POWER SPECTRUM OF RADIATION FROM A GAUSSIAN SOURCE MICROLENSED BY A POINT MASS: ANALYTIC RESULTS

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Gravitational lensing deals with general-relativistic effects in the propagation of electromagnetic radiation. We consider wavelength-dependent contributions in case of a (micro)lensing of an extended Gaussian source by a point mass under standard assumptions about the incoherent emission of different source elements. Analytical expressions for the power spectrum of a microlensed radiation, which are effective in case of a large source, are obtained. If the source center, the mass, and an observer are on a straight line, the power spectrum is found in a closed form in terms of a hypergeometric function. In the case of general locations of the lens and the source, the result is presented in the form of a series. Approximate analytic expressions for the power spectrum in the case of a large source and high frequencies are obtained.

1. Introduction

Effects of the propagation of electromagnetic radiation in a curved space-time, which is described within the General Relativity, is referred to as the gravitational lensing. This direction has an important meaning for investigations of mass distributions in the Universe, for probing the planetary and solar-mass objects in the Milky Way and for investigations of the innermost structures of quasars [1, 13, 23, 27, 29, 30]. One of the most vivid examples of gravitational lensing applications is the confirmation of the dark matter existence in the Bullet Cluster [7]. The term "microlensing" is used when the radiation from a remote source is deflected by the gravitational fields of stellar and/or planetary mass objects [13, 23, 29].

All the effects observed so far in the gravitational lens systems (GLSs) are well described within geometric optics approximations, none of these problems deals with the wave optics. The detection of wavelength-dependent effects in GLSs could open the way to entirely new tests of electrodynamics in a curved space-time. On the other hand, the effects of the physical optics might give an independent information about the lensing objects. Related problems have been studied theoretically starting from the early papers [8, 16, 24]; for the later progress see [12, 18–21, 31, 32]; a complete bibliography can be found in monographs [3, 22, 23].

All the papers deal with different theoretical aspects of the wave optics in weak gravitational fields; there are no observational results. The reason is that the wave effects in GLSs are either very small or very improbable. Nevertheless, Heyl [10, 11] turned attention recently that the wave optics effects having a gravitational origin can be measured in some future in the case of the microlensing by planetary masses and planetesimals. A number of authors (see, e.g., [21, 26] and references therein) study diffractive effects in GLSs in connection with the expected detection of gravitational waves.

Observational signatures of the microlensed radiation from real sources are blurred because of coherence properties; as those in the usual optics, they strongly depend on the size of an extended source. The diffractive microlensing of extended sources has been studied in caustic crossing events [12, 31] and in the case of point mass lensing [21]; the mutual coherence of different lensed images of an extended source near caustics has been estimated in a series of papers by Mandzhos [18–20]. It should be pointed out that, in GLSs, the diffraction effects go side by side with the interference between different images of every source point. The occurrence of different images leads to additional maxima or deformations of the autocorrelation function of the microlensed radiation (see, e.g., [4–6, 33]). Both diffraction and interference in GLSs can be considered on an equal basis by means of a power spectrum of the microlensed radiation.

Most researches on this problem involve numerical calculations. Nevertheless, it is desirable to have analytic results at least for some simple problems. In this paper, we found such a result in the case of a Gaussian source; this result has been unknown earlier in spite of rather a long history of investigations in this field. We derive the power spectrum of the radiation from a Gaussian source, which is microlensed by one point mass under standard assumptions about incoherent source elements. In Section 2, the radiation field is obtained in a standard way using the Kirchhoff integral. In Section 3 and Appendix A, we derive a closed relation in terms of a hypergeometric function for the power spectrum, when the lensing mass is projected onto the center of the source. In Section 4, we use this relation in the case of a general source disposition to obtain approximations for the power spectrum for a sufficiently small mass and a large source. As distinct from the earlier method used in this problem [21], our analytic approach is convenient in the case of a sufficiently large source size as compared to the Einstein radius projected onto the source plane. We also present the first terms of an expansion in powers of $1/\omega$ in the high-frequency case.

2. Basic relations

In this section, we formulate common relations used below. Following [8, 21] and many other authors, we use standard considerations of the diffraction theory and the gravitational lensing (see, e.g., [3, 23]); respectively, we neglect the polarization. The calculations of the field are performed in the flat space-time background. This is relevant, e.g., in the case of the Milky Way systems; however, the results can be easily extended to the case of extragalactic GLSs after some redefinition of distances in a curved space-time.

