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ENTANGLEMENT DYNAMICS INDUCED BY A THREE-LEVEL LASER COUPLED TO THE SQUEEZED VACUUM RESERVOIR

We have studied the squeezing and entanglement properties of the light induced by a three-level laser coupled to the squeezed vacuum reservoir. Applying the pertinent master equation, we have obtained the evolution equations for the cavity mode variables. Using the solutions of the resulting equations, the squeezing properties, entanglement amplification, and the normalized second-order correlation function of the cavity radiation are described. We have seen that the light generated by a three-level laser is in a squeezed state, and the squeezing occurs in the plus quadrature. In addition, we have also established that the effect of the squeezed parameter increases the mean and variance of the photon number. It is found that the squeezing and entanglement in the two-mode light are directly related. As a result, an increase in the degree of squeezing directly leads to an increase in the degree of entanglement and vice versa. This shows that, whenever there is squeezing in the two-mode light, there exists an entanglement in the system.

Keywords: operator dynamics, photon statistics, quadrature squeezing, second-order correlations, photon entanglement.

1. Introduction

Entanglement is one of the fundamental tools for the quantum information processing and communication protocols. The generation and manipulation of the entanglement has attracted a great deal of interest with wide applications in quantum teleportation, quantum dense coding, quantum computation, quantum error correction, and quantum cryptography [1–5]. Recently, much attention is given to the generation of a continuous-variable entanglement to manipulate the discrete counterparts and quantum bits and to perform the quantum the information processing. In general, the degree of entanglement decreases, when it interacts with the environment. But, the quantum information processing efficiency highly depends on the degree of entanglement. Therefore, it is necessary to generate strongly entangled states which can survive under the external noise. In general, due to the strong correlation between the cavity modes, a two-mode squeezed state violates certain classical inequalities and then can be used in preparing the Einstein–Podolsky–Rosen-type (EPR) entanglement [6]. The steady-state entanglement in a nondegenerate three-level laser has been studied, when the atomic coherence is induced by initially prepared atoms in a coherent superposition of the top an bottom levels [7–15] and when the top and bottom levels of three-level atoms injected into a cavity are coupled by coherent light [16–21]. In addition, the squeezed states of light play a crucial role in the development of quan-

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Цитування: Алему М. Динаміка переплутування, індукована трирівневим лазером, що взаємодіє з квантовим резервуаром в умовах стискання. *Укр. фіз. журн.* **68**, №5, 295 (2023).

tum physics. Squeezing is one of the nonclassical features of light that has been extensively studied by several authors [22–24]. In squeezed light, the noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [10]. Squeezed light has potential applications in the low-noise optical communication and the weak-signal detection [4, 5]. Squeezed light can be generated by various quantum optical processes such as subharmonic generations [1–5], four-wave mixing [25], resonance fluorescence [9, 10], and second-harmonic oscillator [26–28].

Moreover, Fesseha [9,10] studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms placed in a closed cavity and pumped by the electron bombardment. It was shown that the maximum quadrature squeezing of the light generated by the laser, operating below the threshold, is 50% below the vacuum-state level. Moreover, Fesseha also found that the quadrature squeezing of the output light is equal to that of the cavity light. On the other hand, this study shows that the local quadrature squeezing is greater than the global quadrature squeezing. He also found that a large part of the total mean photon number is confined in a relatively small frequency interval. In addition, Fesseha [10] studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms placed in a closed cavity and pumped by coherent light. He showed that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found with electron bombardment.

In this paper, we will analyze the squeezing and entanglement properties of light emitted by three-level atoms available in an open cavity coupled to a squeezed vacuum reservoir via a single port-mirror and pumped to the top level by the electron bombardment. Thus, considering the interaction of three-level atoms with a resonant cavity light and the damping of the cavity light by a vacuum reservoir, we will obtain the photon statistics, the quadrature variance, the quadrature squeezing, and entanglement of the cavity light. Furthermore, applying the same solutions, we will also obtain the normalized second-order correlation function for the two-mode light.

2. Operator Dynamics

Now, we consider the case where N three-level atoms in a cascade configuration and available in an open cavity. We denote the top, intermediate, and bottom levels of these atoms by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. We prefer to call the light emitted from the top level as light mode a and the one emitted from the intermediate level as light mode b. We carry out our analysis with light modes a and b having the same or different frequencies. In addition, we assume that light modes a and b to be at resonance with the two transitions $|a\rangle_k \rightarrow |b\rangle_k$ and $|b\rangle_k \rightarrow |c\rangle_k$, with the direct transition between $|a\rangle_k$ and $|c\rangle_k$ to be electric-dipole forbidden. The interaction of a three-level atoms with cavity modes a and b can be described at resonance by the Hamiltonian [10]

$$\hat{H} = ig\left(\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\right), \tag{1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{kk}\langle a|, \tag{2}$$

and

$$\hat{\sigma}_b^k = |c\rangle_{kk}\langle b| \tag{3}$$

are lowering atomic operators, $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes, g is the coupling constant between the atom and the cavity modes. Equation (1) represents the quantum Hamiltonian describing the interaction between a three-level atom and the cavity mode in the dipole and rotating-wave approximations. This Hamiltonian is used to obtain the quantum Langevin equations and equations of evolution for the cavity mode operators. We assume that the cavity modes are coupled to a two-mode squeezed vacuum reservoir via a single-port mirror (Fig. 1).

The quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ are given by [9, 10]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}] + \hat{F}_a(t), \tag{4}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}] + \hat{F}_b(t), \tag{5}$$

where κ is the cavity damping constant, and $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise operators associated with the squeezed vacuum reservoir and having the following correlation properties:

$$\langle \hat{F}_a(t)\rangle = \langle \hat{F}_b(t)\rangle = 0, \tag{6}$$

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle = \langle \hat{F}_b^\dagger(t)\hat{F}_b(t')\rangle = \kappa N_0\delta(t-t'), \tag{7}$$

$$\langle \hat{F}_a(t)\hat{F}_a^\dagger(t')\rangle = \langle \hat{F}_b(t)\hat{F}_b^\dagger(t')\rangle = \kappa(N_0+1)\delta(t-t'), \tag{8}$$

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a^\dagger(t')\rangle = \langle \hat{F}_b^\dagger(t)\hat{F}_b^\dagger(t')\rangle = \langle \hat{F}_a(t)\hat{F}_a(t')\rangle = = \langle \hat{F}_b(t)\hat{F}_b(t')\rangle = -\kappa M_0\delta(t-t'). \tag{9}$$

With the aid of Eqs. (1), (4), and (5), one can easily establish that

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \hat{F}_a(t), \tag{10}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k + \hat{F}_b(t). \tag{11}$$

