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DIATOMIC MOLECULES WITH THE IMPROVED DEFORMED GENERALIZED DENG–FAN POTENTIAL PLUS DEFORMED ECKART POTENTIAL MODEL THROUGH THE SOLUTIONS OF THE MODIFIED KLEIN–GORDON AND SCHRÖDINGER EQUATIONS WITHIN NCQM SYMMETRIES

In this study, the deformed Klein–Gordon equation and Schrödinger equations were solved with the improved deformed generalized Deng–Fan potential plus the deformed Eckart potential (IDGDFDE-P, in short) model using Bopp's shift and standard perturbation theory methods in the symmetries of extended quantum mechanics. By employing the improved approximation to the centrifugal term, the relativistic and nonrelativistic bound-state energies are obtained for some selected diatomic molecules such as N_2, I_2, HCl, CH, LiH, and CO. The relativistic energy shift $\Delta E_{dfe}^{tot}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ and the perturbative nonrelativistic corrections $\Delta E_{dfe}^{nr}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ appeared as functions of the parameters (α,c,b,V_0,V_1,V_2) and the parameters of noncommutativity (Θ,σ,χ), in addition to the atomic quantum numbers (n,j,l,s,m). In both relativistic and nonrelativistic problems, we show that the corrections to the energy spectrum are smaller than for the main energy in the ordinary cases of RQM and NRQM. A straightforward limit of our results to ordinary quantum mechanics shows that the present results under the IDGDFDE-P model is are consistent with what is obtained in the literature. In the new symmetries of noncommutative quantum mechanics (NCQM), it is not possible to get the exact analytical solutions for $l = 0$ and $l \neq 0$. Only the approximate ones can be obtained. We have clearly shown that the Schrödinger and Klein–Gordon equations in the new symmetries can physically describe two Dirac equations and the Duffin–Kemmer equation within the IDGDFDE-P model in the extended symmetries.

Keywords: Klein–Gordon equation, Schrödinger equation, deformed generalized Deng–Fan potential, deformed Eckart potential, diatomic molecules, noncommutative geometry, Bopp's shift method, star products.

1. Introduction

The fundamental equations have been a powerful tool for researchers. This enables one to provide the means for knowing all the necessary information on matter by the wave and energy function and then to deduce all other information about it. However, these equations are considered a scientific revolution that has successfully enabled the initiative to discover more information. Since the early years of the discovery of both the relativistic Klein–Gordon and the nonrelativistic Schrödinger equations, the researchers have been keenly interested in applying them to the study of molecules using different potentials. Among these possibilities is the Morse potential [1], Yukawa potential [2], Manning–Rosen potential [3], Woods–Saxon potential [4, 5], Pöschl–Teller potential [6, 7], Deng–Fan potential [8], Hulthén potential [9, 10], etc. In this work, we will shed light on two potentials. The first one is the Deng–Fan potential [11] which can be used in chemical physics, molecular spectroscopy, molecular physics, and related fields [12], while the second is the Eckart potential [13]. This potential is related to the diatomic molecular potential model that is used very widely in physical chemistry and biophysics [14–17]. It was first introduced by Eckart [13] in 1930. Due to the importance of wide applications of this potential, as previously indicated, a considerable number of researchers have devoted knowledge of it in the nonrelativistic Schrödinger equation [18–24]m relativistic Dirac equation [25–27], and Klein–Gordon equation [28,29] within the two s and l waves. Recently, Ikot *et al.*, by using the Nikiforov–Uvarov functional analysis method, obtained a new approach for exponential-type potentials including the Eckart potential in the context of the Schrödinger equation [30]. Many researchers interested in combining two or more potentials to expand the field of applications to include new ones. A vital example of a system of combined potentials is the deformed generalized Deng–Fan potential and deformed Eckart potential. In 2013, Awogaa *et al.* studied the Schrödinger equation with these combined potentials in D-dimensions [31]. Very recently, Hatami *et al.* [12] considered the Klein–Gordon equation under the linear combination of the deformed generalized Deng–Fan potential and deformed Eckart potential using the Nikiforov–Uvarov and the q-deformed version of the approximation scheme proposed in [32], and obtained the relativistic energy spectrum for any l-state and the corresponding radial wave functions. This new combination is useful in studying the atomic interaction in diatomic molecules such as H_2, CO, ScN, and ScF [33]. It is worth to point out that Dong *et al.* (2005) proposed a new anharmonic oscillator $\left(\frac{1}{2}\mu\omega^2 r^2 + \frac{\hbar}{2\mu}\frac{\alpha}{r^2}\right)$ and presented the exact solutions of the Schrödinger equation with this oscillator including the ring-shaped potential $\frac{\hbar}{2\mu}\frac{\beta\cos^2(\theta)}{r^2\sin^2(\theta)}$ and established the ladder operators directly from the normalized radial wave functions and used them to evaluate the closed expressions of matrix elements for some connected functions [34]. Furthermore, in 2006, Dong and Cassou obtained the exact solutions of the Klein–Gordon equation with equal scalar and vector $\left(-\frac{\alpha}{r}\right)$ with the ring-shaped potential $\frac{\beta\cos^2(\theta)}{r^2\sin^2(\theta)}$. It should be noted that the energy obtained is very complex [35]. Moreover, Nath and Roy considered two physically important potentials (Manning–Rosen and Pöschl–Teller ones) for the ro-vibrational energy in diatomic molecules using the combined Greene–Aldrich- and Pekeris-type approximations within the Nikiforov–Uvarov framework. This employs a recently proposed scheme [36].

As a result of several considerations and many physical problems arising at the level of the non-renormalizable electroweak interaction, the non-regularization of quantum field theories, quantum gravity, string theory, where the idea of non-commutativity resulting from properties of a deformation of space-space (W. Heisenberg in 1930 was the first to suggest the idea, and then it was developed by Snyder in 1947) was one of the major solutions of these problems. In the past two decades, in particular, it has attracted the great attention of researchers [37–52]. Naturally, the topographical properties of the noncommutativity space-space and phase-phase have a clear effect on the various physical properties of quantum systems, and this has been a very interesting subject in many fields of physics, as mentioned previously.

The above-mentioned works inspired us to investigate the approximate solutions of the 3-dimensional deformed Klein–Gordon equation (DKGE, in short) and the deformed Schrödinger equation (DSE, in short) for the improved deformed generalized potential Deng–Fan plus deformed Eckart potential (IDGDFDE-P) model proposed by Hatami *et al.* in the relativistic regime and by Awoga and his coworkers in the nonrelativistic regime [12, 31] in the context of ordinary quantum mechanics. The potential under study can be applied to some selected diatomic molecules such as H_2, I_2, HCl, CH, LiH, and CO in RNCQM and NRNCQM symmetries. We hope to discover more investigations on the microscopic scale and to achieve more scientific knowledge of elementary particles on the nano-scales. The relativistic and nonrelativistic energy levels under the deformed generalized Deng–Fan potential plus the deformed Eckart potential have not been obtained yet in the RNCQM and NRNCQM symmetries. We hope to find new applications and profound physical interpretations using a new model of the improved deformed generalized Deng–Fan potential plus the deformed Eckart potential, this potential modeled in the new symmetries of NCQM as follows:

$$\begin{cases} V_{dfe}(r_{nc}) = V_{dfe}(r) - \dfrac{\partial V_{dfe}(r)}{\partial r}\dfrac{\mathbf{L}\Theta}{2r} + O(\Theta^2), \\ S_{dfe}(r_{nc}) = S_{dfe}(r) - \dfrac{\partial S_{dfe}(r)}{\partial r}\dfrac{\mathbf{L}\Theta}{2r} + O(\Theta^2), \end{cases} \tag{1}$$

where

$$\begin{cases} V_{dfe}(r) = V_0\left(c - \dfrac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right) - \\ - \dfrac{V_1 e^{-\alpha r}}{1-qe^{-\alpha r}} + \dfrac{V_2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2} \\ S_{dfe}(r) = S_0\left(c - \dfrac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right) - \\ - \dfrac{S_1 e^{-\alpha r}}{1-qe^{-\alpha r}} + \dfrac{S_2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2}, \end{cases} \tag{2}$$

where the parameter $b = \exp(\alpha r_e) - 1$, q is the deformation parameter, (c, α and b) are adjustable constants, (V_0, S_0), (V_1, S_1) (V_2, S_2) are the potential depths, r_{nc} and r represent the distances between the two particles in NCQM and QM symmetries. The coupling $\mathbf{L}\Theta$ equals $L_x\ \Theta_{12} + L_y\ \Theta_{23} + L_z\ \Theta_{13}$ with L_x, L_y and L_z are represent the usual components of the angular momentum operator $\mathbf{L}$ in RQM and NRQM while the new noncommutativity parameter Θ_{ij} equals $\theta_{ij}/2$. The new algebraic structure of covariant noncommutative canonical commutations relations (NCNCCRs) in the three representations of Schrödinger, Heisenberg, and interactions pictures, in the new symmetry of NCQM, is as follows [53–63]:

$$\begin{aligned} &\left[x_\mu^{(\mathrm{S,H,I})}, p_\nu^{(\mathrm{S,H,I})}\right] = i\hbar\delta_{\mu\nu} \Longrightarrow \\ &\Longrightarrow \left[\hat{x}_\mu^{(\mathrm{S,H,I})} \overset{*}{,} \hat{p}_\nu^{(\mathrm{S,H,I})}\right] = i\hbar_{\mathrm{eff}}\delta_{\mu\nu} \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\left[x_\mu^{(\mathrm{S,H,I})}, x_\nu^{(\mathrm{S,H,I})}\right] = 0 \Longrightarrow \\ &\Longrightarrow \left[\hat{x}_\mu^{(\mathrm{S,H,I})} \overset{*}{,} \hat{x}_\nu^{(\mathrm{S,H,I})}\right] = i\theta_{\mu\nu} \end{aligned} \tag{4}$$

with $\hat{x}_\mu^{(\mathrm{S,H,I})} = \left(\hat{x}_\mu^{\mathrm{S}}, \hat{x}_\mu^{\mathrm{H}}, \hat{x}_\mu^{\mathrm{I}}\right)$ and $\hat{p}_\mu^{(\mathrm{S,H,I})} = \left(\hat{p}_\mu^{\mathrm{S}}, \hat{p}_\mu^{\mathrm{H}}, \hat{p}_\mu^{\mathrm{I}}\right)$. It is worth to note that Eq. (4) is a covariant equation (the same behavior of x^μ) under the Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. We generalize the NCNCCRs to include Heisenberg and interaction pictures. It should be noted that, in our calculation, we have used natural units $\hbar = c = 1$. Here, $\hbar_{\mathrm{eff}} \cong \hbar$ is the effective Planck constant, $\theta_{\mu\nu} = \epsilon_{\mu\nu}\theta$ (θ is the noncommutative parameter) which is an infinitesimal parameter, if compared to the energy values and elements of antisymmetric (3×3) real matrices, and $\delta_{\mu\nu}$ is the Kronecker symbol. The symbol $*$ represents the Weyl–Moyal star product, which is generalized between two ordinary functions $f(x)$ and $h(x)$ to the new deformed form $f(x) * h(x)$, which is expressed in NCQM symmetries as follows [60–75]:

$$\begin{aligned} &(f * h)(x) = \exp\left(i\epsilon^{\mu\nu}\theta\partial_\mu^x\partial_\mu^x\right)(fh)(x) \approx (fh)(x) - \\ &- \frac{i\epsilon^{\mu\nu}\theta}{2}\partial_\mu^x f\partial_\mu^x h\rfloor_{x^\mu = x^\nu} + O\left(\theta^2\right). \end{aligned} \tag{5}$$