Leaving aside the polarization effects, we describe the radiation field by means of one scalar function $\varphi(t, \mathbf{r})$. Furthermore, we work in the Cartesian coordinates; the observer, the lens, and the source center are situated in a neighborhood of the Z-axis in the planes $z = 0$, $z = D_d$ and $z = D_s$, respectively. As we neglect the polarization, we describe the source "current" with a scalar function $\mathrm{j}(t, \mathbf{y})$, by assuming that this source lies completely in the plane $\mathbf{r} = (\mathbf{y}, D_s)$, $\mathbf{y} \in \mathbf{R}^2$. We also assume that $\mathrm{j}(t, \mathbf{y})$ is a stochastic process having the correlation properties

$$\langle \mathrm{j}(t, \mathbf{r})\mathrm{j}(t', \mathbf{r}')\rangle = \delta(\mathbf{y} - \mathbf{y}')I(t - t', \mathbf{y}); \tag{1}$$

this relation presumes that different points of the source are incoherent; $\langle \ldots \rangle$ represent an ensemble average.

For the Fourier transform $\tilde{\mathrm{j}}(k, \mathbf{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{j}(t, \mathbf{y}) \times$ $\times e^{i\omega t} dt$, we have

$$\langle \tilde{\mathrm{j}}(\omega, \mathbf{y})\tilde{\mathrm{j}}^*(\omega', \mathbf{y}')\rangle = \delta(\mathbf{y} - \mathbf{y}')\delta(\omega - \omega')\,\tilde{I}(\omega, \mathbf{y}), \tag{2}$$

where $\tilde{I}(\omega, \mathbf{y}) = \int dt\, I(t, \mathbf{y})e^{i\omega t}$ is an intensity of an element at a point $\mathbf{y}$ for the frequency ω.

The Helmholtz equation for $\tilde{\varphi}(\omega, \mathbf{r})$ is

$$\Delta\tilde{\varphi} + \omega^2\tilde{\varphi} = -4\pi\tilde{j} \quad (c = 1).$$

Consider a solution $\tilde{\varphi}(\omega, \mathbf{r})$ of this equation describing the radiation from the source plane. Let $D_{ds} = D_s - D_d$ be the distance from the lens plane to the source plane; the source is situated near the Z-axis: $|\mathbf{y}| \ll D_{ds}$. Before the lens plane $\mathbf{r}' = (\mathbf{y}', D_d + 0)$ near the Z-axis ($|\mathbf{y}'| \ll D_{ds}$), the solution is obtained as that in the flat-space diffraction theory [14, 15]:

$$\tilde{\varphi}_b(\omega, \mathbf{y}') =$$

$$= \frac{e^{i\omega D_{ds}}}{D_{ds}} \int d^2\mathbf{y}\; \tilde{\mathrm{j}}(\omega, \mathbf{y}) \exp\left[\frac{i\omega}{2D_{ds}}(\mathbf{y}' - \mathbf{y})^2\right]. \tag{3}$$

In order to calculate the field after passing the lens plane, we use the well-known approach of the phase screen (see, e.g., [3, 22, 23]), by assuming that the radiation gains an additional phase shift ωt_{grav} on the lens plane, where $t_{\mathrm{grav}}(\mathbf{y}')$ is the general relativistic time delay for signals crossing the lens plane at $\mathbf{y}'$. This delay is the same for all frequencies. Then the solution just after passing the lens plane reads

$$\tilde{\varphi}_a(\omega, \mathbf{y}') = e^{-i\omega t_{\mathrm{grav}}(\mathbf{y}')}\tilde{\varphi}_b(\omega, \mathbf{y}').$$