Furthermore, the master equation for a three-level atom interacting with the squeezed vacuum reservoir is given by [2]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}\Big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \\ &- \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\Big] + \\ &+ \frac{\kappa}{2}(N_0+1)\Big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big] + \\ &+ \frac{\kappa}{2}N_0\Big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\Big] + \\ &+ \frac{\kappa}{2}M_0\Big[2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\Big],\end{aligned} \tag{12}$$

where

$$N_0 = \sinh^2(r), \tag{13}$$

in which r is the squeezing parameter, is the mean photon number of the two-mode squeezed vacuum reservoir,

$$M_0 = \sinh(r)\cosh(r) = \sqrt{N_0(N_0+1)} \tag{14}$$

is the intermodal correlations associated with the reservoir, and γ is the spontaneous emission decay constant. We can rewrite Eq. (12) as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}\Big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \\ &- \hat{\rho}\hat{\eta}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k\Big] + \\ &+ \frac{\kappa}{2}(N_0+1)\Big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big] + \\ &+ \frac{\kappa}{2}N_0\Big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\Big] + \\ &+ \frac{\kappa}{2}M_0\Big[2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\Big],\end{aligned} \tag{15}$$

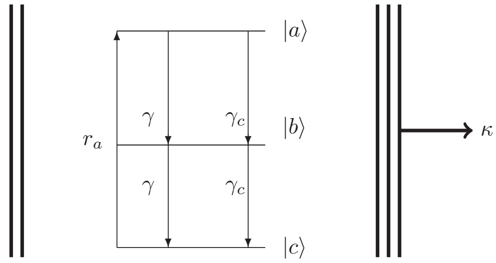

Fig. 1. Schematic representation of a three-level laser coupled to a two-mode squeezed vacuum reservoir

where

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \tag{16}$$

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|. \tag{17}$$

Using Eq. (1), we can put Eq. (15) in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g\Big[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \\ &- \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\Big] + \\ &+ \frac{\gamma}{2}\Big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k\Big] + \\ &+ \frac{\kappa}{2}(N_0+1)\Big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big] + \\ &+ \frac{\kappa}{2}N_0\Big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\Big] + \\ &+ \frac{\kappa}{2}M_0\Big[2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\Big].\end{aligned} \tag{18}$$

Applying the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{19}$$

along with Eq. (18), we can easily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\gamma\langle\hat{\sigma}_a^k\rangle + g\left[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle\right], \tag{20}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^k\rangle + g\left[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle\right], \tag{21}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c^k\rangle + g\left[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle\right], \tag{22}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma\langle\hat{\eta}_a^k\rangle + g\left[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\right], \tag{23}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = \gamma\left[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle\right] +$$
$$+ g\left[\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\right], \tag{24}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \gamma\langle\hat{\eta}_b^k\rangle - g\left[\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle\right], \tag{25}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{kk}\langle a|, \tag{26}$$

and

$$\hat{\eta}_c^k = |c\rangle_{kk}\langle c|. \tag{27}$$

We see that Eqs. (20)–(25) are nonlinear differential equations. Hence, it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [13]. Then using this approximation scheme, we get, from Eqs. (10) and (11), the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2}{\kappa}\hat{F}_a(t), \tag{28}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2}{\kappa}\hat{F}_b(t). \tag{29}$$

Evidently, these would turn out to be exact relations at a steady state. Now, combining Eqs. (28) and (29) with Eqs. (20)–(25), we get

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle\hat{\sigma}_a^k\rangle +$$
$$+ \frac{2g}{\kappa}\left[\langle\hat{\eta}_b^k\hat{F}_a(t)\rangle - \langle\hat{\eta}_a^k\hat{F}_a(t)\rangle + \langle\hat{F}_b^\dagger(t)\hat{\sigma}_c^k\rangle\right], \tag{30}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle\hat{\sigma}_b^k\rangle +$$
$$+ \frac{2g}{\kappa}\left[\langle\hat{\eta}_c^k\hat{F}_b(t)\rangle - \langle\hat{\eta}_b^k\hat{F}_b(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_c^k\rangle\right], \tag{31}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle\hat{\sigma}_c^k\rangle +$$
$$+ \frac{2g}{\kappa}\left[\langle\hat{\sigma}_b^k\hat{F}_a(t)\rangle - \langle\hat{\sigma}_a^k\hat{F}_b(t)\rangle\right], \tag{32}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle\hat{\eta}_a^k\rangle +$$
$$+ \frac{2g}{\kappa}\left[\langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\right], \tag{33}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -\left[\gamma + \gamma_c\right]\langle\hat{\eta}_b^k\rangle + \left[\gamma + \gamma_c\right]\langle\hat{\eta}_a^k\rangle + \frac{2g}{\kappa}\times$$
$$\times\Big[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle -$$
$$- \langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\Big], \tag{34}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \left[\gamma + \gamma_c\right]\langle\hat{\eta}_b^k\rangle -$$
$$- \frac{2g}{\kappa}\left[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle\right], \tag{35}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{36}$$

is the stimulated emission decay constant.

We next proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eqs. (30)–(35). To this end, after removing the angular brackets, Eq. (33) can be rewritten as

$$\frac{d}{dt}\hat{\eta}_a^k = -\left[\gamma + \gamma_c\right]\hat{\eta}_a^k +$$
$$+ \frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}\hat{F}_a(t) + \hat{F}_a^\dagger(t)\hat{\sigma}_a^k\right] + \hat{f}_a(t), \tag{37}$$

where $\hat{f}_a(t)$ is the noise operator associated with $\hat{\eta}_a$. A formal solution of this equation can be written as

$$\hat{\eta}_a^k(t) = \hat{\eta}_a^k(0)e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\times$$
$$\times\left[\frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t') + \hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\right] + \hat{f}_a(t')\right]dt'. \tag{38}$$

Multiplying Eq. (38) on the right by $\hat{F}_a(t)$ and taking the expectation value of the resulting equation, we have

$$\langle\hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = \langle\hat{\eta}_a^k(0)\hat{F}_a(t)\rangle e^{-(\gamma+\gamma_c)t} +$$
$$+ \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\Bigg[\frac{2g}{\kappa}\Big[\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle +$$
$$+ \langle\hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle\Big] + \langle\hat{f}_a(t')\hat{F}_a(t)\rangle\Bigg]dt'. \tag{39}$$

Ignoring the noncommutativity of the atomic and noise operators and neglecting the correlation between $\hat{F}_a(t)$ and $\hat{\sigma}_a^k(t')$, assumed to be considerably small [6], one can write the approximately valid relations

$$\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^{\dagger k}(t')\rangle\langle\hat{F}_a(t')\hat{F}_a(t)\rangle = 0, \tag{40}$$

$$\langle \hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle = \langle \hat{\sigma}_a^k(t')\rangle\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle = 0, \quad (41)$$

$$\langle \hat{f}_a(t')\hat{F}_a(t)\rangle = \langle \hat{f}_a(t')\rangle\langle \hat{F}_a(t)\rangle = 0. \quad (42)$$

Now, accounting for these approximately valid relations along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier time, Eq. (39) takes the form

$$\langle \hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = 0. \quad (43)$$