The indices $(\mu, \nu = 1, 2, 3)$, and $O\left(\theta^2\right)$ stand for the second- and higher-order terms of the NC parameter. Physically, the second term in Eq. (5) presents the effects of space-space noncommutativity. Furthermore, it is possible to unify the operators $\widehat{\omega}_\mu^{\mathrm{H}}(t) = \left(\hat{x}_\mu^{\mathrm{H}} \vee \hat{p}_\mu^{\mathrm{H}}\right)(t)$ and $\widehat{\omega}_\mu^{\mathrm{I}}(t) = \left(\hat{x}_\mu^{\mathrm{I}} \vee \hat{p}_\mu^{\mathrm{I}}\right)(t)$ in the Heisenberg and interaction pictures using the following projection relations, respectively:

$$\begin{aligned} &\widehat{\omega}_\mu^{\mathrm{H}}(t) = \exp\left(\frac{i}{\hbar}\widehat{\mathrm{H}}_{rnc}^{dfe}T\right)\widehat{\omega}_\mu^{\mathrm{S}}\exp\left(-\frac{i}{\hbar}\widehat{\mathrm{H}}_{rnc}^{dfe}T\right), \\ &\widehat{\omega}_\mu^{\mathrm{I}}(t) = \exp\left(\frac{i}{\hbar_{\mathrm{eff}}}\widehat{\mathrm{H}}_{onc}^{dfe}T\right)\widehat{\omega}_\mu^{S}\exp\left(-\frac{i}{\hbar_{\mathrm{eff}}}\widehat{\mathrm{H}}_{onc}^{dfe}T\right), \end{aligned} \tag{6a}$$

where

$$\begin{aligned} &\omega_\mu^H(t) = \exp\left(\frac{i}{\hbar}\widehat{H}_{rdfe}T\right)\zeta_\mu^S\exp\left(-\frac{i}{\hbar}\widehat{H}_{rdfe}T\right), \\ &\omega_\mu^I(t) = \exp\left(\frac{i}{\hbar_{\mathrm{eff}}}\widehat{H}_{odfe}T\right)\omega_\mu^S(t)\exp\left(-\frac{i}{\hbar_{\mathrm{eff}}}\widehat{H}_{odfe}T\right), \end{aligned} \tag{6b}$$

where $\omega_\mu^{\mathrm{S}} = x_\mu^{\mathrm{S}} \vee p_\mu^{\mathrm{S}}$ is the unified two operators in the Schrödinger picture, $\omega_\mu^{\mathrm{H}} = \left(x_\mu^{\mathrm{H}} \vee p_\mu^{\mathrm{H}}\right)(t)$ and $\omega_\mu^{\mathrm{I}} = \left(x_\mu^{\mathrm{I}} \vee p_\mu^{\mathrm{I}}\right)(t)$ are the Heisenberg and interaction pictures in the ordinary QM symmetries. Moreover, the dynamics of new systems $\frac{d\widehat{\omega}_\mu^{\mathrm{H}}(t)}{dt}$ can be described by the following motion equations in the deformed

Heisenberg picture as follows:

$$\frac{d\omega_{\mu}^{\mathrm{H}}(t)}{dt}=-\frac{i}{\hbar}\left[\omega_{\mu}^{\mathrm{H}}(t),\widehat{\mathrm{H}}_{rdfe}\right]+\frac{\partial\omega_{\mu}^{\mathrm{H}}(t)}{\partial t}\Longrightarrow$$
$$\Longrightarrow\frac{d\widehat{\omega}_{\mu}^{\mathrm{H}}(t)}{dt}=-\frac{i}{\hbar_{\mathrm{eff}}}\left[\widehat{\omega}_{\mu}^{\mathrm{H}}(t)\stackrel{*}{,}\widehat{H}_{rnc}^{dfe}\right]+\frac{\partial\widehat{\omega}_{\mu}^{\mathrm{H}}(t)}{\partial t}. \quad (7)$$

Here, ($\widehat{H}_{onc}^{dfe}$ and $\widehat{H}_{rnc}^{dfe}$) are the free and total Hamiltonian operators for the deformed generalized Deng–Fan potential plus the deformed Eckart potential while ($\widehat{\mathrm{H}}_{odfe}$ and $\widehat{\mathrm{H}}_{rdfe}$) are the corresponding Hamiltonians in the NCQM symmetries. The present investigation aims at constructing a relativistic noncommutative effective scheme for the deformed generalized Deng–Fan potential plus the deformed Eckart potential model. It should be noted that the anticommutators of the generators L_x, L_y and L_z in the new symmetries are modified to become as follows:

$$[L_\alpha,L_\beta]=i\varepsilon_{\alpha\beta}^{\gamma}L_\gamma\Longrightarrow[L_\alpha\stackrel{*}{,}L_\beta]=i\varepsilon_{\alpha\beta}^{\gamma}L_\gamma. \quad (8)$$

These generators form a three-dimensional modified Lie algebra of the extended group SO(3). The new bilinear product $[L_\alpha\stackrel{*}{,}L_\beta]$ will satisfy the modified antisymmetries and Jacobi identity properties:

$$[L_\alpha,L_\beta]=-[L_\beta,L_\alpha]\Longrightarrow[L_\alpha\stackrel{*}{,}L_\beta]=-[L_\beta\stackrel{*}{,}L_\alpha] \quad (9)$$

and

$$[L_\gamma\stackrel{*}{,}[L_\alpha\stackrel{*}{,}L_\beta]]+[L_\beta\stackrel{*}{,}[L_\gamma\stackrel{*}{,}L_\alpha]]+[L_\alpha\stackrel{*}{,}[L_\beta\stackrel{*}{,}L_\gamma]]=0. \quad (10)$$

On the other hand, the choice of combined Eckart potentials stems from the fact that it exhibits almost exact behavior similar to the Morse [1] and Deng–Fan [8] potentials and so considers it an excellent choice for the study of atomic interactions for diatomic molecules such as H_2, I_2, HCl, CH, LiH, and CO. Our current work is structured in six sections. The first one includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows. A review of the relativistic KGE with a generalized Deng–Fan potential plus deformed Eckart potential is presented in Sect. 2. Sect. 3 is devoted to studying the DRKGE by applying the ordinary Bopp's shift method and an improved approximation of the centrifugal term to obtain the effective potential of the deformed generalized Deng–Fan potential plus the deformed Eckart potential. In addittion, via perturbation theory, we find the expectation values of some radial terms to calculate the energy shift produced by the perturbed effective deformed generalized Deng–Fan potential plus the deformed Eckart potential. Sect. 4 will consider the global energy shift and the global energy spectra produced with the deformed generalized Deng–Fan potential plus the deformed Eckart potential in the RNCQM symmetries. In Sect. 5, we will determine the energy spectra of some selected diatomic molecules such as H_2, I_2, HCl, CH, LiH, and CO under the deformed generalized Deng–Fan potential plus the deformed Eckart potential in the RNCQM. In Sect. 6, the summary and conclusions are presented.

2. Revised of RKGE under Deformed Generalized Deng–Fan Potential Plus the Deformed Eckart Potential Model

The vector and scalar deformed generalized Deng–Fan potentials plus the deformed Eckart potential in the symmetries of ordinary quantum mechanics are given by [12, 31]:

$$V_{dfe}(r)=V_0\left(c-\frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2-\frac{V_1e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{V_2e^{-\alpha r}}{(1-qe^{-\alpha r})^2} \quad (11)$$

and

$$S_{dfe}(r)=S_0\left(c-\frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2-\frac{S_1e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{S_2e^{-\alpha r}}{(1-qe^{-\alpha r})^2}. \quad (12)$$

The first two terms are the deformed generalized Deng–Fan potential, while the third and fourth terms are the deformed Eckart potential. For the diatomic molecule with the reduced mass M and wave function $\Psi(r,\theta,\varphi)$, the 3-dimensional relativistic Klein–Gordon equation (RKGE) with a scalar potential $S_{dfe}(r)$ and a vector potential $V_{dfe}(r)$ is given as

$$\left(-\Delta+(M+S_{dfe}(r))^2-(E_{nl}-V_{dfe}(r))^2\right)\Psi(r,\theta,\varphi)=0. \quad (13)$$

The vector potential $V_{dfe}(r)$ due to the four-vector linear momentum operator A^μ $(V_{dfe}(r),\mathbf{A}=\mathbf{0})$ and space-time scalar potential $S_{dfe}(r)$ due to mass, E_{nl}, represents the three-dimensional relativistic energy eigenvalues, and l represents the principal and orbital quantum numbers. The deformed generalized Deng–Fan potential and the deformed Eckart potential have spherical symmetry, allowing the solutions of the time-independent RKGE of the known form $\Psi(r,\theta,\varphi) = \frac{U_{nl}(r)}{r} Y_l^m(\theta,\varphi)$ to separate the radial $U_{nl}(r)$ and angular parts $Y_l^m(\theta,\varphi)$ of the wave function $\Psi(r,\theta,\varphi)$, and Δ is the ordinary 3-dimensional Laplacian operator. Thus, Eq. (13) becomes:

$$\left(\frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl}V_{dfe}(r) + MS_{dfe}(r)\right) + \right.$$
$$\left. + V_{dfe}^2(r) - S_{dfe}^2(r) - \frac{l(l+1)}{r^2}\right) U_{nl}(r) = 0. \qquad (14)$$