The radiation comes to an observer at the origin $\mathbf{r} = \mathbf{0}$. The field $\tilde{\varphi}(\omega, \mathbf{0})$ is calculated by means of the Kirchhoff–Sommerfeld method [15]:

$$\tilde{\varphi}(\omega, \mathbf{0}) = \frac{\omega e^{i\omega D_d}}{2\pi i\, D_d} \int d^2\mathbf{y}' \tilde{\varphi}_a(\omega, \mathbf{y}') \exp\left[\frac{i\omega}{2D_d}\mathbf{y}'^2\right] =$$

$$= \frac{e^{i\omega D_s}}{2i\, D_d D_{ds}} \int d^2\mathbf{y}\, \tilde{\mathrm{j}}(\omega, \mathbf{y})\phi(\omega, \mathbf{y}), \tag{4}$$

where

$$\phi(\omega, \mathbf{y}) =$$

$$= \frac{\omega}{\pi} \int \exp\left\{ i\omega \left[\frac{(\mathbf{x}-\mathbf{y})^2}{2D_{ds}} + \frac{\mathbf{x}^2}{2D_d} - t_{\rm grav}(\mathbf{x}) \right] \right\} d^2\mathbf{x} \,. \quad (5)$$

Using (2, 4), we obtain

$$\langle \tilde{\varphi}(\omega, \mathbf{0}) \tilde{\varphi}^*(\omega', \mathbf{0}) \rangle = \delta(\omega - \omega') P(\omega)$$

with the power spectrum

$$P(\omega) = \left(\frac{1}{2\,D_d D_{ds}} \right)^2 \int d^2\mathbf{y}\, \tilde{I}(\omega, \mathbf{y})\, |\phi(\omega, \mathbf{y})|^2 \,. \quad (6)$$

3. Central Gaussian Source and Point Lensing Mass

Relation (6) is written for arbitrary sources and gravitational time delays. Further, we consider the case of one point microlens at $\mathbf{r} = (\mathbf{0}, D_d)$ with the delay time (e.g., [3, 22, 23])

$$t_{\rm grav}(\mathbf{y}) = 2r_g \ln(|\mathbf{y}|/L), \quad r_g = 2Gm; \quad (7)$$

here, m is the microlens mass, L is a dimensional parameter which disappears in final calculations; further, it is omitted.

We assume the Gaussian brightness distribution over a source for the intensity $\tilde{I}$ from Eq. (2):

$$I(\omega, \mathbf{y}, \mathbf{r}_0) = \frac{f(\omega)}{\pi R^2} \exp\left(-\frac{(\mathbf{y}-\mathbf{r}_0)^2}{R^2} \right). \quad (8)$$

Here, $\mathbf{r}_0$ is the source center in the source plane, the function $f(\omega)$ is supposed to be the same for all source points and determines the coherence properties of emitting source elements.

Integral (5) can be calculated exactly [3, 8, 21] in terms of the confluent hypergeometric function $\Phi(a, c; x)$ [2]:

$$\phi(\omega, \mathbf{y}) =$$

$$= \frac{\omega}{\pi} \int \exp\left\{ i\omega \left[\frac{(\mathbf{x}-\mathbf{y})^2}{2D_{ds}} + \frac{\mathbf{x}^2}{2D_d} - 2r_g \ln |\mathbf{x}|) \right] \right\} d^2\mathbf{x} =$$

$$= \omega^{i\sigma} e^{\frac{i\omega \mathbf{y}^2}{2D_s}} \Gamma(1 - i\sigma) \left(\frac{2D_{ds} D_d i}{D_s} \right)^{1 - i\sigma} \times$$

$$\times \Phi\left(i\sigma, 1;\ i\sigma y^2 / R_{E,s}^2 \right), \quad (9)$$

where we used the Kummer transformation [2]; $\sigma = \omega r_g$, $R_{E,s} = [2r_g D^*]^{1/2}$ is the Einstein radius projected onto the source plane, and $D^* = D_{ds} D_s / D_d$; $y = |\mathbf{y}|$.

For the Gaussian source, the power spectrum (6) is as follows:

$$P(\omega, \mathbf{r}_0) = \left(\frac{1}{2\,D_d D_{ds}} \right)^2 \frac{f(\omega)}{\pi R^2} \times$$

$$\times \int d^2\mathbf{y} \exp\left[-\frac{(\mathbf{y}-\mathbf{r}_0)^2}{R^2} \right] |\phi(\omega, \mathbf{y})|^2 \,. \quad (10)$$

The microlensing effect can be described by the ratio

$$\Upsilon = P(\omega, \mathbf{r}_0) / P_0(\omega), \quad (11)$$

where P_0 is the power spectrum in the absence of the microlensing ($\sigma = 0$).