Following a similar procedure, one can also check that

$$\langle \hat{\eta}_b^k(t)\hat{F}_a(t)\rangle = 0, \quad (44)$$

$$\langle \hat{\eta}_c^k(t)\hat{F}_b(t)\rangle = 0, \quad (45)$$

$$\langle \hat{\eta}_b^k(t)\hat{F}_b(t)\rangle = 0, \quad (46)$$

$$\langle \hat{F}_a^\dagger(t)\hat{\sigma}_a^k(t)\rangle = 0, \quad (47)$$

$$\langle \hat{F}_b^\dagger(t)\hat{\sigma}_b^k(t)\rangle = 0. \quad (48)$$

We also take

$$\langle \hat{F}_a^\dagger(t)\hat{\sigma}_c^k(t)\rangle = \langle \hat{F}_b^\dagger(t)\hat{\sigma}_c^k(t)\rangle = 0. \quad (49)$$

With the aid of Eqs. (43)–(49), we rewrite Eqs. (30), (31), (33), (34), and (35) as

$$\frac{d}{dt}\langle \hat{\sigma}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\sigma}_a^k\rangle, \quad (50)$$

$$\frac{d}{dt}\langle \hat{\sigma}_b^k\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle \hat{\sigma}_b^k\rangle. \quad (51)$$

$$\frac{d}{dt}\langle \hat{\eta}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\eta}_a^k\rangle, \quad (52)$$

$$\frac{d}{dt}\langle \hat{\eta}_b^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\eta}_b^k\rangle + \left[\gamma + \gamma_c\right]\langle \hat{\eta}_a^k\rangle, \quad (53)$$

$$\frac{d}{dt}\langle \hat{\eta}_c^k\rangle = \left[\gamma + \gamma_c\right]\langle \hat{\eta}_b^k\rangle. \quad (54)$$

We note that Eqs. (50)–(54) represent the equation of evolution for the atomic operators in the absence of the pumping process. The pumping process must surely affect the dynamics of $\langle \hat{\eta}_a^k\rangle$ and $\langle \hat{\eta}_c^k\rangle$. We seek here to pump the atoms by the electron bombardment. If r_a represents the rate at which a single atom is pumped from the bottom to the top level, then $\langle \hat{\eta}_a^k\rangle$ increases at the rate of $r_a\langle \hat{\eta}_c^k\rangle$, and $\langle \hat{\eta}_c^k\rangle$ decreases at the same rate. In view of this, we rewrite Eqs. (52) and (54) as

$$\frac{d}{dt}\langle \hat{\eta}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\eta}_a^k\rangle + r_a\langle \hat{\eta}_c^k\rangle, \quad (55)$$

$$\frac{d}{dt}\langle \hat{\eta}_c^k\rangle = \left[\gamma + \gamma_c\right]\langle \hat{\eta}_b^k\rangle - r_a\langle \hat{\eta}_c\rangle. \quad (56)$$

We next sum Eqs. (50), (51), (53), (55), and (56) over the N three-level atoms, so that

$$\frac{d}{dt}\langle \hat{m}_a\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{m}_a\rangle, \quad (57)$$

$$\frac{d}{dt}\langle \hat{m}_b\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle \hat{m}_b\rangle, \quad (58)$$

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{N}_a\rangle + r_a\langle \hat{N}_c\rangle, \quad (59)$$

$$\frac{d}{dt}\langle \hat{N}_b\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{N}_b\rangle + \left[\gamma + \gamma_c\right]\langle \hat{N}_a\rangle, \quad (60)$$

$$\frac{d}{dt}\langle \hat{N}_c\rangle = \left[\gamma + \gamma_c\right]\langle \hat{N}_b\rangle - r_a\langle \hat{N}_c\rangle, \quad (61)$$

in which

$$\hat{m}_a = \sum_{k=1}^{N}\hat{\sigma}_a^k, \quad (62)$$

$$\hat{m}_b = \sum_{k=1}^{N}\hat{\sigma}_b^k, \quad (63)$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \quad (64)$$

$$\hat{N}_b = \sum_{k=1}^{N}\hat{\eta}_b^k, \quad (65)$$

$$\hat{N}_c = \sum_{k=1}^{N}\hat{\eta}_c^k, \quad (66)$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \quad (67)$$

we easily arrive at

$$\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle + \langle \hat{N}_c\rangle = N. \quad (68)$$

Furthermore, applying the definition given by Eq. (2) and setting, for any k,

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \quad (69)$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \quad (70)$$

Following the same procedure, one can also check that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{71}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{72}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{73}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{74}$$

$$\hat{N}_c = N|c\rangle\langle c|, \tag{75}$$

where

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k. \tag{76}$$

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{77}$$

and accounting for Eqs. (70)–(75), it can be readily established that

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{78}$$

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{79}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{80}$$

With the aid of Eq. (68), one can put Eq. (59) in the form

$$\frac{d}{dt}\langle \hat{N}_a \rangle = -\left[\gamma + \gamma_c + r_a\right]\langle \hat{N}_a \rangle + r_a[N - \langle \hat{N}_b \rangle]. \tag{81}$$

Applying the large-time approximation scheme to Eq. (60), we get

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle. \tag{82}$$

Thus, in view of this result, Eq. (81) can be written as

$$\frac{d}{dt}\langle \hat{N}_a \rangle = -\left[\gamma + \gamma_c + 2r_a\right]\langle \hat{N}_a \rangle + N r_a. \tag{83}$$

The steady-state solution of Eq. (83) is expressible as

$$\langle \hat{N}_a \rangle = \frac{r_a N}{\gamma + \gamma_c + 2r_a}. \tag{84}$$

Using the steady-state solution of Eq. (61) along with Eq. (82), we have

$$\langle \hat{N}_c \rangle = \frac{\gamma + \gamma_c}{r_a}\langle \hat{N}_a \rangle. \tag{85}$$

With regard for Eq. (84), Eq. (85) takes the form

$$\langle \hat{N}_c \rangle = \frac{(\gamma + \gamma_c)N}{\gamma + \gamma_c + 2r_a}. \tag{86}$$

For $r_a = 0$, we see that $\langle \hat{N}_a \rangle = \langle \hat{N}_b \rangle = 0$ and $\langle \hat{N}_c \rangle = = N$. This result holds whether the atoms are initially in the top or bottom level.

