ON $T(n,4)$ TORUS KNOTS AND CHEBYSHEV POLYNOMIALS

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The Alexander polynomials $\Delta_{n,3}(t)$ and $\Delta_{n,4}(t)$ are presented as a sum of the Alexander polynomials $\Delta_{k,2}(t)$. These polynomials are also expressed in the form of a sum of Chebyshev polynomials of the second kind. These expansions allow one to introduce the "coordinates" in corresponding bases, which are proposed to be the numerical invariants characterizing links and knots.

1. Introduction

The knot history began in 1867, when Lord Kelvin (W.H. Thomson) suggested to describe the atoms as knotted vortex tubes in the ether [1]. In 1975, L.D. Faddeev proposed that knot-like solitons could exist in a modified sigma model extended to the three-dimensional space [2]. But the second advent of the knots into physics [3–6] began only in 1997 after the article by Faddeev and Niemi [7]. They made first attempts of a numerical construction of the solitons with minimal energy in the form of knots. Increasing the computer power demonstrated that a number of linked and knotted configurations do exist in the Faddeev model, which are the solutions characterized by local or global energy minima.

Powerful tools in the knot theory are polynomial invariants. In this paper, we concentrate on studying the Alexander polynomial invariants for torus knots $T(n,l)$, l=$2,3,4$, and on their connection with the Chebyshev polynomials [8] of the second kind. The Alexander polynomials $\Delta_{n,3}(t)$ and $\Delta_{n,4}(t)$ are presented firstly as a sum of the Alexander polynomials $\Delta_{k,2}(t)$ and, secondly, through a sum of Chebyshev polynomials of the second kind. These expansions allow us to introduce the "coordinates" in corresponding bases, which are proposed, in the case of finding similar expansions for all links and knots, to be numerical invariants characterizing links and knots. The obtained expansions can be also used for the investigation of baryon masses [9].

2. Alexander Polynomial Invariants for Torus Knots

The skein relation for the Alexander polynomials $\Delta(t)$ for knots and links,

$$\Delta_+(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_O(t) + \Delta_-(t) \tag{1}$$

together with the condition for unknots,

$$\Delta_{\mathrm{unknot}} = 1 \tag{2}$$

gives an axiomatic definition of the Alexander polynomials. Using (1) and (2), one can find the Alexander polynomial for any knot or link with the help of the so-called surgery operations of "elimination" and "switching". We denote torus knots as $T(n,l)$, where n and l are coprime positive integers, and the corresponding Alexander polynomial as $\Delta_{n,l}(t)$.

As known [10, 11], the Alexander polynomial for the torus knot $T(n,l)$ is given by the formula

$$\Delta_{n,l}(t) = \frac{(t^{\frac{nl}{2}} - t^{-\frac{nl}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(t^{\frac{l}{2}} - t^{-\frac{l}{2}})} \tag{3}$$

and has the form of the Laurent polynomial with the highest positive degree

$$m = \frac{1}{2}(n-1)(l-1)\,. \tag{4}$$

For $l = 2$, (3) gives

$$\Delta_{n,2}(t) = \frac{t^{\frac{n}{2}} + t^{-\frac{n}{2}}}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}}\,. \tag{5}$$

Some first examples of $\Delta_{n,2}(t)$ from (5) are

$$\Delta_{1,2}(t) = 1, \quad \Delta_{3,2}(t) = t - 1 + t^{-1},$$
$$\Delta_{5,2}(t) = t^2 - t + 1 - t^{-1} + t^{-2}\,. \tag{6}$$

Below, we will find an expansion of the Alexander polynomials for two sets of torus knots, $T(n,3)$ and $T(n,4)$, through the Alexander polynomials for torus knots $T(k,2)$.