Using the shortened notation $E_{\text{eff}}^{dfe} = M^2 - E_{nl}^2$ and $V_{\text{eff}}^{dfe}(r) = 2\left(E_{nl}V_{dfe}(r) + MS_{dfe}(r)\right) - V_{dfe}^2(r) + S_{dfe}^2(r) + \frac{l(l+1)}{r^2}$, we obtain the following second-order Schrödinger-like equation:

$$\left(\frac{d^2}{dr^2} - \left(E_{\text{eff}}^{dfe} + V_{\text{eff}}^{dfe}(r)\right)\right) U_{nl}(r) = 0, \qquad (15)$$

when the vector potential equals the scalar potential $V_{mp}(r) = S_{mp}(r)$, the effective potential of the deformed generalized Deng–Fan potential plus the deformed Eckart potential model is as follows:

$$V_{\text{eff}}^{dfe}(r) = 2\left(E_{nl} + M\right)\left(V_0\left(c - \frac{be^{-\alpha r}}{1 - qe^{-\alpha r}}\right)^2 - \right.$$
$$\left. - \frac{V_1 e^{-\alpha r}}{1 - qe^{-\alpha r}} + \frac{V_2 e^{-\alpha r}}{\left(1 - qe^{-\alpha r}\right)^2}\right) + \frac{l(l+1)}{r^2}. \qquad (16)$$

Hamati *et al.* [12] derived analytical expressions for the wave function and the corresponding energy values for the deformed generalized Deng–Fan potential plus the deformed Eckart potential using the Nikiforov–Uvarov method and employing the approximation scheme for the centrifugal term in the relativistic regime as

$$\Psi(r,\theta,\varphi) = \frac{N_{nl}}{r} s^{\epsilon_{nl}} (1 - qs)^{1/2 + v_{nl}} Y_l^m(\theta,\varphi) \times$$
$$\times\, {}_2F_1\left(-n, 2\varepsilon_{nl} + 2v_{nl} + n + 1; 1 + 2\varepsilon_{nl}; qs\right) \qquad (17)$$

and

$$\varepsilon_{nl}^q = \frac{2\left(E_{nl} + M\right)\left(2V_0bc + V_1\right)/\alpha^2}{\Upsilon_1\left(\alpha, E_{nl}, V_0, V_2\right)} -$$
$$- \frac{2\left(E_{nl} + M\right)\left(2V_0bc + V_1\right)/\alpha^2}{\Upsilon\left(\alpha, E_{nl}, V_0, V_2\right)} -$$
$$- \frac{\Upsilon_2}{\Upsilon_1\left(\alpha, E_{nl}, V_0, V_2\right)}, \qquad (18)$$

where $s = \exp(-\alpha r)$, ${}_2F_1(-n, 2\varepsilon_{nl} + 2v_{nl} + n + 1; 1 + 2\varepsilon_{nl}; qs)$ are the hypergeometric polynomials, while N_{nl} is the normalization constant, $\Upsilon_1\left(\alpha, E_{nl}, V_0, V_2\right)$, Υ_2 and v_{nl} are given by:

$$\Upsilon_1\left(\alpha, E_{nl}, V_0, V_2\right) = q(2n+1) +$$
$$+ \sqrt{\begin{matrix} q^2 + 8q\left(E_{nl} + M\right)V_2/\alpha^2 + \\ + 4\left(\frac{2(E_{nl}+M)V_0b^2}{\alpha^2} + l(l+1)\right)\end{matrix}},$$

$$\Upsilon_2 = 2n^2 \sqrt{\begin{matrix} q^2 + 8q\left(E_{nl} + M\right)V_2/\alpha^2 + \\ + 4\left(\frac{2(E_{nl}+M)V_0b^2}{\alpha^2} + l(l+1)\right)\end{matrix}},$$

$$\alpha^2\varepsilon_{nl}^2 = M^2 - E_{nl}^2 - 2\left(E_{nl} + M\right)V_0c^2,$$

$$v_{nl} = 1/2\left[1 + 4\frac{E_{nl} + M}{\alpha^2 q} V_2 + \right.$$
$$\left. + 4\left(\frac{2\left(E_{nl} + M\right)V_0b^2}{\alpha^2 q^2} + \frac{l(l+1)}{q^2}\right)\right]^{1/2}.$$

3. The Solution of DRKGE under the IDGDFDE-P Model in RNCQM Symmetries

3.1. Review of Bopp's shift method

At the beginning of this subsection, we give and define the formula of the IDGDFDE-P model in the symmetries of relativistic noncommutative three-dimensional real space RNCQM symmetries. To achieve this goal, it is useful to write the DKGE by applying the notion of the Weyl–Moyal star product which has been seen previously in Eqs. (3)–(10) on the differential equation that is satisfied by the radial wave function $U_{nl}(r)$ in Eq. (15). Thus, the radial wave function in RNCQM symmetries becomes as follows [76–82]:

$$\left(\frac{d^2}{dr^2} - \left(E_{\text{eff}}^{dfe} + V_{\text{eff}}^{dfe}(r)\right)\right) * U_{nl}(r) = 0. \qquad (19)$$

It is established extensively in the literature and a basic text that star products can be simplified by Bopp's shift method [68, 78–84]. The physicist Fritz Bopp was the first to consider pseudodifferential operators obtained from a symbol by the quantization rules $x \to \to x - \frac{i}{2}\frac{\partial}{\partial p}$ and $p \to p + \frac{i}{2}\frac{\partial}{\partial x}$ instead of the ordinary correspondence $x \to x$ and $p \to \frac{i}{2}\frac{\partial}{\partial x}$ [83, 84]. In the physics literature, this is known as Bopp's shifts method. This quantization procedure is called Bopp quantization. Specialists know that Bopp's shift method [65, 80, 81], has been applied effectively and has succeeded in simplifying the three basic equations: DSE [49, 53–55, 62–64], DKGE [48, 52, 74, 75, 78, 80–82], and deformed Dirac equation (DDE) [51, 58] with the notion of star product to the Schrödinger equation (SE), KGE, and Dirac equation (DE) with the notion of ordinary product, respectively. Thus, Bopp's shift method is based on reducing second-order linear differential equations (DSE, DKGE, and DDE) with star product to second-order linear differential equations (SE, KGE, and DE) without star product with simultaneous translation in the space-space. The CNCCRs with star product in Eqs. (3) and (4) become new CNCCRs without the notion of star product as follows (see, e.g., [44–54]):

$$\left[\hat{x}_\mu^{(\mathrm{S,H,I})}, \hat{p}_\nu^{(\mathrm{S,H,I})}\right] = \hat{x}_\mu^{(\mathrm{S,H,I})}\hat{p}_\nu^{(\mathrm{S,H,I})} - - \hat{p}_\nu^{(\mathrm{S,H,I})}\hat{x}_\mu^{(\mathrm{S,H,I})} = i\hbar_{\mathrm{eff}}\delta_{\mu\nu} \quad (20)$$

and

$$\left[\hat{x}_\mu^{(\mathrm{S,H,I})}, \hat{x}_\nu^{(\mathrm{S,H,I})}\right] = \hat{x}_\mu^{(\mathrm{S,H,I})}\hat{x}_\nu^{(\mathrm{S,H,I})} - - \hat{x}_\nu^{(\mathrm{S,H,I})}\hat{x}_\mu^{(\mathrm{S,H,I})} = i\theta_{\mu\nu}. \quad (21)$$

The generalized positions and momentum coordinates $\hat{x}_\mu^{(\mathrm{S,H,I})} = \left(\hat{x}_\mu^{\mathrm{S}}, \hat{x}_\mu^{\mathrm{H}}, \hat{x}_\mu^{\mathrm{I}}\right)$ and $\hat{p}_\mu^{(\mathrm{S,H,I})} = = \left(\hat{p}_\mu^{\mathrm{S}}, \hat{p}_\mu^{\mathrm{H}}, \hat{p}_\mu^{\mathrm{I}}\right)$, in the symmetries of RNCQM are defined in terms of the corresponding coordinates in the symmetries of RQM $x_\mu^{(\mathrm{S,H,I})} = \left(x_\mu^{\mathrm{S}}, x_\mu^{\mathrm{H}}, x_\mu^{\mathrm{I}}\right)$ and $p_\mu^{(\mathrm{S,H,I})} = \left(p_\mu^{\mathrm{S}}, p_\mu^{\mathrm{H}}, p_\mu^{\mathrm{I}}\right)$ via, respectively [40–50]:

$$\hat{x}_\mu^{(\mathrm{S,H,I})} = x_\mu^{(\mathrm{S,H,I})} - \sum_{\nu=1}^{3}\frac{\theta_{\mu\nu}}{2}p_\nu^{(\mathrm{S,H,I})}, \quad (22)$$

$$\hat{p}_\mu^{(\mathrm{S,H,I})} = p_\mu^{(\mathrm{S,H,I})}. \quad (23)$$

This allows us to find the operator r_{nc}^2 equal to $r^2 - - \mathbf{L}\Theta$ (see in the Introduction) in NCQM symmetries [71–74].