In the presence of N three-level atoms, we rewrite Eq. (10) as [10]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda \hat{m}_a + \beta \hat{F}_a(t), \tag{87}$$

in which λ and β are constants whose values remain to be fixed. Applying Eq. (28), we get

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{4g^2}{\kappa^2}(\hat{\eta}_b^k - \hat{\eta}_a^k) + \frac{4}{\kappa^2}[F_a, F_a^\dagger], \tag{88}$$

and, on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{4g^2}{\kappa^2}(\hat{N}_b - \hat{N}_a) + \frac{4N}{\kappa^2}[F_a, F_a^\dagger], \tag{89}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N} [\hat{a}, \hat{a}^\dagger]_k \tag{90}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^\dagger$, when light mode a is interecting with all N three-level atoms. On the other hand, applying the large-time approximation to Eq. (87), one can easily find

$$[\hat{a}, \hat{a}^\dagger] = N\frac{4\lambda^2}{\kappa^2}(\hat{N}_b - \hat{N}_a) + \frac{4\beta^2}{\kappa^2}[F_a, F_a^\dagger]. \tag{91}$$

Thus, accounting for Eqs. (89) and (91), we see that

$$\lambda = \pm\frac{g}{\sqrt{N}}, \tag{92}$$

$$\beta = \pm\sqrt{N}. \tag{93}$$

In view of Eqs. (92) and (93), Eq. (87) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \sqrt{N}\hat{F}_a(t). \tag{94}$$

Following a similar procedure, one can also readily establish that

$$[\hat{b}, \hat{b}^\dagger] = \frac{4g^2}{\kappa^2}(\hat{N}_c - \hat{N}_b) + \frac{4N}{\kappa^2}[F_b, F_b^\dagger], \tag{95}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \sqrt{N}\hat{F}_b(t). \tag{96}$$

Furthermore, in order to include the effect of pumping, we rewrite Eqs. (57) and (58) as

$$\frac{d}{dt}\hat{m}_a = -\frac{\mu}{2}\hat{m}_a + \hat{G}_a(t), \tag{97}$$

$$\frac{d}{dt}\hat{m}_b = -\frac{\mu}{2}\hat{m}_b + \hat{G}_b(t) \tag{98}$$

in which $\hat{G}_a(t)$ and $\hat{G}_b(t)$ are noise operators with vanishing mean, and μ is a parameter whose value remains to be determined. Employing the relation

$$\frac{d}{dt}\left\langle \hat{m}_a^\dagger \hat{m}_a \right\rangle = \left\langle \frac{d\hat{m}_a^\dagger}{dt}\hat{m}_a \right\rangle + \left\langle \hat{m}_a^\dagger \frac{d\hat{m}_a}{dt} \right\rangle \tag{99}$$

along with Eq. (97), we easily find

$$\frac{d}{dt}\langle \hat{m}_a^\dagger \hat{m}_a\rangle = -\mu\langle \hat{m}_a^\dagger \hat{m}_a\rangle + \\ + \langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle, \tag{100}$$

which yields

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\mu\langle \hat{N}_a\rangle + \\ + \frac{1}{N}\left[\langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle\right]. \tag{101}$$

The comparison of Eqs. (83) and (101) shows that

$$\mu = \gamma + \gamma_c + 2r_a \tag{102}$$

and

$$\left\langle \hat{m}_a^\dagger \hat{G}_a(t)\right\rangle + \left\langle \hat{G}_a^\dagger(t)\hat{m}_a\right\rangle = r_a N^2. \tag{103}$$

We observe that Eq. (103) is equivalent to

$$\left\langle \hat{G}_a^\dagger(t)\hat{G}_a(t')\right\rangle = r_a N^2 \delta(t-t'). \tag{104}$$

One can also easily verify that

$$\left\langle \hat{G}_b(t)\hat{G}_b^\dagger(t')\right\rangle = (\gamma+\gamma_c)N^2\delta(t-t'). \tag{105}$$

Furthermore, adding Eqs. (57) and (58), we have

$$\frac{d}{dt}\langle \hat{m}\rangle = -\frac{1}{2}\left[\gamma+\gamma_c\right]\langle \hat{m}\rangle - \frac{1}{2}\left[\gamma+\gamma_c\right]\langle \hat{m}_a\rangle, \tag{106}$$

where $\hat{m}$ is given by Eq. (77). Upon casting Eq. (106) into the form

$$\frac{d}{dt}\hat{m} = -\frac{\mu}{2}\hat{m} - \frac{\mu}{2}\hat{m}_a + \hat{G}(t), \tag{107}$$

one can verify that μ has the value given by Eq. (102) and

$$\left\langle \hat{G}^\dagger(t)\hat{G}(t')\right\rangle = r_a N^2 \delta(t-t'). \tag{108}$$

On the other hand, assuming the atoms to be initially on the bottom level, the expectation value of the solution of Eq. (97) happens to be

$$\langle \hat{m}_a(t)\rangle = 0. \tag{109}$$

Hence, the expectation value of the solution of Eq. (94) turns out to be

$$\langle \hat{a}(t)\rangle = 0. \tag{110}$$

In view of Eqs. (94) and (110), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. One can verify that

$$\langle \hat{b}(t)\rangle = 0. \tag{111}$$

Then, accounting for Eqs. (96) and (111), we realize that $\hat{b}(t)$ is a Gaussian variable with zero mean.

Furthermore adding Eqs. (110) and (111), we obtain

$$\langle \hat{c}\rangle = 0, \tag{112}$$

where

$$\hat{c} = \hat{a} + \hat{b}. \tag{113}$$

In addition, adding Eqs.((94) and (96), we get

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m} + \sqrt{N}\hat{F}_c(t), \tag{114}$$

where

$$\hat{F}_c(t) = \hat{F}_a(t) + \hat{F}_b(t), \tag{115}$$

and $\hat{m}$ is given by Eq. (77). One can easily check that

$$\langle \hat{F}_c(t)\rangle = 0, \tag{116}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c(t')\rangle = 2N_0\kappa\delta(t-t'), \tag{117}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c^\dagger(t')\rangle = \langle \hat{F}_c(t)\hat{F}_c(t)\rangle = -2\kappa M_0\delta(t-t'), \tag{118}$$

$$\langle \hat{F}_c(t)\hat{F}_c^\dagger(t')\rangle = 2\kappa(N_0+1)\delta(t-t'). \tag{119}$$

In view of Eqs. (112) and (114), we see that $\hat{c}$ is a Gaussian variable with zero mean.

3. Photon Statistics

In this section, we will calculate the mean and variance of the photon number for the two-mode cavity light at a steady state. To this end, using the relation

$$\frac{d}{dt}\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle = \left\langle \frac{d\hat{c}^\dagger(t)}{dt}\hat{c}(t)\right\rangle + \left\langle \hat{c}^\dagger(t)\frac{d\hat{c}(t)}{dt}\right\rangle \tag{120}$$

along with Eq. (114), we find

$$\frac{d}{dt}\langle \hat{c}^\dagger t)\hat{c}(t)\rangle = -\kappa\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle + \frac{g}{\sqrt{N}}\left[\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle + \right.$$
$$\left. + \langle \hat{m}^\dagger(t)\hat{c}(t)\rangle\right] + \sqrt{N}\left[\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle + \langle \hat{c}^\dagger(t)\hat{F}_c(t)\rangle\right]. \tag{121}$$

Now, we will evaluate $\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle$. Applying the large-time approximation, we get, from Eq. (114), the approximately valid relation

$$\hat{c}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m} + \frac{2\sqrt{N}}{\kappa}\hat{F}_c(t). \tag{122}$$