If $l = 3$, (3) looks as

$$\Delta_{n,3}(t) = \frac{t^n + 1 + t^{-n}}{t^1 + 1 + t^{-1}}\,. \qquad (7)$$

Some examples of (7) are

$$\Delta_{1,3}(t) = 1, \quad \Delta_{2,3}(t) = t - 1 + t^{-1},$$
$$\Delta_{4,3}(t) = t^3 - t^2 + 1 - t^{-2} + t^{-3},$$
$$\Delta_{5,3}(t) = t^4 - t^3 + t - 1 + t^{-1} - t^{-3} + t^{-4}\,. \qquad (8)$$

For $l = 4$, relation (3) yields

$$\Delta_{n,4}(t) = \frac{t^{\frac{3n}{2}} + t^{\frac{n}{2}} + t^{-\frac{n}{2}} + t^{-\frac{3n}{2}}}{t^{\frac{3}{2}} + t^{\frac{1}{2}} + t^{-\frac{1}{2}} + t^{-\frac{3}{2}}}\,. \qquad (9)$$

The first examples of (9) are as follows:

$$\Delta_{1,4}(t) = 1, \quad \Delta_{3,4}(t) = t^3 - t^2 + 1 - t^{-2} + t^{-3},$$
$$\Delta_{5,4}(t) = t^6 - t^5 + t^2 - 1 + t^{-2} + t^{-5} + t^{-6}\,. \qquad (10)$$

3. On the Expression of $\Delta_{n,3}(t)$ through $\Delta_{k,2}(t)$

In this section, we find the formula expressing any Alexander polynomial $\Delta_{n,3}(t)$ as an algebraic sum of the Alexander polynomials $\Delta_{k,2}(t)$ [12].

Proposition 1. *The Alexander polynomial for any torus knot $T(n,3)$ is presented as an algebraic sum of the Alexander polynomials for torus knots $T(k,2)$ with the help of the formula*

$$\Delta_{n,3}(t) = \sum_{j=0}^{d_1} \Delta_{2n-1-6j,\,2}(t) - \sum_{i=0}^{d_2} \Delta_{2n-5-6i,\,2}(t)\big)\,, \quad (11)$$

where $d_1 = \left[\frac{2n-1}{6}\right]$ *and* $d_2 = \left[\frac{2n-5}{6}\right]$.

The proof of (11) follows from the relation

$$\Delta_{n,3}(t) - \Delta_{n-3,3}(t) = \Delta_{2n-1,2}(t) - \Delta_{2n-5,2}(t)\,, \qquad (12)$$

which can be proved, in turn, with the help of (5) and (7).

Equation (11) can be rewritten as

$$\begin{aligned}&\Delta_{n,3}(t) \equiv \Delta_{3d+r,3}(t) = \\ &= \sum_{j=0}^{d} \Delta_{2n-1-6j,2}(t) - \sum_{i=0}^{d-1} \Delta_{2n-5-6i,2}(t)\,,\end{aligned} \qquad (13)$$

where $r = 1, 2$, and $d = 0, 1, 2, 3 \ldots$. For n given, d is the integer part of $\frac{n}{3}$: $d = \left[\frac{n}{3}\right]$.

Using (11), we give a few examples expressing the Alexander polynomials $\Delta_{n,3}(t)$ as a sum of the Alexander polynomials $\Delta_{k,2}(t)$:

$$\Delta_{1,3}(t) = \Delta_{1,2}(t)\,, \quad \Delta_{2,3}(t) = \Delta_{3,2}(t)\,,$$
$$\Delta_{4,3}(t) = \Delta_{7,2}(t) - \Delta_{3,2}(t) + \Delta_{1,2}(t)\,,$$
$$\Delta_{5,3}(t) = \Delta_{9,2}(t) - \Delta_{5,2}(t) + \Delta_{3,2}(t)\,,$$
$$\Delta_{7,3}(t) = \Delta_{13,2}(t) - \Delta_{9,2}(t) + \Delta_{7,2}(t) -$$
$$-\Delta_{3,2}(t) + \Delta_{1,2}(t)\,,$$
$$\Delta_{8,3}(t) = \Delta_{15,2}(t) - \Delta_{11,2}(t) + \Delta_{9,2}(t) -$$
$$-\Delta_{5,2}(t) + \Delta_{3,2}(t)\,. \qquad (14)$$

It is easy to see that the expansion of $\Delta_{n,3}(t)$ includes $2d+1$ terms like $\Delta_{k,2}(t)$.