If the source center is at the origin ($\mathbf{r}_0 = 0$), integral (10) can be estimated in terms of hypergeometric functions [28]; the derivation (see Appendix A) yields $\Upsilon = \Upsilon_0(\alpha, \sigma)$, where

$$\Upsilon_0(\alpha, \sigma) \equiv \exp[2\sigma \arctan(\beta)]\, |\Gamma(1 - i\sigma)|^2 \times$$

$$\times F\left(i\sigma, -i\sigma;\, 1\,; (1 + \beta^2)^{-1} \right); \quad (12)$$

$\beta = \alpha/\sigma$, $\alpha = R_{E,s}^2 / R^2$, $R > 0$; and $F(a, b; c; x)$ is the hypergeometric function. Note that $\sqrt{\beta} = R^{-1}\sqrt{\lambda D^* / \pi}$ plays the role of the ratio of the Fresnel zone size to the source size.

For large $\alpha \gg 1$ and $\sigma \sim O(1)$, the argument of the hypergeometric function is small, and we have $\Upsilon_0(\alpha, \sigma) \approx 2\pi\sigma$.

For $\alpha \ll 1$, i.e., $R_{E,s} \ll R$, and bounded $\sigma \sim O(1)$, it is convenient to use the expansion of the hypergeometric function $F(a, b; c; x)$ near the point $x = 1$ for an integer c. Then formula (12) takes the form

$$\Upsilon_0(\alpha, \sigma) = \exp[2\sigma \arctan(\beta)] \times$$

$$\left\{ 1 - \sigma^2 s \sum_{n=0}^{\infty} \left| \frac{\Gamma(n + 1 + i\sigma)}{\Gamma(1 + i\sigma)} \right|^2 \frac{s^n [k_n(\sigma) - \ln s]}{(n + 1)(n!)^2} \right\}, \quad (13)$$

where

$$k_n(\sigma) = 2\psi(n + 1) - \psi(n + 1 + i\sigma) - \psi(n + 1 - i\sigma) +$$

$$+ \frac{1}{n + 1},$$

$$s = \alpha^2 / (\alpha^2 + \sigma^2), \quad \psi(x) \equiv d \ln \Gamma(x) / dx.$$

We write the expansion in α up to the terms $\sim \alpha^2$ and $\alpha^2 \ln\alpha$ that depend on the frequency. In this approximation, Eq. (13) can be written as

$$\Upsilon_0(\alpha,\sigma) = 1 + 2\alpha + 2\alpha^2 - \alpha^2[k_0 - 2\ln(\alpha/\sigma)] \quad . \quad (14)$$

Expansion (13) can be rewritten in the form, which is convenient to look for the asymptotic expansions at large frequencies:

$$\Upsilon_0(\alpha,\sigma) = \exp[2\sigma \arctan(\alpha/\sigma)]\times$$

$$\times\left\{1 - \sum_{n=0}^{\infty} C_n(\sigma) \frac{\tilde{s}^{n+1}[\tilde{k}_n(\sigma) - \ln\tilde{s}]}{(n+1)(n!)^2}\right\}, \quad (15)$$

where $\tilde{s} = \sigma^2 s = \alpha^2/(1+\beta^2)$,

$\tilde{k}_n(\sigma) = k_n(\sigma) + 2ln(\sigma)$,

$$C_n(\sigma) = \left|\frac{\Gamma(n+1+i\sigma)}{\Gamma(1+i\sigma)\sigma^n}\right|^2 = \prod_{m=0}^{n}\left(1 + \frac{m^2}{\sigma^2}\right). \quad (16)$$

Using the asymptotic relations for the function ψ (e.g., [2]) at large arguments, the coefficients $\tilde{k}_n(\sigma)$ can be expanded in powers of σ^{-2}. Therefore, we can write

$$\Upsilon_0(\alpha,\sigma) = \sum_{n=0}^{m} \sigma^{-2n}\Upsilon_0^{(n)}(\alpha) + O(\sigma^{-2(m+1)}). \quad (17)$$

Up to the terms $\sim \sigma^{-2}$, we have

$$\tilde{k}_n(\sigma) = \tilde{k}_n(\infty) - \frac{1}{\sigma^2}\left[n(n+1) + \frac{1}{6}\right] + O(\sigma^{-4}), \quad (18)$$

