system of equations (4)–(7) can be reduced to an equation for an effective particle driven by a force

$$\frac{d^2 \langle r^2 \rangle}{dt^2} = \text{const} \cdot V_E(\langle r^2 \rangle). \quad (8)$$

In the next section, the solution of Eqs. (4)–(7) is compared with the result of a numerical simulation.

4. Comparison with Simulation

In Fig. 1, the Lagrangian velocity correlation function obtained as a solution of Eqs. (4)–(7) is shown in comparison with simulation data. It reproduces the simulation curve; however, the approximation given by Eq. (7) at the initial stage is not too much accurate.

More interesting is the plot of the Lagrangian velocity correlation function for a longer time (Fig. 2), where the model solution reproduces the guiding line of fluctuations obtained in the simulation. It is of importance that the Lagrangian velocity correlation function found as a solution of the model equation tends to zero, being negative. The negative tail of the correlation function reflects the effect of particle trapping and infinitely long correlation.

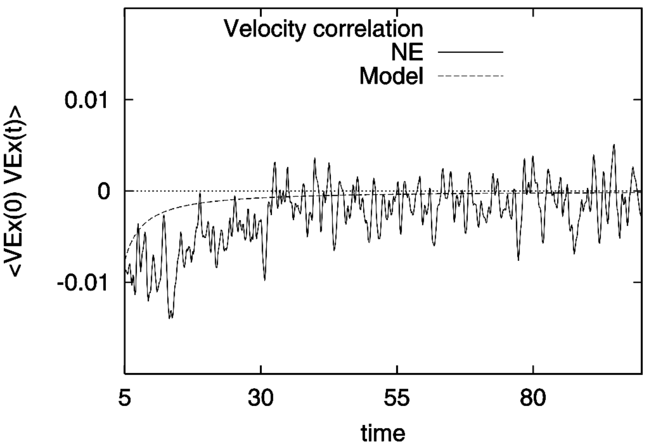

Fig. 2. Velocity correlation function for large time

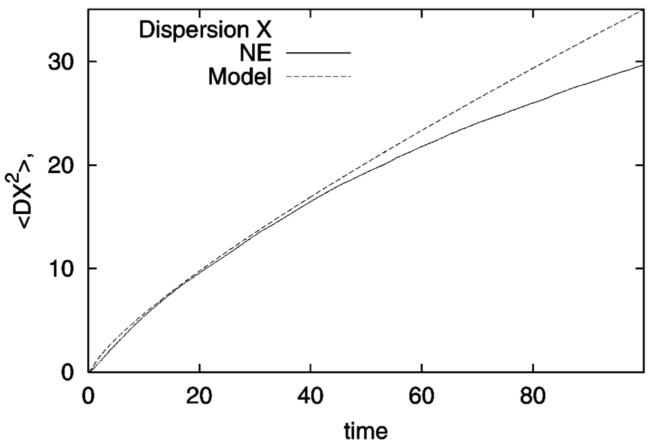

Fig. 4. Dispersion of coordinate

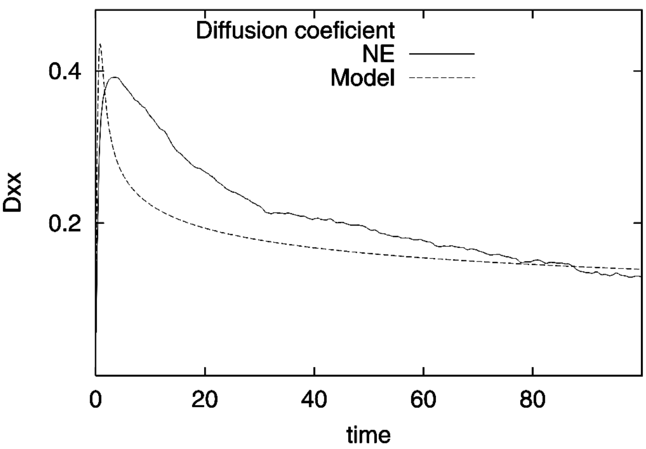

Fig. 3. Running diffusion coefficient

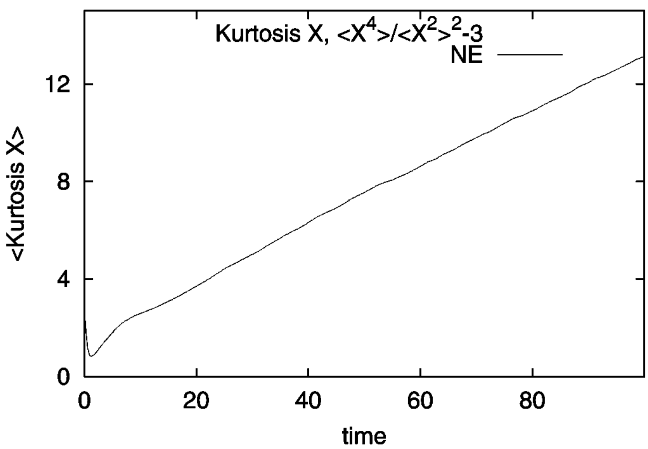

Fig. 5. Kurtosis obtained in simulation

Plots for the diffusion coefficient are given in Fig. 3, and those for the dispersion (second moment of the distribution function) in Fig. 4. The solution of the analytical model shows an agreement with the results of the simulation.

In addition, the evolution of the kurtosis (fourth irreducible moment of the distribution function), which was obtained in the simulation, is shown in Fig. 5. For the Gaussian distribution, the kurtosis would be strictly zero. Its growth indicates a deviation of the distribution function from the Gaussian form.

5. Discussion and Conclusions

The solution of the model equations (4)–(7) recovers the temporal dependence of the Lagrangian velocity correlation function, diffusion coefficient, and coordinate dispersion (Figs. 1–4). Not too much accurate in some details, it gives, nevertheless, in general a reasonable quantitative agreement. It should be noted that no fitting parameters were used, and the only assumption was the transition from the Eulerian to Lagrangian velocity correlation function given by Eq. (7). This closure of the model equations is strictly different from the Corrsin approximation.

The growth of the kurtoisis indicates that a deviation of the distribution function from the Gaussian form increases with time. Nevertheless, the model that was formulated in terms of the dispersion shows its ability to reproduce such important effects as particle trapping and infinitely long correlation. The trapping effect terminates the particle spread and leads to a subdiffusive form of dispersion. This implies that the diffusion coefficient should tends asymptotically to zero. In turn, this means that the Lagrangian velocity correlation function should have a negative tail. Such negative tail that was observed in the simulation is recovered by the solution of the model equation.

The work is partly supported by DFFD-RFFI project F40.2/108.

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Received 23.09.11

ДИФУЗІЯ В ЗАМОРОЖЕНОМУ ВИПАДКОВОМУ ПОЛІ ШВИДКОСТЕЙ

В.І. Засенко, А.Г. Загородній, О.М. Черняк

Р е з ю м е

Дифузію частинок у замороженому ізотропному двовимірному випадковому полі швидкостей, що спостерігалась у числовому моделюванні, порівняно з передбаченнями простої моделі. Модель враховує захоплення частинок і нескінченно довгі часові кореляції.