4. On "Coordinates" of $\Delta_{n,3}(t)$ in the $\Delta_{k,2}(t)$-Basis

From (24), it follows that the Alexander polynomials are orthonormal due to the orthonormality of the Chebyshev polynomials (with a definite weight on the interval $[-1, 1]$).

Thus, we can introduce an orthonormal basis consisting of $\Delta_{k,2}(t)$ and find an expansion of any $\Delta_{n,3}(t)$ in this basis. We may consider k as the "coordinates" of a knot $T(n,3)$ in the $\Delta_{k,2}(t)$-basis. From (14), we have the presentation of the particular torus knots using such "coordinates":

$$\Delta_{1,3}(t)=(1), \quad \Delta_{2,3}(t)=(3), \quad \Delta_{4,3}(t)=(7;-3;1),$$
$$\Delta_{5,3}(t)=(9;-5;3), \quad \Delta_{7,3}(t)=(13;-9;7;-3;1),$$
$$\Delta_{8,3}(t)=(15;-11;9;-5;3),$$
$$\Delta_{10,3}(t)=(19;-15;13;-9;7;-3;1)\,. \qquad (15)$$

Could we find a similar representation for an arbitrary knot, we gain the numerical invariant for knots (and links).

5. On the Expression of $\Delta_{n,4}(t)$ through $\Delta_{k,2}(t)$

In this section, we present any Alexander polynomial $\Delta_{n,4}(t)$ as an algebraic sum of Alexander polynomials $\Delta_{k,2}(t)$ of $T(k,2)$ torus knots.

Proposition 2. *The Alexander polynomial for any torus knot $T(n,4)$ is presented as a sum of the Alexander polynomials for torus knots $T(k,2)$ by the formula*

$$\begin{aligned}&\Delta_{n,4}(t) = \sum_{j=0}^{d_1} (-1)^j \Delta_{3n-2-4j,2}(t) + \\ &+ \sum_{i=0}^{d_2} (-1)^i \Delta_{n-2-4i,2}(t)\,,\end{aligned} \qquad (16)$$

where $d_1 = \left[\frac{3n-2}{4}\right]$ *and* $d_2 = \left[\frac{n-2}{4}\right]$.

To prove (16), we use

$$\Delta_{n,4}(t) + \Delta_{n-4,4}(t) = \Delta_{3n-2,2}(t) - \Delta_{3n-6,2}(t) + \\ +\Delta_{3n-10,2}(t) + \Delta_{n-2,2}(t), \tag{17}$$

which can be proved, in turn, with the help of (7) and (9).

Using (16), we give some first examples of the presentation of the Alexander polynomials $\Delta_{n,4}(t)$ by the a of the Alexander polynomials $\Delta_{k,2}(t)$:

$$\Delta_{1,4}(t) = \Delta_{1,2}(t), \\ \Delta_{3,4}(t) = \Delta_{7,2}(t) - \Delta_{3,2}(t) + \Delta_{1,2}(t), \\ \Delta_{5,4}(t) = \Delta_{13,2}(t) - \Delta_{9,2}(t) + \Delta_{5,2}(t) + \\ +\Delta_{3,2}(t) - \Delta_{1,2}(t), \\ \Delta_{7,4}(t) = \Delta_{19,2}(t) - \Delta_{15,2}(t) + \Delta_{11,2}(t) - \\ -\Delta_{7,2}(t) + \Delta_{5,2}(t) + \Delta_{3,2}(t) - \Delta_{1,2}(t). \tag{18}$$

The expansion of $\Delta_{n,4}(t)$ includes n terms like $\Delta_{k,2}(t)$.