$\tilde{k}_n(\infty) = 2\psi(n+1) + (n+1)^{-1}$. It follows from (16) that

$$C_n(\sigma) = 1 + \frac{n(n+1)(2n+1)}{6\,\sigma^2} + ...$$

Then we have the geometric optics limit ($\sigma \equiv \omega r_g \to \infty$)

$$\Upsilon_0(\alpha,\infty) = \Upsilon_0^{(0)}(\alpha) =$$

$$= \exp(2\alpha)\left\{1 - \alpha^2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}[\tilde{k}_n(\infty) - 2\ln\alpha]}{(n+1)(n!)^2}\right\} =$$

$$= 2\alpha e^{2\alpha} K_1(2\alpha)\,, \quad (19)$$

where we used a representation for the modified Bessel function K_1 (see formula 8.446 in [9]). A direct calculation within geometric optics (Appendix B) is in accordance with this expression.

For the next term of the expansion, we have

$$\Upsilon_0^{(1)}(\alpha) = -\alpha^2 e^{2\alpha}\left\{\frac{2}{3}\alpha + \sum_{n=0}^{\infty} \frac{C_n^{(1)}(\alpha)\alpha^{2n}}{(n+1)(n!)^2}\right\},$$

$$C_n^{(1)}(\alpha) = \alpha^2 - n(n+1) - \frac{1}{6} +$$

$$+ \left(\tilde{k}_n(\infty) - 2\ln\alpha\right)\left[\frac{n}{6}(n+1)(2n+1) - \alpha^2(n+1)\right] .$$

4. Non-Central Source

We now proceed to the derivation of Υ in the case of a non-central source, by using Eqs. (12)–(14). The integration over the angular variable yields

$$P(\omega,\mathbf{r}_0) = \left(\frac{1}{2\,D_d D_{ds}}\right)^2 \frac{2f(\omega)}{R^2} \exp\left(-\frac{\mathbf{r}_0^2}{R^2}\right)\times$$

$$\times \int_0^{\infty} dy\, y \exp\left(-\frac{y^2}{R^2}\right) I_0\left(\frac{2yr_0}{R^2}\right) |\phi(\omega,\mathbf{y})|^2\,, \quad (20)$$

where $r_0 = |\mathbf{r}_0|$. Using the expansion of the modified Bessel function I_0 in a Taylor series and the substitution $y^2 = tR_{E,s}^2$, we get

$$\Upsilon(\alpha,\sigma,r_0) = \alpha \exp\left(-\frac{r_0^2}{R^2} + \pi\sigma\right) |\Gamma(1-i\sigma)|^2 \times$$

$$\times \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^2}\left(\frac{r_0}{R}\right)^{2n} \int_0^{\infty} dt\; e^{-\alpha t} t^n \,|\Phi\,(\,i\sigma, 1;\; i\sigma t)|^2\,. \quad (21)$$

Using the differentiation with respect to the parameter α and taking Eqs. (9) and (12) into account, we obtain

$$\Upsilon(\alpha,\sigma,r_0) = \exp\left(-\frac{r_0^2}{R^2}\right)\times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{n+1}}{(n!)^2}\left(\frac{r_0}{R}\right)^{2n} \frac{\partial^n}{\partial\alpha^n}\left[\frac{\Upsilon_0(\alpha,\sigma)}{\alpha}\right]. \quad (22)$$

Obviously, this representation is workable when r_0/R is not large; this will be supposed further. This excludes, e.g., the case of a point source.

The simple analysis shows that, to obtain the terms up to the orders of α^2 and $\alpha^2 \ln\alpha$ in (22), it is sufficient to use the truncated expression (14). We get

$$\Upsilon(\alpha,\sigma,r_0) = 1 + 2\alpha e^{-r_0^2/R^2} + \alpha^2 e^{-r_0^2/R^2}\left\{2g\left(\frac{r_0^2}{R^2}\right) - -2\frac{r_0^2}{R^2} + \left(1-\frac{r_0^2}{R^2}\right)\left[2 - k_0(\sigma) - 2\ln(\sigma/\alpha)\right]\right\}, \quad (23)$$

where

$$g(x) = \sum_{n=2}^{\infty} \frac{x^n}{n\,(n-1)\,n!} =$$

$$= (x-1)[Ei(x) - C - \ln x] - e^x + 1 + 2x,$$

$Ei(x)$ is the integral exponent, and C is the Euler constant.