Multiplying the adjoint of Eq. (122) on the right by $\hat{m}(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}[\langle \hat{N}_a(t)\rangle + \langle \hat{N}_b(t)\rangle] +$$
$$+ \frac{2\sqrt{N}}{\kappa}\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle. \tag{123}$$

We now proceed to evaluate $\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle$. To this end, a formal solution of Eq. (107) can be written as

$$\hat{m}(t) = \hat{m}(0)e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')} \times$$
$$\times \left[-\frac{\mu}{2}\hat{m}_a(t') + \hat{G}(t')\right]dt'. \tag{124}$$

Multiplying Eq. (124) on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle = \langle \hat{F}_c^\dagger(t)\hat{m}(0)\rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')} \times$$
$$\times\left[-\frac{\mu}{2}\langle \hat{F}_c^\dagger(t)\hat{m}_a(t')\rangle + \langle \hat{F}_c^\dagger(t)\hat{G}(t')\rangle\right]dt'. \tag{125}$$

Taking into account that the noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier times and assuming that the cavity mode and atomic mode operators are not correlated, we get

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle = 0. \tag{126}$$

In view of this result, Eq. (123) takes the form

$$\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}[\langle \hat{N}_a(t)\rangle + \langle \hat{N}_b(t)\rangle]. \tag{127}$$

Now, we evaluate $\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle$. To this end, a formal solution of Eq. (114) can be written as

$$\hat{c}(t) = \hat{c}(0)e^{-\frac{\kappa}{2}t} +$$
$$+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\hat{m}(t') + \sqrt{N}\hat{F}_c(t')\right]dt'. \tag{128}$$

Multiplying Eq. (128) on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle = \langle \hat{F}_c^\dagger(t)\hat{c}(0)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')} \times$$
$$\times\left[\frac{g}{\sqrt{N}}\langle \hat{F}_c^\dagger(t)\hat{m}(t')\rangle + \sqrt{N}\langle \hat{F}_c^\dagger(t)\hat{F}_c(t')\rangle\right]dt'. \tag{129}$$

In view of Eqs. (117) and (126) along with the fact that the noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier times, Eq. (129) becomes

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle = \sqrt{N}N_0\kappa. \tag{130}$$

Now, with regard for Eqs. (127) and (130) along with their complex conjugates, we can rewrite Eq. (121) as

$$\frac{d}{dt}\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle = -\kappa\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle +$$
$$+ \frac{4g^2}{\kappa}\left[\langle \hat{N}_a(t)\rangle + \langle \hat{N}_b(t)\rangle\right] + 2\sqrt{N}N_0\kappa. \tag{131}$$

The steady-state solution of this equation is expressible as

$$\langle \hat{c}^\dagger\hat{c}\rangle = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle\right] + 2NN_0. \tag{132}$$

Following a similar procedure, one can establish that

$$\langle \hat{c}\hat{c}^\dagger\rangle = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_c\rangle + \langle \hat{N}_b\rangle\right] + 2N(N_0+1), \tag{133}$$

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \langle \hat{m}_c \rangle - 2M_0 N. \tag{134}$$

In view of Eqs. (82), (84), and (86), Eqs. (132) and (133) can be rewritten as

$$\langle \hat{c}^\dagger \hat{c} \rangle = \frac{\gamma_c}{\kappa} \left[\frac{2Nr_a}{\gamma + \gamma_c + 2r_a} \right] + 2NN_0, \tag{135}$$

$$\langle \hat{c}\hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa} \left[\frac{\gamma + \gamma_c + r_a}{\gamma + \gamma_c + 2r_a} \right] N + 2N(N_0 + 1). \tag{136}$$

In the absence of a squeezed parameter ($r = 0$), the mean photon number of the two-mode cavity light has the form

$$\bar{n}_c = \frac{\gamma_c}{\kappa} \left(\frac{2Nr_a}{\gamma + \gamma_c + 2r_a} \right). \tag{137}$$

It can be seen from the plots in Figs. 2 and 3 that the mean photon number of two- mode cavity light is greater in the presence of a squeezed parameter than in its absence. In other words, the effect of the squeezed parameter is to increase the mean photon number. In addition, the plots in Figs. 2 and 3 indicate that the maximum mean photon number is 44.51 for $\gamma = 0$ and $r_a = 10$.

Furthermore, the variance of the photon number for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle (\hat{c}^\dagger \hat{c})^2 \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{138}$$

Using the fact that $\hat{c}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^{2\dagger} \rangle \langle \hat{c}^2 \rangle. \tag{139}$$

We now proceed to calculate the expectation value of the atomic operator $\hat{m}_c$ following the approach presented in [10]. To this end, applying the identity given by Eq. (67), the state vector of a three-level atom can be put in the form

$$|\psi_k\rangle = c_a |a_k\rangle + c_b |b_k\rangle + c_c |c_k\rangle, \tag{140}$$

in which

$$c_a = \langle a_k | \psi_k \rangle, \tag{141}$$

$$c_b = \langle b_k | \psi_k \rangle, \tag{142}$$

$$c_c = \langle c_k | \psi_k \rangle. \tag{143}$$

The state vector described by Eq. (140) can be used to determine the expectation value of an atomic operator formed by a pair of identical energy levels or

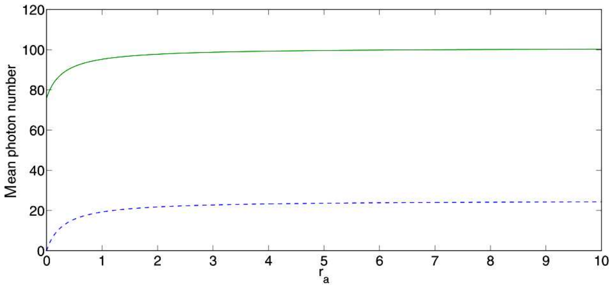

Fig. 2. Plots of the mean photon number for the two-mode cavity light at the steady state [Eq. (135)] for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, $N_0 = 0.2$ (solid curve), $N_0 = 0$ (dashed curve), and $N = 50$

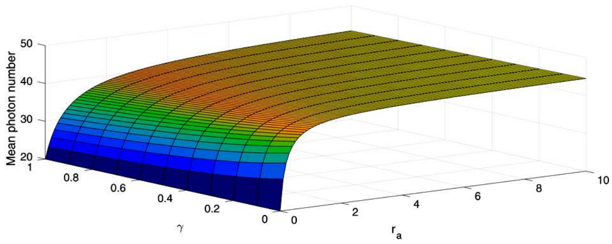

Fig. 3. Plots of the mean photon number for the two-mode cavity light at the steady state [Eq. (135)] for $\kappa = 0.8$, $\gamma_c = 0.4$, $N_0 = 0.2$, and $N = 50$ vs γ and r_a

by two distinct energy levels between which the transition with the emission of a photon is dipole forbidden. One can readily establish that

$$\langle \hat{\eta}_a^k \rangle = c_a c_a^*, \tag{144}$$

$$\langle \hat{\eta}_c^k \rangle = c_c c_c^*, \tag{145}$$

and

$$\langle \hat{\sigma}_c^k \rangle = c_a c_c^*. \tag{146}$$

We see that

$$|\langle \hat{\sigma}_c^k \rangle|^2 = \langle \hat{\eta}_a^k \rangle \langle \hat{\eta}_c^k \rangle, \tag{147}$$

and, taking $|\langle \hat{\sigma}_c^k \rangle|$ to be real, we get

$$|\langle \hat{\sigma}_c^k \rangle| = \sqrt{\langle \hat{\eta}_a^k \rangle \langle \hat{\eta}_{c.}^k \rangle} \tag{148}$$