3.2. The new effective potential of the IDGDFDE-P model in RNCQM symmetries

According to the Bopp shift method, Eq. (19) becomes similar to the following, like the Schrödinger equation (without the notions of star product):

$$\left(\frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - \frac{l\,(l+1)}{r_{nc}^2} - - 2V_{dfe}\,(r_{nc})\,(E_{nl} + M)\right)U_{nl}\,(r) = 0. \quad (24)$$

The new operators $V_{dfe}\,(r_{nc})$ and $\frac{l(l+1)}{r_{nc}^2}$ expressed in RNCQM symmetries are as follows:

$$V_{dfe}\,(r_{nc}) = V_0\left(c - \frac{be^{-\alpha r}}{1 - qe^{-\alpha r}}\right) - - \frac{V_1 e^{-\alpha r}}{1 - qe^{-\alpha r}} + \frac{V_2 e^{-\alpha r}}{(1 - qe^{-\alpha r})^2} - - \frac{\partial V_{dfe}\,(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + O\left(\Theta^2\right), \quad (25)$$

$$\frac{l\,(l+1)}{r_{nc}^2} = \frac{l\,(l+1)}{r^2} + \frac{l\,(l+1)}{r^4}\mathbf{L}\Theta + O\left(\Theta^2\right). \quad (26)$$

Therefore, we can rewrite:

$$2V_{dfe}\,(r_{nc})\,(E_{nl} + M) = = 2V_{dfe}\,(r)\,(E_{nl} + M) - \left(\frac{E_{nl} + M}{r}\right) \times \times \frac{\partial V_{dfe}\,(r)}{\partial r}\mathbf{L}\Theta + O(\Theta^2). \quad (27)$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to introduce the symbol $V_{nc-\mathrm{eff}}^{dfe}\,(r)$. Thus, the radial equation (24) becomes:

$$\left(\frac{d^2}{dr^2} - \left(E_{\mathrm{eff}}^{dfe} + V_{nc-\mathrm{eff}}^{dfe}\,(r)\right)\right)U_{nl}\,(r) = 0, \quad (28)$$

with

$$V_{nc-\mathrm{eff}}^{dfe}\,(r) = V_{\mathrm{eff}}^{dfe}\,(r) + V_{\mathrm{pert}}^{dfe}\,(r). \quad (29)$$

Moreover, $V_{\mathrm{pert}}^{dfe}\,(r)$ is given by the following relation:

$$V_{\mathrm{pert}}^{dfe}\,(r) = = \left(\frac{l\,(l+1)}{r^4} - \frac{E_{nl} + M}{r}\frac{\partial V_{dfe}\,(r)}{\partial r}\right)\mathbf{L}\Theta + O\left(\Theta^2\right). \quad (30)$$

After straightforward calculations, we obtain

$$\frac{\partial V_{dfe}(r)}{\partial r}=\frac{\beta_1 e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{\beta_2 e^{-2\alpha r}}{\left(1-qe^{-\alpha r}\right)^2}+ +\frac{\beta_3 e^{-3\alpha r}}{\left(1-qe^{-\alpha r}\right)^3}+\frac{\beta_4 e^{-\alpha r}}{\left(1-qe^{-\alpha r}\right)^2}+ +\frac{\beta_5 e^{-2\alpha r}}{\left(1-qe^{-\alpha r}\right)^4}, \quad (31)$$

where $\beta_1=(2V_0bc+V_1)\,\alpha$, $\beta_2=2bV_0cq\alpha-2V_0b^2\alpha+ +V_1q\alpha$, $\beta_3=-2V_0b^2q\alpha$, $\beta_4=-V_2\alpha$ and $\beta_5= =-V_2q\alpha$. We insert Eq. (31) into Eq. (30), which allows it to be rewritten in the following form:

$$V_{\text{pert}}^{dfe}(r)=\left(\frac{l\,(l+1)}{r^4}-\left(\frac{E_{nl}+M}{r}\right)\times \times\left(\frac{\beta_1 e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{\beta_2 e^{-2\alpha r}}{\left(1-qe^{-\alpha r}\right)^2}+ +\frac{\beta_3 e^{-3\alpha r}}{\left(1-qe^{-\alpha r}\right)^3}+\frac{\beta_4 e^{-\alpha r}}{\left(1-qe^{-\alpha r}\right)^2}+ +\frac{\beta_5 e^{-2\alpha r}}{\left(1-qe^{-\alpha r}\right)^4}\right)\right)\mathbf{L}\Theta+O\left(\Theta^2\right). \quad (32)$$

It should be noted that Eq. (15) with the deformed generalized Deng–Fan potential plus the deformed Eckart potential can be exactly solved for the s-wave ($l=0$), but, in the case $l\neq0$, Hamati *et al.* obtained approximate analytical solutions of the RKGE with the arbitrary $l\neq0$ state using the Nikiforov–Uvarov method and employing the approximation scheme for the centrifugal term. In the new form of the radial-like Schrödinger equation given by Eq. (28), we have new terms including $\frac{1}{r}$, $\frac{1}{r^4}$ and other Coulombic-like terms which make this equation impossible to be solved analytically for $l=0$. For $l\neq0$, it can only be solved approximately. From this point of view, we can consider the improved approximation of the centrifugal term proposed by Badawi *et al.* [85], this method proved its power and efficiency, when compared with the Greene–Aldrich approximation [85]. The approximations of the type suggested by Greene and Aldrich for a short-range potential is an excellent approximation to the centrifugal term and allows us to get a second-order solvable differential equation unlike the following approximation used in the previous works [19, 30, 31, 48, 52, 53, 82]:

$$\frac{1}{r^2}\approx\frac{\alpha^2 e^{-2\alpha r}}{\left(1-qe^{-\alpha r}\right)^2}=\frac{\alpha^2 s^2}{\left(1-qs\right)^2}\Longrightarrow \Longrightarrow\frac{1}{r}\approx\frac{\alpha e^{-\alpha r}}{1-qe^{-\alpha r}}=\frac{\alpha s}{1-qs}. \quad (33)$$

We point out here that the above approximation is only valid for small values of the screening parameter. By considering the transformation of the form $s= =\exp(-\alpha r)$, Eq. (32) now becomes:

$$V_{\text{pert}}^{dfe}(r)=\left(\frac{l\,(l+1)}{r^4}-\left(\frac{E_{nl}+M}{r}\right)\times \times\left(\frac{\beta_1 s}{1-qs}+\frac{\beta_2 s^2}{\left(1-qs\right)^2}+\frac{\beta_3 s^3}{\left(1-qs\right)^3}+ +\frac{\beta_4 s}{\left(1-qs\right)^2}+\frac{\beta_5 s^2}{\left(1-qs\right)^4}\right)\right)\mathbf{L}\Theta+O\left(\Theta^2\right). \quad (34)$$

We now use the approximation noted above, Eq. (33), to find an approximation for the effective potential $V_{\text{pert}}^{dfe}(s)$ that is applied to a small value of the screening parameter:

$$V_{\text{pert}}^{dfe}(s)=\left(\frac{\beta\,(n,l)\,s^4}{\left(1-qs\right)^4}-\alpha\left(E_{nl}+M\right)\times \times\left(\frac{\beta_1 s^2}{\left(1-qs\right)^2}+\frac{\beta_2 s^3}{\left(1-qs\right)^3}+ +\frac{\beta_4 s^2}{\left(1-qs\right)^3}+\frac{\beta_5 s^3}{\left(1-qs\right)^5}\right)\right)\mathbf{L}\Theta+O\left(\Theta^2\right), \quad (35)$$

with $\beta\,(n,l)=\alpha^4 l\,(l+1)-\alpha\,(E_{nl}+M)\,\beta_3$. We have applied the approximations by Greene and Aldrich to the term $\frac{l(l+1)}{r^4}$. The deformed generalized potential Deng–Fan plus deformed Eckart potential is extended by including new terms proportional to the radial terms $\frac{s^2}{(1-qs)^4}$, $\frac{s^2}{(1-qs)^2}$, $\frac{s^3}{(1-qs)^3}$, $\frac{s^2}{(1-qs)^3}$ and $\frac{s^3}{(1-qs)^5}$ to become the IDGDFDE-P model in RNCQM symmetries. The produced new effective potential $V_{nc-\text{eff}}^{dfe}(s)$ is also proportional to the infinitesimal vector Θ. This allows us to consider the additive part $V_{\text{pert}}^{dfe}(s)$ as a perturbation potential compared with the main potential $V_{\text{eff}}^{dfe}(s)$ (parent potential operator in the symmetries of RNCQM;

i.e., the inequality $V_{\text{pert}}^{dfe}(s) \ll V_{\text{eff}}^{dfe}(s)$ has become achieved. Hence, all physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized excited states.

3.3. The expectation values in RNCQM symmetries

In this subsection, we want to apply perturbative theory. In the case of RNCQM, we find the expectation values of the radial terms ($\frac{s^2}{(1-qs)^4}$, $\frac{s^2}{(1-qs)^2}$, $\frac{s^3}{(1-qs)^3}$, $\frac{s^2}{(1-qs)^3}$ and $\frac{s^3}{(1-qs)^5}$) and consider the unperturbed wave function seen previously in Eq. (17). Thus, after straightforward calculations, we obtain the following results:

$$\left\langle \frac{s^2}{(1-qs)^4} \right\rangle_{(nlm)} = \\ = N_{nl}^2 \int_0^{+\infty} \frac{s^2 dr}{(1-qs)^4} s^{2\epsilon_{nl}} (1-qs)^{1+2\nu_{nl}} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2, \quad (36a)$$

$$\left\langle \frac{s^2}{(1-qs)^2} \right\rangle_{(nlm)} N_{nl}^2 = \\ = \int_0^{+\infty} \frac{s^2}{(1-qs)^2} s^{2\epsilon_{nl}} (1-qs)^{1+2\nu_{nl}} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 dr, \quad (36b)$$

$$\left\langle \frac{s^3}{(1-qs)^3} \right\rangle_{(nlm)} = \\ = N_{nl}^2 \int_0^{+\infty} \frac{s^3}{(1-qs)^3} s^{2\epsilon_{nl}} (1-qs)^{1+2\nu_{nl}} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 dr, \quad (36c)$$

$$\left\langle \frac{s^2}{(1-qs)^3} \right\rangle_{(nlm)} = \\ = N_{nl}^2 \int_0^{+\infty} \frac{s^2}{(1-qs)^3} s^{2\epsilon_{nl}} (1-qs)^{1+2\nu_{nl}} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 dr, \quad (36d)$$

$$\left\langle \frac{s^3}{(1-qs)^5} \right\rangle_{(nlm)} = \\ = N_{nl}^2 \int_0^{+\infty} \frac{s^3}{(1-qs)^5} s^{2\epsilon_{nl}} (1-qs)^{1+2\nu_{nl}} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 dr. \quad (36e)$$