Thus, relation (18) yields the "coordinates" of corresponding $T(n,4)$ knots (i.e., the "coordinates" of a knot $T(n,4)$ in the $\Delta_{k,2}(t)$-basis)

$$\Delta_{1,4}(t)=(1), \quad \Delta_{3,4}(t)=(7;-3;1), \\ \Delta_{5,4}(t)=(13;-9;5;3;-1), \\ \Delta_{7,4}(t)=(19;-15;11;-7;5;3;-1), \\ \Delta_{9,4}(t)=(25;-21;17;-13;9;7;-5;-3;1). \tag{19}$$

In the following section, we are going to present the Alexander polynomials $\Delta_{n,2}(t)$, $\Delta_{n,3}(t)$, and $\Delta_{n,4}(t)$ in terms of the Chebyshev polynomials of the second kind $V_n(x)$, $x = t + t^{-1}$.

6. Alexander Polynomials in Terms of Chebyshev Polynomials

The monic Chebyshev polynomials of the second kind, which have unit coefficients of x^n, can be defined as

$$V_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad 2\cos\theta = x \tag{20}$$

or by the recurrence relation

$$V_{n+1} = xV_n - V_{n-1}, \quad V_0 = 1, \quad V_1 = x. \tag{21}$$

Some first low-order Chebyshev polynomials of the second kind are:

$$V_0 = 1, \quad V_1 = x, \quad V_2 = x^2 - 1, \quad V_3=x^3-2x, \\ V_4 = x^4 - 3x^2 + 1, \quad V_5=x^5-4x^3+3x. \tag{22}$$

By analogy to the "coordinates" of a knot in the $\Delta_{k,2}(t)$-basis, we introduce the "coordinates" of a knot in the $\widetilde{V}_k(x)$-basis, where $\widetilde{V}_k(x)$ is a certain combination of Chebyshev polynomials of the second kind, namely

$$\widetilde{V}_k(x) = V_k(x) - V_{k-1}(x) - V_{k-2}(x)+ \\ +V_{k-3}(x)+V_{k-4}(x)-V_{k-5}(x)-\cdots V_0(x)= \\ = \sum_{j=1}^{k+1}(-1)^{[\frac{j}{2}]}V_{k+1-j}(x). \tag{23}$$

For example, the $\widetilde{V}_7(x)$-basis looks as

$$\widetilde{V}_7(x) = V_7(x) - V_6(x) - V_5(x) + V_4(x) + V_3(x)- \\ -V_2(x) - V_1(x) + V_0(x).$$

The Alexander polynomials $\Delta_{n,2}(t)$ satisfy relation [9, 12]

$$\Delta_{n,2}(t) \equiv \Delta_{2m+1,2}(t) = V_m(x) - V_{m-1}(x), \\ x = t + t^{-1}. \tag{24}$$

From (23) and (24), we have

$$\Delta_{n,2}(t) \equiv \Delta_{2m+1,2}(t)=\widetilde{V}_m(x)+\widetilde{V}_{m-2}(x), \; x=t+t^{-1}. \tag{25}$$

Whence, we obtain the "coordinates" of $\Delta_{n,2}(t)$ in $\widetilde{V}$-basis:

$$\Delta_{2m+1,2}(t) = (m;\; m-2)_{\widetilde{V}}, \tag{26}$$

which can be written as

$$\Delta_{n,2}(t) = \widetilde{V}_{\frac{n-1}{2}} + \widetilde{V}_{\frac{n-5}{2}} = \left(\frac{n-1}{2};\; \frac{n-5}{2}\right)_{\widetilde{V}}. \tag{27}$$