So, by summing over k from 1 up to N, we get

$$\langle \hat{m}_c \rangle = \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \tag{149}$$

Now, Eq. (134) takes the form

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle} - 2M_0 N. \tag{150}$$

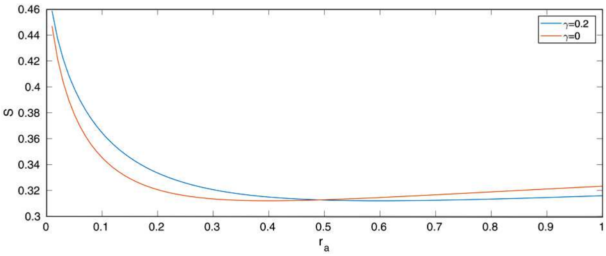

Fig. 4. Plot of the quadrature squeezing at the steady state, [Eq. (168)] versus r_a for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, $M_0 = 0.59$, and for $N_0 = 0.27$

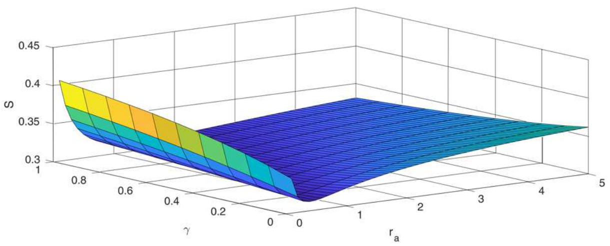

Fig. 5. Plot of the quadrature squeezing at the steady state, [Eq. (168)] versus r_a and γ for $\kappa = 0.8$, $\gamma_c = 0.4$, $M_0 = 0.59$, and $N_0 = 0.27$

By using Eq. (85), we have

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle - 2M_0 N. \tag{151}$$

In view of Eqs. (132), (133), and (151), Eq. (139) becomes

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\left(\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle\right) + 2NN_0\right)\times$$
$$\times\left(\frac{\gamma_c}{\kappa}\left(\langle \hat{N}_b\rangle + \langle \hat{N}_c\rangle\right) + 2N(N_0+1)\right) +$$
$$+\left(\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a\rangle - 2NM_0\right)^2. \tag{152}$$

Finally, with regard for Eqs. (82), (84), (85), and (86) along with Eq. (152), we arrive at

$$(\Delta n)^2 = \frac{1}{4}\bar{n}^2\left(\frac{3(\gamma+\gamma_c)}{r_a} + 2\right) +$$
$$+2\bar{n}N\left(1 - N_0 - \frac{\gamma+\gamma_c}{r_a}N_0 - \sqrt{\frac{\gamma+\gamma_c}{r_a}}M_0\right) +$$
$$+N^2\left(\frac{\gamma+\gamma_c}{r_a}N_0^2 - 2\sqrt{\frac{\gamma+\gamma_c}{r_a}}N_0M_0 + 4M_0^2\right). \tag{153}$$

In the absence of a squeezed parameter ($r = 0$), we readily find

$$(\Delta n)^2 = \frac{1}{4}\bar{n}^2(3\eta + 2) + 2N\bar{n}, \tag{154}$$

where

$$\eta = \frac{\gamma+\gamma_c}{r_a}. \tag{155}$$

Now, the inspection of Eq. (153) indicates that $(\Delta n)^2 > \bar{n}$. Hence, the photon statistics of the two-mode cavity light is super-Poissonian. Our result shows that the photon number variance of the two-mode cavity light is greater than the one obtained by Menisha [17]. This must be due to the reservoir noise operators.

4. Quadrature Squeezing

We proceed to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval. To this end, the squeezing properties of the two-mode cavity light are described by two quadrature operators defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{156}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{157}$$

It can be readily established that [15]

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}(\langle \hat{N}_a\rangle - \langle \hat{N}_c\rangle) - 4Ni. \tag{158}$$

It follows that [16]

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}(\langle \hat{N}_c\rangle - \langle \hat{N}_a\rangle) + 2N. \tag{159}$$

Setting $r_a = 0$, we see that

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}N + 2N. \tag{160}$$

This represents the quadrature variance for two-mode vacuum state. The variance of the quadrature operator is expressible as

$$(\Delta c_\pm)^2 = \pm\langle(\hat{c}^\dagger \pm \hat{c})^2\rangle \mp [\langle \hat{c}^\dagger + \hat{c}\rangle]^2. \tag{161}$$

Accounting for Eq. (112), we have

$$(\Delta c_\pm)^2 = \langle \hat{c}^\dagger\hat{c}\rangle + \langle \hat{c}\hat{c}^\dagger\rangle \pm \langle \hat{c}^{\dagger 2}\rangle \pm \langle \hat{c}^2\rangle. \tag{162}$$

Now, employing Eqs. (68), (132), (133), and (151), we arrive at

$$(\Delta c_+)^2 = \frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle + 2\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle\right) + \\ + 4NN_0 + 2N - 4M_0N, \tag{163}$$

$$(\Delta c_-)^2 = \frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle - 2\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle\right) + \\ + 4NN_0 + 2N + 4M_0N. \tag{164}$$

Moreover, by setting r_a and $N_0 = M_0 = 0$ in Eqs. (163) and (164), we get

$$(\Delta c_+)^2_v = (\Delta c_-)^2_v = \frac{\gamma_c}{\kappa}N + 2N. \tag{165}$$

This represents the quadrature variance of a two-mode vacuum state. We seek to calculate the quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode cavity vacuum state. We define the quadrature squeezing of the two-mode cavity light by

$$S = \frac{(\Delta c_+)^2_v - (\Delta c_+)^2}{(\Delta c_+)^2_v}. \tag{166}$$

Now, employing Eqs. (163) and (165), one can put Eq. (166) in the form

$$S = \frac{4\kappa(M_0 - N_0)}{(\gamma_c + 2\kappa)} - \frac{\frac{\gamma_c}{\kappa}\left(2\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle + \langle \hat{N}_a \rangle\right)}{\frac{\gamma_c}{\kappa}N + 2N}. \tag{167}$$

In view of Eq. (84), Eq. (167) takes the form

$$S = \frac{4\kappa(M_0 - N_0)}{\gamma_c + 2\kappa} - \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\eta}+1}{\eta+2}\right). \tag{168}$$

We note that, unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light is independent of the number of photons. The plot in Fig. 4 indicates that the maximum quadrature squeezing is 45.88% below the vacuum-state level, and this occurs, when the three-level laser is operating below the threshold. The plots in Figs. 4 and 5 indicate that the quadrature squeezing is greater for $\gamma = 0.2$ than that for $\gamma = 0$ for $0.01 < r_a < 0.49$ and is smaller for $\gamma = 0.2$ than that for $\gamma = 0$ for $0.49 < r_a < 1$. In addition, from the plots, we see that the maximum quadrature squeezing is 45.88% for $\gamma = 0.2$ and 44.7% for $\gamma = 0$. This occurs, when the three-level laser is operating at $r_a = 0.01$.