We now obtain the asymptotic expansion for $\Upsilon(\alpha,\sigma,r_0)$ at large frequencies. The substitution of (17) into (22) leads to the asymptotic series in powers of σ^{-2}:

$$\Upsilon(\alpha,\sigma,r_0) = \sum_{m}^{M} \sigma^{-2m}\Upsilon^{(m)}(\alpha,r_0) + O(\sigma^{-2(M+1)}) \quad (24)$$

with the coefficients

$$\Upsilon^{(m)}(\alpha,r_0) = \exp\left(-\frac{r_0^2}{R^2}\right)\times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{n+1}}{(n!)^2}\left(\frac{r_0}{R}\right)^{2n} \frac{\partial^n}{\partial\alpha^n}\left[\frac{\Upsilon_0^{(m)}(\alpha)}{\alpha}\right]. \quad (25)$$

Some caution is needed when we differentiate the asymptotic expansion (17) term by term; this can be easily justified in view of the explicit structure of the functions involved. The geometric optics limit yields

$$\Upsilon(\alpha,\infty,r_0) = \Upsilon^{(0)}(\alpha,r_0), \quad (26)$$

where $\Upsilon^{(0)}(\alpha,r_0)$ is given by (25) for $m=0$. The direct calculation of the amplification using the standard gravitational lensing theory yields the same result (see Appendix B).

The coefficient $\Upsilon^{(1)}(\alpha)$, which describes the first correction to (26) with an accuracy up to $\sim \alpha^2$, can be obtained either from (25) or from (23):

$$\Upsilon^{(1)}(\alpha) = \frac{1}{6}\alpha^2 \exp\left(-\frac{r_0^2}{R^2}\right)\left(1-\frac{r_0^2}{R^2}\right). \quad (27)$$

5. Discussion

We have obtained the analytic expressions for the radiation power spectrum for an extended Gaussian source microlensed by a point mass under standard assumptions about the incoherence of different source elements. Our results allow one to treat the cases of wide and narrow bandwidth receptions on the same basis by an appropriate choice of the function $f(\omega)$ in Eq. (8). If the source center, the lensing mass, and the observer are situated on one straight line, the power spectrum is given by Eq. (12) in terms of a hypergeometric function. In the case of a general arrangement, the result is presented in the form of the functional series (22). This representation of the power spectrum enables us to derive approximations up to any accuracy in the case of sufficiently small $\alpha = (R_{E,s}/R)^2$. The representation is efficient under the condition that the distance r_0 between the lens projection onto the source plane and the source center is comparable/less than the source size. The opposite case ($r_0 \gg R$) can be treated by the method developed in [21], where the expansion of the magnification around the source position is used. Representations (12) and (22) have been used to derive the lowest orders of the asymptotic expressions (23) and (27) in the cases of a small lens and high frequencies. We have shown by means of a direct calculation that the high frequency limit yields exactly the expressions of the geometric optics. As we see from (14) and (27), the first nontrivial wave optics contribution, which is dependent on the frequency, appears in the terms $\sim \alpha^2$ (though the first geometric optics lensing contribution has order $\sim \alpha$); it disappears for very large sources. At high frequencies, this contribution behaves itself as $\sim \omega^{-2}$.

In what follows, we propose some estimates for the radio waveband; though much smaller wavelengths also can be of interest (cf., e.g., [25]). For the effects of the wave optics to be significant, one must have $\sigma = r_g\omega \sim 1$ and $\alpha \sim 1$. For a wavelength $\sim$ 1–10 cm, the first condition is fulfilled if the microlens mass is of the order of $10^{-5}M_\odot$. Such planetary mass objects must be common in the Milky Way, and there is a number of observational confirmations of exosolar planets, including microlensing observations (e.g., [29]). Heyl [10] pointed out that the signatures of the diffractive gravitational microlensing can be detected during the occultation of distant stars by Kuiper-Belt and Oort cloud objects. However, in this case, the treatment must be modified by the introduction of an opaque screen that describes the lensing object (cf. [3]).