Let us write the "V-coordinates" for $\Delta_{n,3}(t)$. It follows from (14) and (24) that

$$\Delta_{n,3}(t) = V_{n-1}(x)+ \\ +\sum_{k=0}^{d}\big(-V_{n-2-3k}(x)-V_{n-3-3k}(x)+2V_{n-4-3k}(x)\big)= \\ = \sum_{j=0}^{d_1}\widetilde{V}_{n-1-3j}(x) - \sum_{i=0}^{d_2}\widetilde{V}_{n-5-3i}(x), \tag{28}$$

where $d=\left[\frac{n-2}{3}\right]$, $d_1=\left[\frac{n-1}{3}\right]$, $d_2=\left[\frac{n-5}{3}\right]$, $x=t+t^{-1}$. Below, we give some examples of (28):

$$\begin{aligned}
&\Delta_{1,3}(t) = V_0(x) = \widetilde{V}_0(x) = (0)_{\widetilde{V}},\\
&\Delta_{2,3}(t) = V_1(x) - V_0(x) = \widetilde{V}_1(x) = (1)_{\widetilde{V}},\\
&\Delta_{4,3}(t) = V_3(x) - V_2(x) - V_1(x) + 2V_0(x) =\\
&= \widetilde{V}_3(x) - \widetilde{V}_0(x) = (3;\, -0)_{\widetilde{V}},\\
&\Delta_{5,3}(t) = V_4(x) - V_3(x) - V_2(x) + 2V_1(x) - V_0(x) =\\
&= \widetilde{V}_4(x) + \widetilde{V}_1(x) - \widetilde{V}_0(x) = (4;\, 1;\, -0)_{\widetilde{V}}\,. \qquad (29)
\end{aligned}$$

Proposition 3. *The Alexander polynomial for the torus knot $T(n,4)$ is expressed as a sum of Chebyshev polynomials of the second kind by the relation*

$$\begin{aligned}
&\Delta_{n,4}(t) =\\
&= \sum_{j=1}^{\frac{3n-1}{2}} (-1)^{[\frac{j}{2}]} V_{\frac{3n-1}{2}-j}(x) + \sum_{i=1}^{\frac{n-1}{2}} (-1)^{[\frac{i}{2}]} V_{\frac{n-1}{2}-i}(x) =\\
&= \widetilde{V}_{\frac{3n-3}{2}} + \widetilde{V}_{\frac{n-3}{2}} = \left(\frac{3n-3}{2};\ \frac{n-3}{2}\right)_{\widetilde{V}}, \qquad (30)
\end{aligned}$$

where $x = t + t^{-1}$.

Some first examples of (30) are as follows:

$$\begin{aligned}
&\Delta_{1,4}(t) = V_0(x) = \widetilde{V}_0(x) = (0)_{\widetilde{V}},\\
&\Delta_{3,4}(t) = V_3(x) - V_2(x) - V_1(x) + 2V_0(x) =\\
&= \widetilde{V}_3(x) + \widetilde{V}_0(x) = (3;\, 0)_{\widetilde{V}},\\
&\Delta_{5,4}(t) = V_6(x) - V_5(x) - V_4(x) + V_3(x) +\\
&+ V_2(x) - 2V_0(x) = \widetilde{V}_6(x) + \widetilde{V}_1(x) = (6;\, 1)_{\widetilde{V}}\,. \qquad (31)
\end{aligned}$$

7. Concluding Remarks

Here, we have shown how to introduce the "coordinates" for torus knots and links with the help of corresponding Alexander polynomials. As a basis, we can choose the simplest Alexander polynomials $\Delta_{k,2}(t)$ or the Chebyshev polynomials of the second kind. We have found the above-mentioned expansions for the torus knots $\Delta_{n,l}(t)$, $l=2,3,4$. By finding such expansions for all (torus) knots and links, we have obtained the numerical invariants for them.

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ПРО ТОРИЧНІ ВУЗЛИ $T(n,4)$ І ПОЛІНОМИ ЧЕБИШОВА

А.М. Павлюк

Р е з ю м е

Поліноми Александера $\Delta_{n,3}(t)$ і $\Delta_{n,4}(t)$ представлено як суму поліномів Александера $\Delta_{k,2}(t)$. Ці поліноми також виражено через суму поліномів Чебишова другого роду. Отримані розклади дозволяють ввести "координати" щодо вказаних базисів, які, як передбачається, є числовими інваріантами вузлів і зачеплень.