5. Entanglement Properties of the Two-Mode Light

Here, we proceed to study the entanglement condition of the two modes in the cavity. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles [16]-[20]. If the density operator for the combined state cannot be described as a combination of the product of density operators of the constituents,

$$\hat{\rho} \neq \sum_j P_j \hat{\rho}^1_j \bigotimes \hat{\rho}^2_j, \tag{169}$$

where $P_j \geq 0$, and $\sum_j P_j = 1$ is set to ensure the normalization of the combined density of states. Nowadays, a lot of criteria have been developed to measure, detect, and manipulate the entanglement generated by various quantum optical devices. According to DGCZ, the quantum state of a system is said to be entangled, if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 < 2N, \tag{170}$$

where

$$\hat{u} = \hat{x}_a - \hat{x}_b, \tag{171}$$

$$\hat{v} = \hat{p}_a + \hat{p}_b, \tag{172}$$

where $\hat{x}_a = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$, $\hat{x}_b = (\hat{b}^\dagger + \hat{b})/\sqrt{2}$ $\hat{p}_a = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, $\hat{p}_b = i(\hat{b}^\dagger - \hat{b})/\sqrt{2}$, are quadrature operators for modes $\hat{a}$ and $\hat{b}$. Accounting for (171) and (172), (170) yields

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 = 2\frac{\gamma_c}{\kappa}[N + \langle \hat{N}_b \rangle - \langle \hat{m}_c \rangle]. \tag{173}$$

Thus, in view of Eq. (173) together with (163) and (164), the sum of the variances of $\hat{u}$ and $\hat{v}$ can be expressed as

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 = 2\Delta c_+^2, \tag{174}$$

where Δc_+^2 is given by (163). One can readily see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-

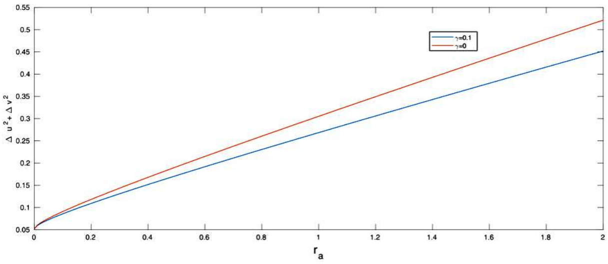

Fig. 6. Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity light versus r_a for $\gamma_c = 0.4$ and for different values of γ

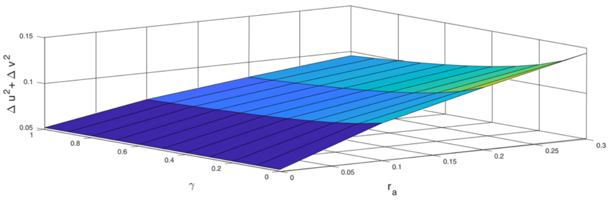

Fig. 7. Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity light versus r_a and γ for $\gamma_c = 0.4$

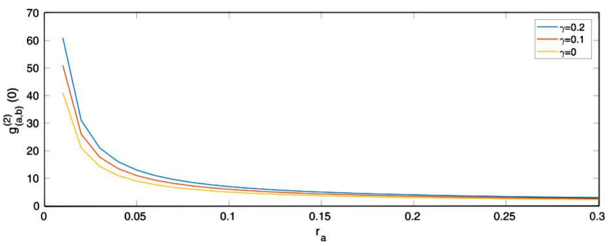

Fig. 8. Plots of $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light versus r_a and for $\gamma_c = 0.4$ and different values of γ

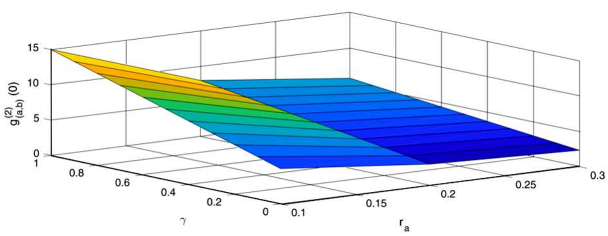

Fig. 9. Plots of $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light versus r_a and γ for $\gamma_c = 0.4$

mode light. One can immediately notice that this particular entanglement measure is directly related to the two-mode squeezing. This direct relationship shows that, whenever there is a two-mode squeezing in the system, there will be the entanglement in the system as well. It is worth to note that the entanglement disappears, when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing, as given by (163). It also follows that, like the mean photon number and quadrature variance, the degree of entanglement depends on the number of atoms. With the help of criterion (170), we get that a significant entanglement occurs between the states of the light generated in the cavity. This is due to the strong correlation between the radiation emitted, when the atoms decay from the upper energy level to the lower one via the intermediate level. In the following, the sum of the variances of a pair of EPR-type operators $\Delta\hat{u}^2 + \Delta\hat{v}^2$ is plotted against the pumping rate, so that the available entanglement is clearly evident for various values of the spontaneous emission rate, γ.

6. Normalized Second-Order Correlation Functions

The second-order correlation function for a superposition of the two modes of the cavity radiation at equal times can also be investigated, by using [18–21]:

$$g^{(2)}_{(a,b)}(0) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}. \tag{175}$$

Since $\hat{a}$ and $\hat{b}$ are Gaussian variables with vanishing means, the normalized second-order correlation function for the two-mode light takes, at the steady-state, the form

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle \hat{b}\hat{a} \rangle \langle \hat{a}^\dagger \hat{b}^\dagger \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}. \tag{176}$$

It follows that

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle \hat{m}_c \rangle^2}{\langle \hat{N}_a \rangle \langle \hat{N}_b \rangle}. \tag{177}$$

In view of (82), (85), and (149), we obtain

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\gamma_c + \gamma}{r_a}. \tag{178}$$

It can be seen from this result that the second-order correlation function of the two-mode light does not depend on the number of atoms.

Figures 8 and 9 show that the second-order correlation function for the two-mode light versus r_a in the presence ($\gamma \neq 0$) and the absence ($\gamma = 0$) of the spontaneous emission. One can see from these figures that $g^{(2)}_{a,b}(0)$ decreases, as r_a increases in both cases. It can be observed from Fig. 8 that the second-order correlation function vanishes for $r_a < 0.01$. Moreover, the effect of the spontaneous emission increases the second-order correlation function.