In order that α be not small, the source size R must be of the order or less than $R_{E,s}$. This is true for a wide interval of D^* in the case of the microlensing by the Milky Way objects; however, in this case, the probability of the microlensing of a suitable radio source is small (see, however, [11]). For a distant extragalactic source at $D_s \approx D_{ds} \sim 10^3$ Mpc microlensed by a planet at $D_d = 10$ kpc, we have $R_{E,s} \sim 10^{-2}[M/(10^{-5}M_\odot)]^{1/2}$ pc. The typical size of extragalactic radio sources is larger, though it may have inhomogeneous structures with typical scales $\sim R_{E,s}$. In this case, the wavelength-dependent effects of gravitational lensing will be noticeable.

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APPENDIX A.
Derivation of Eq. (12) in the case of a central source

From Eq. (10), we have

$$|\phi(\omega, \mathbf{y})|^2 = |\Gamma(1-i\sigma)|^2 \left(\frac{2D_{ds}D_d}{D_s}\right)^2 \times$$

$$\times e^{\sigma\pi} \left|\Phi\left(i\sigma, 1;\ i\sigma y^2/R_{E,s}^2\right)\right|^2. \tag{28}$$

For $r_0 = 0$, the power spectrum (10) equals

$$P(\omega, 0) = \left(\frac{1}{2D_dD_{ds}}\right)^2 \frac{2f(\omega)}{R^2} \int\limits_0^\infty dy\, y \exp\left(-\frac{y^2}{R^2}\right) |\phi(\omega, y)|^2, \tag{29}$$

where the integration over the angular variable is performed. With regard for (28) and (29) and by using the substitution $t = y^2$, we obtain ratio (11) corresponding to $\mathbf{r}_0 = 0$:

$$\Upsilon_0(\alpha, \sigma) =$$

$$= e^{\sigma\pi} |\Gamma(1-i\sigma)|^2 \int dt\, e^{-t} |\Phi(i\sigma, 1;\ i\sigma t/\alpha)|^2, \tag{30}$$

where $\alpha = R_{E,i}^2/R^2$. The integral in formula (30) is a special case of the expression that can be obtained by means of formula 6.15.22 in [2]. Here, in order to estimate (30), we calculate directly the integral

$$I(a, \lambda, a', \lambda') = \int\limits_0^\infty dt\, e^{-t} \Phi(a, 1;\ \lambda t)\, \Phi(a', 1;\ \lambda' t), \tag{31}$$

where $0 < \mathrm{Re}(a) < 1$, $0 < \mathrm{Re}(a') < 1$, $|\lambda'| + |\lambda| < 1$. With this aim, we need representations for the confluent hypergeometric function

$$\Phi(a, 1;\ x) = \frac{1}{B(a, 1-a)} \int\limits_0^1 du\, \frac{e^{xu} u^{a-1}}{(1-u)^a} \tag{32}$$

and for the hypergeometric function

$$F(a, b; 1;\ x) = \frac{1}{\Gamma(b)\Gamma(1-b)} \int\limits_0^1 du\, \frac{t^{b-1}}{(1-zt)^a (1-t)^b} =$$

$$= \frac{1}{B(b, 1-b)} \int\limits_0^\infty d\tau\, \frac{\tau^{b-1}(1+\tau)^{a-1}}{[1+(1-z)\tau]^a}, \tag{33}$$

where $B(a, b)$ is the Beta-function, and $t = \tau/(1+\tau)$.

We use (32) for $\Phi(a, 1;\ \lambda t)$ and $\Phi(a', 1;\ \lambda' t)$ in (31) and integrate over t after the change of the integration order:

$$I(a, \lambda, a', \lambda') = \frac{1}{B(a, 1-a) B(a', 1-a')} \times$$

$$\times \int\limits_0^1 du\, u^{a-1}(1-u)^{-a} \int\limits_0^1 dv\, \frac{v^{a'-1}(1-v)^{-a'}}{1-\lambda u - \lambda' v}.$$

The integration over dv after the substitution $v \to \xi$ ($v = \tau/(1+\tau)$ and $\tau = \xi(1-\lambda u)/(1-\lambda-\lambda u)$) is easily carried out with the use of an integral representation for the Beta-function.