We have used useful abbreviations $\langle X \rangle_{(nlm)} = \langle n, l, m\ X\ n, l, m \rangle$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\theta,\varphi) Y_{l'}^{m'}(\theta,\varphi) \sin(\theta)\, d\theta d\varphi = \delta_{ll'}\delta_{mm'}$. We have $s = \exp(-\alpha r)$, which allows us to obtain $dr = -\frac{1}{\alpha}\frac{ds}{s}$. From the asymptotic behavior of $s = \exp(-\alpha r)$, when $(r \to 0)$ $(s \to +1)$ and $(r \to +\infty)$ $(s \to 0)$, this allows us to reformulate Eqs. (31)-(36) as follows:

$$\left\langle \frac{s^2}{(1-qs)^4} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha} \int_0^{+q} s^{2\epsilon_{nl}+1} (1-qs)^{2\nu_{nl}-3} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 ds, \quad (37a)$$

$$\left\langle \frac{s^2}{(1-qs)^2} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha} \int_0^{+q} s^{2\epsilon_{nl}+1} (1-qs)^{2\nu_{nl}-1} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 ds, \quad (37b)$$

$$\left\langle \frac{s^3}{(1-qs)^3} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha} \int_0^{+q} s^{2\epsilon_{nl}+2} (1-qs)^{2\nu_{nl}-2} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 ds, \quad (37c)$$

$$\left\langle \frac{s^2}{(1-qs)^3} \right\rangle_{(nlm)} \frac{N_{nl}^2}{\alpha} \int_0^{+q} s^{2\epsilon_{nl}+1} (1-qs)^{2\nu_{nl}-2} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 ds, \quad (37d)$$

$$\left\langle \frac{s^3}{(1-qs)^5} \right\rangle_{(nlm)} \frac{N_{nl}^2}{\alpha} \int_0^{+q} s^{2\epsilon_{nl}+2} (1-qs)^{2\nu_{nl}-4} \times \\ \times \left[{}_2F_1(-n, 2\varepsilon_{nl}+2\nu_{nl}+n+1; 1+2\varepsilon_{nl}; qs) \right]^2 ds. \quad (37e)$$

When the deformation parameter equals one, we can use the method proposed by Dong *et al.* [86] and applied by Zhang [87] and can calculate the integrals in Eqs. $(37, i = \overline{1,6})$. With the help of the special integral formula

$$\int_0^{+1} s^{\xi-1}(1-s)^{\sigma-1}\left[{}_2F_1(c_1,c_2;c_3;s)\right]^2 ds =$$

$$= \frac{\Gamma(\xi)\Gamma(\sigma)}{\Gamma(\xi+\sigma)}\, {}_3F_2(c_1,c_2,\sigma;c_3,\sigma+\xi;1), \tag{38}$$

where ${}_3F_2(c_1,c_2,\sigma;c_3,\sigma+\xi;1)$. Is obtained from the generalized hypergeometric function ${}_3F_2(\alpha_1,\alpha_2,...,\alpha_p;\beta_1,\beta_2,...,\beta_q;1)$ for $p = 3$ and $q = 2$, while $\Gamma(\sigma)$ denoting the usual gamma function, we obtain, from Eqs. $(37, i = \overline{1,6})$, the following results:

$$\left\langle \frac{s^2}{(1-qs)^4} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha}\frac{\Gamma(2\epsilon_{nl}+2)\Gamma(2\nu_{nl}-2)}{\Gamma(\Lambda)} \times$$

$$\times\, {}_3F_2(-n,\Lambda+n+1,2\nu_{nl}-2;1+2\varepsilon_{nl},\Lambda;1), \tag{39a}$$

$$\left\langle \frac{s^2}{(1-qs)^2} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha}\frac{\Gamma(2\epsilon_{nl}+2)\Gamma(2\nu_{nl})}{\Gamma(\Lambda+2)} \times$$

$$\times\, {}_3F_2(-n,\Lambda+n+1,2\nu_{nl};1+2\varepsilon_{nl},\Lambda+2;1), \tag{39b}$$

$$\left\langle \frac{s^3}{(1-qs)^3} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha}\frac{\Gamma(2\epsilon_{nl}+3)\Gamma(2\nu_{nl}-1)}{\Gamma(\Lambda+2)} \times$$

$$\times\, {}_3F_2(-n,\Lambda+n+1,2\nu_{nl}-1;1+2\varepsilon_{nl},\Lambda+2;1), \tag{39c}$$

$$\left\langle \frac{s^2}{(1-qs)^3} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha}\frac{\Gamma(2\epsilon_{nl}+2)\Gamma(2\nu_{nl}-1)}{\Gamma(\Lambda+1)} \times$$

$$\times\, {}_3F_2(-n,\Lambda+n+1,2\nu_{nl}-1;1+2\varepsilon_{nl},\Lambda+1;1), \tag{39d}$$

$$\left\langle \frac{s^3}{(1-qs)^5} \right\rangle_{(nlm)} = \frac{N_{nl}^2}{\alpha}\frac{\Gamma(2\epsilon_{nl}+3)\Gamma(2\nu_{nl}-3)}{\Gamma(\Lambda)} \times$$

$$\times\, {}_3F_2(-n,\Lambda+n+1,2\nu_{nl}-3;1+2\varepsilon_{nl},\Lambda;1), \tag{39e}$$

where $\Lambda(\alpha,c,b,V_0,V_1,V_2) = 2\epsilon_{nl} + 2\nu_{nl}$.

3.4. The energy shift for the IDGDFDE-P model in RNCQM symmetries

The global relativistic energy shift for the IDGDFDE-P model in RNCQM symmetries is composed of three principal parts. The first one is produced from the effect of the generated spin-orbit effective potential. This effective potential is obtained by replacing the coupling of the angular momentum operator and the noncommutative vector $\mathbf{L}\Theta$ with the new equivalent coupling $\Theta\mathbf{LS}$ (with $\Theta^2 = \Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2$). This degree of freedom comes considering that the infinitesimal noncommutative vector Θ is arbitrary. We have chosen the noncommutativity vector to become parallel with the spin-$\mathbf{S}$ of the diatomic molecules under a deformed generalized Deng–Fan potential plus the deformed Eckart potential. Furthermore, we replace the new spin-orbit coupling $\Theta\mathbf{LS}$ with the corresponding physical form $(\Theta/2)\,\mathbf{G}^2$, with $\mathbf{G}^2 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$. Moreover, in quantum mechanics, the operators ($\widehat{H}_{rnc}^{dfe}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physical quantities, the eigenvalues of the operator $\mathbf{G}^2$ are equal to the values $k(j,l,s) = [j(j+1) - l(l+1) - s(s+1)]/2$, with $|l-s| \le j \le |l+s|$. As a direct consequence, the partial energy shift $\Delta E_{dfe}^{so}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,j,l,s)$ due to the perturbed effective potential $V_{\text{pert}}^{dfe}(s)$ produced for the n^{th} excited state, in RNCQM symmetries as follows:

$$\Delta E_{dfe}^{so}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,j,l,s) =$$

$$= \Theta\left(j(j+1) - l(l+1) - s(s+1)\right) \times$$

$$\times \langle F\rangle_{(nlm)}^{Rdfe}(n,\alpha,c,b,V_0,V_1,V_2). \tag{40}$$

The global expectation value $\langle F\rangle_{(nlm)}^{Rdfe}(n,\alpha,c,b,V_1,V_2)$ is determined from the following expression:

$$\langle F\rangle_{(nlm)}^{Rdfe}(n,\alpha,c,b,V_1,V_2) =$$

$$= \beta(n,l)\left\langle \frac{s^4}{(1-qs)^4}\right\rangle_{(nlm)} - \alpha(E_{nl}+M) \times$$

$$\times\left(\beta_1\left\langle \frac{s^4}{(1-qs)^4}\right\rangle_{(nlm)} + \beta_2\left\langle \frac{s^3}{(1-qs)^3}\right\rangle_{(nlm)} + \right.$$

$$\left. + \beta_4\left\langle \frac{s^2}{(1-qs)^3}\right\rangle_{(nlm)} + \beta_5\left\langle \frac{s^3}{(1-qs)^5}\right\rangle_{(nlm)}\right). \tag{41}$$

The second part is obtained from the magnetic effect of the perturbative effective potential $V_{\text{pert}}^{dfe}(s)$ under the IDGDFDE-P model. This effective potential is achieved, when we replace both ($\mathbf{L}\Theta$ and Θ_{12}) by ($\sigma B L_z$ and σB), respectively. Here, B and σ symbolize the intensity of the magnetic field induced by the effect of a deformation of the space-space geometry and a new infinitesimal noncommutativity parameter so that the physical unit of the original noncommutativity parameter Θ_{12} (length)^2 is the same unit of σB. We have also need to apply $\langle n', l', m' L_z n, l, m\rangle = m\delta_{m'm}\delta_{l'l}\delta_{n'n}$ ($-l' \leq m' \leq l$ and $-l \leq m \leq l$). All of these data allow the discovery of the new energy shift $\Delta E_{dfe}^{mag}(n, \alpha, c, b, V_0, V_1, V_2, \sigma, m)$ due to the perturbed Zeeman effect which was created by the influence of the IDGDFDE-P model for the n^{th} excited state in RNCQM symmetries as follows:

$$\Delta E_{dfe}^{mag}(n, \alpha, c, b, V_0, V_1, V_2, \sigma, j, l, s) = \\ = \sigma B \langle F \rangle_{(nlm)}^{Rdfe}(n, \alpha, c, b, V_0, V_1, V_2)\, m. \quad (42)$$

For our purposes, we are interested in finding a new third automatically important symmetry for the IDGDFDE-P model at the zero temperature in the RNCQM symmetries. This physical phenomenon is induced automatically by the influence of a perturbed effective potential $V_{\text{pert}}^{dfe}(s)$ which is seen in Eq. (35). We discover these important physical phenomena, when our studied system consists of noninteracting particles and is considered a Fermi gas. It is formed from all the particles in their gaseous state (H_2, I_2, HCl, CH, LiH, and CO) undergoing the rotation with angular velocity $\boldsymbol{\Omega}$, if we make the following two simultaneous transformations to ensure that the previous calculations are not repeated:

$$\Theta \rightarrow \chi\boldsymbol{\Omega} \quad \text{and } \mathbf{L}\Theta \rightarrow \chi\mathbf{L}\boldsymbol{\Omega}. \quad (43)$$