7. Conclusion

In this paper we have studied the squeezing and entanglement properties of the light generated by three-level atoms available in an open cavity and pumped to the top level by the electron bombardment at a constant rate. Applying the large-time approximation scheme, we have obtained the steady-state solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators.

Making the use of the steady-state solutions of atomic and cavity mode operators, the quadrature variance, quadrature squeezing, and entanglement for the two-mode cavity light, at the steady state, are determined. In addition, the normalized second-order correlation function is obtained for a superposition of the two modes. It is found that the squeezing and entanglement in the two-mode light are directly related to each other. As a result, an increase in the degree of squeezing directly leads to an increase in the degree of entanglement and vice versa. This shows that, whenever there is the squeezing in the two-mode light, there exists an entanglement in the system. In addition, it is shown that the photons in the laser cavity are highly correlated, and the degree of photon number correlation increases with the spontaneous emission decay constant, γ. Therefore, the presence of the spontaneous emission leads to an increase in the photon number correlation.

1. M.O. Scully, K. Wodkiewicz, M.S. Zubairy, J. Bergou, N. Lu, J. Meyer ter Vehn. Two-photon correlated-spontaneous-emission laser. Quantum noise quenching and squeezing. *Phys. Rev.* **60**, 1832 (1988).
2. F. Kassahun. *Fundamental of Quantum Optics* (Lulu, 2010).
3. N. Lu, F. Zhao, J. Bergou. Nonlinear theory of a two-photon correlated-spontaneous-emission laser: A coherently pumped two-level-two-photon laser. *Phys. Rev. A* **39**, 5189 (1989).
4. N.A. Ansari. Effect of atomic coherence on the second and higher-order squeezing in a two-photon three-level cascade atomic system. *Phys. Rev. A* **48**, 4686 (1993).
5. Y. Tewodros. Darge and Fesseha Kassahun. Coherently driven degenerate three-level laser with parametric amplifier. *PMC Physics B* 1 (2010).
6. E. Alebachew, K. Fesseha. A degenerate three-level laser with a parametric amplifier. *Opt. Commun.* **265**, 314 (2006).
7. The Quantum Analysis of Nonlinear Optical Parametric Processes with Thermal Reservoirs. *Int. J. Opt.* **2020**, 11 (2020).
8. N.A. Ansari, J. Gea-Banacloche, M.S. Zubairy. Phase-sensitive amplification in a three-level atomic system. *Phys. Rev. A* **41**, 5179 (1990).
9. F. Kassahun. Three-level laser dynamics with the atoms pumped by electron bombardment. https://arXiv:1105.1438v3 quant-ph (2012).
10. Fesseha Kassahun. *Refined Quantum Analysis of Light, Revised Edition* (Create Space Independent Publishing Platform, 2016).
11. C.A. Blockely, D.F. Walls. Intensity fluctuations in a frequency down-conversion process with three-level atoms. *Phys. Rev. A* **43**, 5049 (1991).
12. K. Fesseha. Three-level laser dynamics with squeezed light. *Phys. Rev.* **63**, 33811 (2001).
13. K. Fesseha. Stimulated emission by two-level atoms pumped to the upper level. *Opt. Commun.* **284**, 1357 (2011).
14. K. Fesseha. The commutation relation for cavity mode operators. ArXiv.1611.01003v2 quant-Ph 29 March (2018).
15. Tamirat Abebe. Coherently driven nondegenerate three-level laser with noiseless vacuum reservoir. *Bulg. J. Phys.* **45**, 357 (2018).
16. S. Tesfa. Entanglement amplification in a nondegenerate three-level cascade laser. *Phys. Rev. A* **74**, 043816 (2006).
17. Menisha Alemu. Spontaneous emission by three-level atoms pumped by electron bombardment. *Universal J. Phys. Appl.* **14** (1), 11 (2020).
18. C.G. Feyisa, T. Abebe, N. Gemechu. Continuous-variable entanglement adjustable by phase fluctuation and classical pumping field in a correlated emission laser seeded with squeezed light. *J. Russ. Laser. Res.* **41**, 563 (2020).
19. M.O. Scully, M.S. Zubairy. *Quantum Optics* (Cambridge University Press, 1997).
20. C. Gerry, P.L. Knight. *Introductory Quantum Optics* (Cambridge University Press, 2005).
21. T. Abebe. Enhancement of squeezing and entanglement in a non-degenerate three-level cascade laser with coherently driven cavity. *Ukr. J. Phys.* **63**, 733 (2018).
22. Swapan Mandal. On the possibilty of continuous generation of squeezed states in a quartic anharmonic oscillator. *J. Phys. B: Atomic, Molecular and Optical Phys.* **33**, 1029 (2000).
23. Swapan Mandal. Squeezing, higher order squeezing, photon bunching and photon antibunching in a quadratic Hamiltonian. *Modern Phys. Lett. B* **16**, 963 (2002).
24. N. Lu, S. Zhu. Quantum theory of two-photon correlated spontaneous-emission lasers. Exact atom-field interaction Hamiltonian approach. *Phys. Rev. A* **40**, 5735 (1989).
25. R.E. Slusher, L.W. Hollberg, B. Yurke, J.C. Mertz, J.F. Valley. Observation of squeezed states generated by four-wave mixing in an optical cavity. *Phys. Rev. Lett.* **55**, 2409 (1985).
26. E. Alebachew. A coherently driven two-level atom inside a parametric oscillator. *J. Mod. Opt.* **55**, 1159 (2008).
27. E. Alebachew. Bright entangled light from two-mode cascade laser. *Opt. Commun.* **281**, 6124 (2008).

28. D. Ayehu. Squeezing and entanglement properties of the cavity light with decoherence in a cascade three-level laser. *J. Russ. Laser Res.* **42**, 136 (2021).

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ДИНАМІКА ПЕРЕПЛУТУВАННЯ, ІНДУКОВАНА ТРИРІВНЕВИМ ЛАЗЕРОМ, ЩО ВЗАЄМОДІЄ З КВАНТОВИМ РЕЗЕРВУАРОМ В УМОВАХ СТИСКАННЯ

Вивчаються властивості стискання та переплутування світла, індукованого трирівневим лазером, що взаємодіє з вакуумним резервуаром в умовах стискання. Отримано рівняння еволюції змінних, які характеризують моди випромінювання із порожнини лазера, та розраховано кореляційну функцію другого порядку для цього випромінювання. Доведено, що світло лазера знаходиться у стиснутому стані, і стискання відбувається з позитивною квадратурою. Знайдено, що стискання визначає зростання середнього значення і дисперсії числа фотонів. Показано, що стискання і переплутування для двомодового світла взаємозалежні.

К л ю ч о в і   с л о в а: динаміка операторів, статистика фотонів, квадратурне стискання, кореляції другого порядку, переплутування фотонів.