Then the substitution $u = \eta/(1-\lambda+\eta)$ yields

$$I(a, \lambda, a', \lambda') =$$

$$= \frac{(1-\lambda)^{-a}(1-\lambda')^{-a'}}{B(a, 1-a)} \int\limits_0^\infty d\eta\, \frac{\eta^{a-1}(1+\eta)^{a'-1}}{[1+(1-z)\eta]^{a'}},$$

where $z = \lambda\lambda'(1-\lambda)^{-1}(1-\lambda')^{-1}$. Therefore, in view of representation (33), we get the final relation

$$I(a, \lambda, a', \lambda') = \frac{F(a', a;\ 1;\ z)}{(1-\lambda)^a (1-\lambda')^{a'}}. \tag{34}$$

All calculations are fulfilled in the domain of the parameters a, λ, a', λ', where the integrals involved are convergent. However, relation (34) can be analytically continued to a wider domain, which includes the values $a = i\sigma$, $a' = -i\sigma$, $\lambda = i\sigma/\alpha$, $\lambda' = -i\sigma/\alpha$. This yields Eq. (12) of the main text.

APPENDIX B.
Gaussian source amplification in geometric optics

The amplification of a point source by a point mass lens within the geometric optics in variables of the source plane is well known [3, 23]:

$$\mu(y) = \frac{y^2 + 2R_{E,s}^2}{y\sqrt{y^2 + 4R_{E,s}^2}}. \tag{35}$$

In our case, this must be convolved with the Gaussian brightness distribution (8) yielding the total amplification

$$A = \frac{1}{\pi R^2} \int d^2\mathbf{y}\, \mu(y) e^{-(\mathbf{y}-\mathbf{r_0})^2/R^2} =$$

$$= \frac{2}{R^2} e^{-\frac{r_0^2}{R^2}} \int\limits_0^\infty dy\, y\, \mu(y) e^{-\frac{y^2}{R^2}} I_0\left(\frac{2yr_0}{R^2}.\right)$$

Taking (35) into account and using the Taylor expansion of I_0 and the substitution $y^2 = tR_{E,s}^2$, we have

$$A = A(\alpha, r_0) =$$

$$= e^{-\frac{r_0^2}{R^2}} \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{(n!)^2} \left(\frac{r_0}{R}\right)^{2n} \int_0^\infty dy\, e^{-\alpha t} t^n \frac{t+2}{\sqrt{t^2+4t}} =$$

$$= e^{-\frac{r_0^2}{R^2}} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{n+1}}{(n!)^2} \left(\frac{r_0}{R}\right)^{2n} \frac{\partial^n}{\partial \alpha^n} \int_0^\infty dy\, e^{-\alpha t} \frac{t+2}{\sqrt{t^2+4t}}. \quad (36)$$

The latter integral can be written [9] in terms of the modified Bessel function K_1:

$$A(\alpha, 0) = \alpha \int_0^\infty dt\, e^{-\alpha t} \frac{t+2}{\sqrt{t^2+4t}} = 2\alpha e^{2\alpha} K_1(2\alpha). \quad (37)$$

This is the same as $\Upsilon_0(\alpha, \infty)$ of (19).

For an arbitrary source position, relation (36) yields

$$A(\alpha, r_0) = e^{-\frac{r_0^2}{R^2}} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{n+1}}{(n!)^2} \left(\frac{r_0}{R}\right)^{2n} \frac{\partial^n}{\partial \alpha^n} \left[\frac{A(\alpha, 0)}{\alpha}\right], \quad (38)$$

which is the same as $\Upsilon(\alpha, \infty, r_0)$ of (26).

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СПЕКТР ПОТУЖНОСТІ ВИПРОМІНЮВАННЯ ВІД ГАУСІВСЬКОГО ДЖЕРЕЛА, МІКРОЛІНЗОВАНОГО ТОЧКОВОЮ МАСОЮ: АНАЛІТИЧНІ РЕЗУЛЬТАТИ

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Р е з ю м е

Теорія гравітаційного лінзування вивчає загально-релятивістські ефекти при поширенні електромагнітного випромінювання. У даній роботі розглянуто ефекти, залежні від довжини хвилі, при (мікро)лінзуванні протяжного гаусівського джерела на точковій масі за стандартних припущень щодо некогерентності різних елементів джерела. Отримано аналітичні вирази для спектра потужності мікролінзованого випромінювання, що є ефективними за великого джерела. Коли центр джерела, маса та спостерігач розташовані на одній прямій, знайдено спектр потужності в замкненій формі через гіпергеометричну функцію. У випадку загального розташування цю величину знайдено у формі ряду. Отримано асимптотичні вирази для спектра потужності за великого розміру джерела та за високих частот.