Here, χ is just an infinitesimal real proportional constant. We can express the effective potential $V_{\text{pert}}^{dfe-\text{rot}}(s)$ which induced the rotational movements of the diatomic molecules as follows:

$$V_{\text{pert}}^{dfe-\text{rot}}(s) = \frac{\beta(n,l)\, s^4}{(1-qs)^4}\mathbf{L}\boldsymbol{\Omega} - \alpha\chi(E_{nl}+M) \times \\ \times\left(\frac{\beta_1 s^2}{(1-qs)^2} + \frac{\beta_2 s^3}{(1-qs)^3} + \right. \\ \left. + \frac{\beta_4 s^2}{(1-qs)^3} + \frac{\beta_5 s^3}{(1-qs)^5}\right)\mathbf{L}\boldsymbol{\Omega}. \quad (44)$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\boldsymbol{\Omega}$ parallel to the Oz axis. Then we transform the spin-orbit coupling to the new physical phenomenon as follows:

$$\chi k(s)\mathbf{L}\boldsymbol{\Omega} = \chi k(s)\Omega L_z \quad (45)$$

with

$$k(s) = \frac{\beta(n,l)\, s^4}{(1-qs)^4} - \alpha(E_{nl}+M) \times \\ \times\left(\frac{\beta_1 s^2}{(1-qs)^2} + \frac{\beta_2 s^3}{(1-qs)^3} + \right. \\ \left. + \frac{\beta_4 s^2}{(1-qs)^3} + \frac{\beta_5 s^3}{(1-qs)^5}\right). \quad (46)$$

All of these data allow the discovery of the new energy shift $\Delta E_{dfe}^{f-\text{rot}}(n, \alpha, c, b, V_0, V_1, V_2, \chi, m)$ due to the perturbed Fermi gas effect $V_{\text{pert}}^{dfe-\text{rot}}(r)$ which is generated automatically by the influence of the deformed generalized Deng–Fan potential plus the deformed Eckart potential for the n^{th} excited state in RNCQM symmetries as follows:

$$\Delta E_{dfe}^{f-rot}(n, \alpha, c, b, V_0, V_1, V_2, \chi, m) = \\ = \chi \langle F \rangle_{(nlm)}^{Rdfe}(n, \alpha, c, b, V_0, V_1, V_2, V_2)\, \Omega m. \quad (47)$$

It is worth mentioning that the authors of Refs. [88, 89]studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two- and three-dimensional spaces at the zero temperature. But in that study, the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term $\chi k(s)\mathbf{L}\boldsymbol{\Omega}$ automatically appears due to the large symmetries resulting from the deformation of the space-phase.

4. Results and Discussion

Here, we summarize our obtained results

$\Delta E_{dfe}^{so}(n, \alpha, c, b, V_0, V_1, V_2, j, l, s)$,

$\Delta E_{dfe}^{mag}(n, \alpha, c, b, V_0, V_1, V_2, m)$

and $\Delta E_{dfe}^{f-\text{rot}}(n, \alpha, c, b, V_0, V_1, V_2, m)$

for the n^{th} excited state due to the spin-orbital coupling, modified Zeeman effect, and perturbed Fermi gas potential induced with $V_{\text{pert}}^{dfe}(s)$ on the basis of the superposition principle. This allows us to deduce the additive energy shift

$\Delta E_{dfe}^{\text{tot}}(n,\alpha,c,b,V_0,V_1,V_2,j,l,s,m) \equiv \Delta E_{dfe}^{\text{tot}}$ under the influence of the IDGDFDE-P model in RNCQM symmetries as follows:

$$\Delta E_{dfe}^{\text{tot}} = \langle F\rangle_{(nlm)}^{Rdfe}(n,\alpha,c,b,V_0,V_1,V_2)\times$$
$$\times(\Theta k(j,l,s)+\sigma B+m\chi\Omega m). \qquad (48)$$

The above results present the global energy shift, which was generated with the effect of noncommutativity properties of the space-space; it depends explicitly on the noncommutativity parameters (Θ,σ,χ), the parameters of the generalized Deng–Fan potential plus deformed Eckart potential (α,c,b,V_0,V_1,V_2) in addition to the atomic quantum numbers (j,l,s,m). We observed that the obtained global effective energy under the deformed generalized Deng–Fan potential plus the deformed Eckart potential has a carry a unit energy, because it is combined with the carrier of energy $(M^2-E_{nl}^2)$. As a direct consequence, the energy $E_{r-nc}^{dfe}(n,\alpha,c,b,V_0,V_1,V_2,j,l,s,m)$ produced with the IDGDFDE-P model, in the symmetries of RNCQM is the sum of the root quart of the shift energy $\Delta\left[E_{dfe}^{f-\text{rot}}(n,\alpha,c,b,V_0,V_1,V_2,\chi,m)\right]^{1/2}$ and the relativistic energy E_{nl} produced by the effect due to the deformed generalized Deng–Fan potential plus the deformed Eckart potential in RQM, as follows:

$$E_{r-nc}^{dfe}(n,\alpha,c,b,V_0,V_1,V_2,V_2,\Theta,\sigma,\chi,j,l,s,m)=$$
$$=E_{nl}+\begin{bmatrix}\langle F\rangle_{(nlm)}^{Rdfe}(n,\alpha,c,b,V_0,V_1,V_2)\\(\Theta k(j,l,s)+\sigma Bm+m\chi\Omega m)\end{bmatrix}^{1/2}. \qquad (49)$$

The relativistic energy E_{nl} is determined from the energy equation (18). For the ground state and first excited state, the above equation can be reduced to the following form:

$$E_{r-nc}^{dfe}(n=0,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)=$$
$$=E_{0l}+\begin{bmatrix}\langle F\rangle_{(nlm)}^{Rdfe}(n=0,\alpha,c,b,V_0,V_1,V_2)\\(\Theta k(j,l,s)+\sigma Bm+m\chi\Omega m)\end{bmatrix}^{1/2} \qquad (50)$$

and

$$E_{r-nc}^{dfe}(n=1,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)=$$
$$=E_{1l}+\begin{bmatrix}\langle F\rangle_{(nlm)}^{Rdfe}(n=1,\alpha,c,b,V_0,V_1,V_2)\\(\Theta k(j,l,s)+\sigma Bm+m\chi\Omega m)\end{bmatrix}^{1/2}. \qquad (51)$$

Equation (49) can describe the relativistic energy of some diatomic molecules such as H_2, I_2, HCl, CH, LiH, and CO under the IDGDFDE-P model in the symmetries of relativistic NC quantum mechanics.

4.1. Relativistic particular cases under the IDGDFDE-P model

After examining the bound-state solutions of any l-state deformed Klein–Gordon equation within the IDGDFDE-P model in RNCQM symmetries, our task is now to discuss some particular cases in what follows. By adjusting the potential parameters for each case, some familiar potentials, which are useful for other physical systems, can be obtained.

First: Setting V_1 and V_2 to zero, the potential in Eq. (11) turns to the deformed generalized Deng–Fan potential [8] in RQM symmetries, as follows:

$$V_{df}(r)=V_0\left(c-\frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2. \qquad (52)$$

The perturbed effective potential $V_{\text{pert}}^{dfe}(s)$ in Eq. (35) turns to the perturbed effective potential $V_{\text{pert}}^{dp}(r)$ in the symmetries of RNCQM as follows:

$$V_{\text{pert}}^{dfe}(s)=\left(\frac{\beta(n,l)s^4}{(1-qs)^4}-\alpha(E_{nl}+M)\times\right.$$
$$\left.\times\left(\frac{2V_0\,bc\alpha s^2}{(1-qs)^2}+\frac{\beta_2^n s^3}{(1-qs)^3}\right)\right)\mathbf{L}\Theta+O(\Theta^2), \qquad (53)$$

where $\beta_2^n = 2bV_0cq\alpha - 2V_0b^2\alpha$. In this case, the additive energy shift $\Delta E_{df}^{\text{tot}}(n,\alpha,c,b,V_0,j,l,s,m)\equiv$ $\equiv \Delta E_{df}^{\text{tot}}$ under the influence of the modified equally mixed deformed generalized Deng–Fan potential in RNCQM symmetries is determined from the following formula:

$$\Delta E_{mp}^{\text{tot}}=\langle F\rangle_{(nlm)}^{RDF}(n,\alpha,c,b,V_0)\times$$
$$\times(\Theta k(j,l,s)+\sigma B+m\chi\Omega m). \qquad (54)$$

Thus, the corresponding global expectation value $\langle F\rangle_{(nlm)}^{RDF}(n,\alpha,c,b,V_0)$ is determined from the following expression:

$$\langle F\rangle_{(nlm)}^{RDF}(n,\alpha,c,b,V_0)=$$
$$=\beta(n,l)\left\langle\frac{s^2}{(1-qs)^4}\right\rangle_{(nlm)}-\alpha(E_{nl}+M)\times$$
$$\times\left(2V_0bc\alpha\left\langle\frac{s^2}{(1-qs)^2}\right\rangle_{(nlm)}+\right.$$
$$\left.+\beta_2^n\left\langle\frac{s^3}{(1-qs)^3}\right\rangle_{(nlm)}\right). \qquad (55)$$

The new relativistic energy in Eq. (49) reduces to the new energy $E^{dp}_{r-nc}(n,\alpha,c,b,V_0,\Theta,\sigma,\chi,j,l,s,m)$ under the new deformed generalized Deng–Fan potential in RNCQM, as follows:

$$E^{df}_{r-nc}(n,\alpha,c,b,V_0,\Theta,\sigma,\chi,j,l,s,m)= = E^{dp}_{nl}+\left[\begin{array}{c}\langle F\rangle^{RDF}_{(nlm)}(n,\alpha,c,b,V_0)\\ (\Theta k(j,l,s)+\sigma Bm+m\chi\Omega m)\end{array}\right]^{1/2}. \quad (56)$$

Making the corresponding parameter replacements in Eq. (18), we obtain the energy equation for the deformed generalized Deng–Fan potential in the Klein–Gordon theory with equally mixed potentials, in RQM symmetries as:

$$\varepsilon^q_{nl}=\frac{4V_0bc\left(E^{dp}_{nl}+M\right)/\alpha^2}{\Upsilon_3}- -\frac{4V_0bc\left(E^{dp}_{nl}+M\right)/\alpha^2}{\Upsilon_3}- -\frac{n^2\sqrt{q^2+4\left(\frac{2\left(E^{dp}_{nl}+M\right)V_0b^2}{\alpha^2}+l(l+1)\right)}}{\Upsilon_3}, \quad (57)$$

where

$$-\alpha^2\varepsilon^2_{nl}=M^2-E^{dp2}_{nl}-2\left(E^{dp}_{nl}+M\right)V_0c^2$$

and

$$\Upsilon_3=q(2n+1)+ +\sqrt{q^2+4\left(\frac{2\left(E^{dp}_{nl}+M\right)V_0b^2}{\alpha^2}+l(l+1)\right)}.$$

Second: Setting V_0 to be zero, the potential in Eq. (11) turns to the deformed Eckart potential in RQM symmetries, as follows:

$$V_{ep}(r)=-\frac{V_1e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{V_2e^{-\alpha r}}{(1-qe^{-\alpha r})^2}. \quad (58)$$

The perturbed effective potential $V^{dfe}_{\text{pert}}(s)$ in Eq. (35) turns to the perturbed effective potential $V^{ep}_{\text{pert}}(r)$ in the symmetries of RNCQM as follows:

$$V^{ep}_{\text{pert}}(s)=\left(\frac{\alpha^4l(l+1)s^4}{(1-qs)^4}-\alpha(E_{nl}+M)\times \times\left(\frac{V_1\alpha s^2}{(1-qs)^2}+\frac{V_1q\alpha s^3}{(1-qs)^3}- -\frac{V_2\alpha s^2}{(1-qs)^3}-\frac{V_2q\alpha s^3}{(1-qs)^5}\right)\right)\mathbf{L}\Theta+O\left(\Theta^2\right). \quad (59)$$

In this case, the additive energy shift $\Delta E^{\text{tot}}_{ep}(n,\alpha,V_1,V_2,j,l,s,m)$ under the influence of the modified equally mixed new deformed Eckart potential in RNCQM symmetries is determined from the following formula:

$$\Delta E^{\text{tot}}_{ep}(n,\alpha,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)= =\langle F\rangle^{REP}_{(nlm)}(n,\alpha,V_1,V_2)\times \times(\Theta k(j,l,s)+\sigma B+m\chi\Omega m). \quad (60)$$

Thus, the corresponding global expectation value $\langle F\rangle^{REP}_{(nlm)}(n,\alpha,V_1,V_2)$ is determined from the following expression:

$$\langle F\rangle^{REP}_{(nlm)}(n,\alpha,V_1,V_2)= =\alpha^4l(l+1)\left\langle\frac{s^2}{(1-qs)^4}\right\rangle_{(nlm)}-\alpha^2(E_{nl}+M)\times \times\left(V_1\left\langle\frac{s^2}{(1-qs)^2}\right\rangle_{(nlm)}+qV_1\left\langle\frac{s^3}{(1-qs)^3}\right\rangle_{(nlm)}- -V_2\left\langle\frac{s^2}{(1-qs)^3}\right\rangle_{(nlm)}-qV_2\left\langle\frac{s^3}{(1-qs)^5}\right\rangle_{(nlm)}\right). \quad (61)$$

The new relativistic energy in Eq. (49) reduces to the new energy $E^{ep}_{r-nc}(n,\alpha,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ under new deformed Eckart potential in RNCQM, as follows:

$$E^{ep}_{r-nc}(n,\alpha,V_1,V_2,\Theta,\sigma,\chi,\Theta,\sigma,\chi,j,l,s,m)= =E^{ep}_{nl}+\left[\begin{array}{c}\langle F\rangle^{EP}_{(nlm)}(n,\alpha,V_1,V_2)\\ (\Theta k(j,l,s)+\sigma Bm+m\chi\Omega m)\end{array}\right]^{1/2}. \quad (62)$$

Making the corresponding parameter replacements in Eq. (18), we obtain the energy equation for the deformed Eckart potential in the Klein–Gordon theory with equally mixed potentials, in RQM symmetries, as:

$$\sqrt{\left(E^{ep2}_{nl}-M^2\right)/\alpha^2}= =\frac{2V_1\left(E^{ep}_{nl}+M\right)/\alpha^2-2V_1\left(E^{ep}_{nl}+M\right)/\alpha^2}{q(2n+1)+\sqrt{q^2+8q\left(E^{ep}_{nl}+M\right)V_2/\alpha^2+4l(l+1)}}- -\frac{n^2\sqrt{q^2+8q\left(E^{ep}_{nl}+M\right)V_2/\alpha^2+4l(l+1)}}{q(2n+1)+\sqrt{q^2+8q\left(E^{ep}_{nl}+M\right)V_2/\alpha^2+4l(l+1)}}. \quad (63)$$

5. Nonrelativistic Spectrum under the IDGDFDE-P Model

In this section, we want to derive the nonrelativistic spectrum, which is produced by the effect of the IDGDFDE-P model for diatomic molecules such as H_2, I_2, HCl, CH, LiH, and CO. From Eqs. (1) and (11), we can write this potential in the nonrelativistic noncommutative three-dimensional real space for NRNCQM symmetries as follows:

$$V_{nc}^{dfe}(r) = V_0\left(c - \frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right) - \frac{V_1 e^{-\alpha r}}{1-qe^{-\alpha r}} + + \frac{V_2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2} + V_{nr-\text{pert}}^{dfe}(r). \quad (64)$$

Here, $V_{\text{pert}}^{dfe}(r)$ is the perturbative potential in nonrelativistic noncommutative three-dimensional real space with NRNCQM symmetries:

$$V_{nr-\text{pert}}^{dfe}(r) = \frac{l(l+1)}{r^4}\mathbf{L}\Theta - \frac{\partial V_{nc}^{dfe}(r)}{\partial r}\frac{\mathbf{L}\Theta}{2r} + O(\Theta^2). \quad (65)$$

The first term in Eq. (65) is caused by the centrifuge term $\frac{l(l+1)}{r_{nc}^2}$ in NRNCQM (Eq. (26)), which equals the usual centrifuge term $\frac{l(l+1)}{r^2}$ plus the perturbative centrifugal term $\frac{l(l+1)}{r^4}\mathbf{L}\Theta$, whereas the second term is caused by the effect of the deformed generalized Deng-Fan potential. We used the Greene–Aldrich approximation type for a short-range potential, which is an excellent approximation to the centrifugal term for the Eckart potential, and we calculated $\frac{\partial V_{nc}^{dfe}(r)}{\partial r}$ (see Eq. (31). Now, substituting Eq. (26) into Eq. (65) and applying the approximation in Eq. (33), we get the perturbative potential created with the effect of the IDGDFDE-P model in NRNCQM symmetries as follows:

$$V_{nr-\text{pert}}^{dfe}(r) = \left(\frac{\beta(n,l)s^4}{(1-qs)^4} - - \frac{1}{2}\left(\frac{\beta_1 s^2}{(1-qs)^2} + \frac{\beta_2 s^3}{(1-qs)^3} + + \frac{\beta_4 s^2}{(1-qs)^3} + \frac{\beta_5 s^3}{(1-qs)^5}\right)\right)\mathbf{L}\Theta + O(\Theta^2). \quad (66)$$

To find the nonrelativistic energy corrections produced by the perturbative potential $V_{nr-\text{pert}}^{dfe}(s)$, we need to know the expectation values of $\frac{s^2}{(1-qs)^4}$, $\frac{s^2}{(1-qs)^2}$, $\frac{s^3}{(1-qs)^3}$, $\frac{s^2}{(1-qs)^3}$ and $\frac{s^3}{(1-qs)^5}$. We get the corresponding global expectation values $\langle F\rangle_{(nlm)}^{NRdfe}(n,\alpha,c,b,V_0,V_1,V_2)$ by using the expectations values obtained in Eqs. (39a), (39b), (39c), (39d), and (39e) for the n-th excited state:

$$\langle F\rangle_{(nlm)}^{NRdfe}(n,\alpha,c,b,V_0,V_1,V_2) = = \beta(n,l)\left\langle\frac{s^4}{(1-qs)^4}\right\rangle_{(nlm)} - - \frac{1}{2}\left(\beta_1\left\langle\frac{s^4}{(1-qs)^4}\right\rangle_{(nlm)} + \beta_2\left\langle\frac{s^3}{(1-qs)^3}\right\rangle_{(nlm)} + + \beta_4\left\langle\frac{s^2}{(1-qs)^3}\right\rangle_{(nlm)} + \beta_5\left\langle\frac{s^3}{(1-qs)^5}\right\rangle_{(nlm)}\right). \quad (67)$$

By following the same physical methodology that we devoted in our relativistic previous study, the energy corrections $\Delta E_{dfe}^{nr}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ for the n^{th} excited state due to the spin-orbit coupling, modified Zeeman effect, and nonrelativistic perturbed Fermi gas potential which induced under the influence of the IDGDFDE-P model in NRNCQM symmetries as follows:

$$\Delta E_{dfe}^{nr}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m) = = \langle F\rangle_{(nlm)}^{NRdfe}(n,\alpha,c,b,V_1,V_2)\,(\Theta k(j,l,s) + + \sigma B + m\chi\Omega m). \quad (68)$$

As a direct consequence, the new nonrelativistic energy $E_{nr-nc}^{dfe}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ produced within the IDGDFDE-P model, in the symmetries of NRNCQM for the n^{th} generalized excited states, the sum of the energy corrections $\Delta E_{dfe}^{nr}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ plus the nonrelativistic energy E_{nl}^{nr} produced with the main part of the potential (the deformed generalized Deng–Fan potential plus the deformed Eckart potential) in NRQM as follows:

$$E_{nr-nc}^{dfe}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m) = = E_{nl}^{nr} + \langle F\rangle_{(nlm)}^{NRdfe}(n,\alpha,c,b,V_0,V_1,V_2)\times \times(\Theta k(j,l,s) + \sigma Bm + \chi\Omega m). \quad (69)$$

The nonrelativistic energy E_{nl}^{nr} due to the effect of the deformed generalized Deng–Fan potential plus the deformed Eckart potential is determined directly from the work Awoga *et al.* [31] given by:

$$E_{nl}^{nr} = -\frac{\alpha^2}{2M}\left[\frac{\left(2M/\alpha^2\right)\left(2bcV_0 + l(l+1)\right) + 2Mb^2V_0}{2q^2\alpha^2\left(n+\sigma_q\right)} - \right.$$
$$\left. - \frac{n+\sigma_q}{2}\right]^2 + c^2V_0, \qquad (70)$$

with $\sigma_q = \frac{1}{2}\left[1 + \sqrt{1 + 4\left(2Mb^2V_0/q^2\alpha^2 + \beta/q\right)}\right]$ and $\beta = \frac{2M}{\alpha^2}\left[\left(V_2 + l(l+1)\right)\right]$. Now, considering composite systems such as molecules made of $N = 2$ particles of masses $m_n\,(n = 1, 2)$ in the frame of a noncommutative algebra, it is worth to account for the features of descriptions of the systems in the space. In NRQM symmetries, it was obtained that composite systems with different masses are described with different noncommutative parameters [90–93]:

$$\left[\hat{x}_\mu^{(\mathrm{S,H,I})} \stackrel{*}{,} \hat{x}_\nu^{(\mathrm{S,H,I})}\right] = i\theta_{\mu\nu}^c, \qquad (71)$$

where the noncommutativity parameter $\theta_{\mu\nu}^c$ is given by:

$$\theta_{\mu\nu}^c = \sum_{n=1}^{2} \mu_n^2\,\theta_{\mu\nu}^{(n)}, \qquad (72)$$

with $\mu_n = \frac{m_n}{\sum_n m_n}$, the indices ($n$ = 1,2) label the particle, and $\theta_{\mu\nu}^{(n)}$ is the parameter of noncommutativity, corresponding to the particle of mass m_n. Note that, in the case of a system of two particles with the same mass $m_1 = m_2$ such as the homogeneous (H_2 and I_2) diatomic molecules, the parameter $\theta_{\mu\nu}^{(n)} = \theta_{\mu\nu}$. Thus, the two parameters Θ and σ which appear in Eq. (69) are changed to the new form:

$$\Theta^{c2} = \left(\sum_{n=1}^{2}\mu_n^2\Theta_{12}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu_n^2\Theta_{23}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu_n^2\Theta_{13}^{(n)}\right)^2, \qquad (73)$$

$$\sigma^{c2} = \left(\sum_{n=1}^{2}\mu_n^2\sigma_{12}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu_n^2\sigma_{23}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu_n^2\sigma_{13}^{(n)}\right)^2 \qquad (74)$$

and

$$\chi^{c2} = \left(\sum_{n=1}^{2}\mu\chi_{12}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu\chi_{23}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu\chi_{13}^{(n)}\right)^2. \qquad (75)$$

As was mentioned above, in the case of a system of two particles with the same mass $m_1 = m_2$ such as the homogeneous (H_2 and I_2) diatomic molecules, $\Theta_{\mu\nu}^{(n)} = \Theta_{\mu\nu}$ and $\sigma_{\mu\nu}^{(n)} = \sigma_{\mu\nu}$. Finally, we can generalize the nonrelativistic global energy $E_{nr-nc}^{dfe}\left(n,\alpha,c,b,V_0,V_1,V_2,\Theta^c,\sigma^c,\chi^c,j,l,s,m\right)$ within the IDGDFDE-P model considering that composite systems with different masses are described with different noncommutative parameters for the diatomic HCl, CH, LiH, and CO as:

$$E_{nr-nc}^{dfe}\left(n,\alpha,c,b,V_0,V_1,V_2,\Theta^c,\sigma^c,\chi^c j,l,s,m\right) =$$
$$= E_{nl}^{nr} + \langle F\rangle_{(nlm)}^{NRdfe}\left(n,\alpha,c,b,V_0,V_1,V_2\right)\times$$
$$\times\left(\Theta^c k\left(j,l,s\right) + \sigma^c B + m\chi^c\Omega m\right). \qquad (76)$$

We now take a look at the special cases. This is done by adjusting the constants in the studied potential.

First: Modified Hulthén potential. If we make the choice $V_0 = V_2 = c = 0$ and $q = 1$, the perturbative potential created with the effect of the IDGDFDE-P model in NRNCQM symmetries reduces to the perturbative potential for the modified Hulthén potential as follows:

$$V_{nr-\mathrm{pert}}^{dfe}(s) = \left(\frac{\left(\alpha^4 l\,(l+1) - 2\alpha^2\left(E_{nl}+M\right)V_0b^2\right)s^4}{(1-s)^4} - \right.$$
$$\left. - \frac{V_1\alpha}{2}\left(\frac{s^2}{(1-s)^2} + \frac{s^3}{(1-s)^3}\right)\right)\mathbf{L}\Theta + O\left(\Theta^2\right). \qquad (77)$$

The new relativistic energy in Eq. (69) reduces to the new energy $E_{nr-nc}^{ep}\left(n,\alpha,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m\right)$ for the new modified Hulthén potential in NRNCQM, as follows:

$$E_{nr-nc}^{dfe}\left(n,\alpha,b,V_1,\Theta,\sigma,\chi,j,l,s,m\right) =$$
$$= -\frac{\alpha^2}{2M}\left[\frac{2MV_1}{\alpha^2\,(n+l+1)} - \frac{n+l+1}{2}\right]^2 +$$
$$+ \langle F\rangle_{(nlm)}^{NRHP}\left(n,\alpha,b,V_1\right)\left(\Theta k\left(j,l,s\right) + \sigma B + m\chi\Omega m\right). \qquad (78)$$

The corresponding global expectation value $\langle F\rangle^{NRHP}_{(nlm)}(n,\alpha,b,V_1)$ is determined from the following expression:

$$\langle F\rangle^{NRHP}_{(nlm)}(n,\alpha,b,V_1) = = \begin{pmatrix} \alpha^4 l(l+1) \\ -2\alpha^2(E_{nl}+M)V_0 b^2 \end{pmatrix} \left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(nlm)} - - \frac{V_1\alpha}{2}\left(\left\langle \frac{s^2}{(1-s)^2} \right\rangle_{(nlm)} + \left\langle \frac{s^3}{(1-q)^3} \right\rangle_{(nlm)}\right) \quad (79)$$

with $\sigma_q^h = \frac{1}{2}\left[1+\sqrt{1+\frac{8M}{\alpha^2}(V_2+l(l+1))}\right]$.

Second: If we make the choice $V_1 = V_2 = c = 0$ and $q = 1$, the perturbative potential created with the effect of the IDGDFDE-P model in NRNCQM symmetries reduces to the perturbative potential for the modified Manning–Rosen potential as follows:

$$V^{mrp}_{\text{pert}}(s) = \left(\frac{\left(\alpha^4 l(l+1) + 2\alpha^2(E_{nl}+M)V_0 b^2\right)s^4}{(1-s)^4} + + \frac{V_0 b^2\alpha s^3}{(1-s)^3}\right)\mathbf{L}\Theta + O\left(\Theta^2\right). \quad (80)$$

The new relativistic energy in Eq. (69) reduces to the new energy $E^{mrp}_{nr-nc}(n,\alpha,b,V_0,\Theta,\sigma,\chi,j,l,s,m)$ for a new modified Manning–Rosen potential in NRNCQM, as follows:

$$E^{mrp}_{nr-nc}(n,\alpha,b,V_0,\Theta,\sigma,\chi,j,l,s,m) = = -\frac{\alpha^2}{2M}\left[\frac{Mb^2V_0}{\alpha^2(n+\rho)} - \frac{n+\rho}{2}\right]^2 + + \langle F\rangle^{NRMRP}_{(nlm)}(n,\alpha,b,V_0)\times \times(\Theta k(j,l,s) + \sigma Bm + \chi\Omega m), \quad (81)$$

with $\rho = \frac{1}{2}\left[1+\sqrt{(2l+1)^2+8Mb^2V_0/\alpha^2}\right]$, and the corresponding global expectation value $\langle F\rangle^{NRMRP}_{(nlm)}(n,\alpha,b,V_0)$ is determined from the following expression:

$$\langle F\rangle^{NRMRP}_{(nlm)}(n,\alpha,b,V_0) = \left\langle \frac{s^2}{(1-s)^4} \right\rangle_{(nlm)} \times \times \begin{pmatrix} \alpha^4 l(l+1) \\ +2\alpha^2(E_{nl}+M)V_0 b^2 \end{pmatrix} + V_0 b^2\alpha\left\langle \frac{s^3}{(1-s)^3} \right\rangle_{(nlm)}. \quad (82)$$

The KGE is the most well-known relativistic wave equation describing spin-zero particles, but its extension in RNCQM symmetries, DKGE, under the improved deformed generalized Deng–Fan potential plus the deformed Eckart potential has a physical behavior similar to the Duffin–Kemmer equation for mesons with spin. It can describe the dynamic state of a particle with spin one in the symmetries of relativistic noncommutative quantum mechanics. This is one of the most important new results of this research. Worthwhile, it is better to mention that, for the two simultaneous limits (Θ,σ,χ) and $(\Theta^c,\sigma^c,\chi^c)\to(0,0,0)$, we recover the results in Refs. [12, 31].

6. Summary and Conclusions

In this work, we have found the approximate bound-state solutions of DRKGE and DNRSE using the tool of Bopp's shift and standard perturbation theory methods of the improved deformed generalized Deng–Fan potential plus the deformed Eckart potential in both relativistic and nonrelativistic regimes, which correspond to high- and low-energy physics. We have employed the improved approximation scheme to deal with the centrifugal term to obtain the new relativistic bound-state solutions $E^{dfe}_{r-nc}(n,\alpha,c,b,V_0,V_1,V_2,V_2,\Theta,\sigma,\chi,j,l,s,m)$ corresponding to the generalized n^{th} excited states that appear as a sum of the total energy shift $\Delta E^{\text{tot}}_{dfe}$ and the relativistic energy E_{nl} of the deformed generalized Deng–Fan potential plus the deformed Eckart potential. Furthermore, we have obtained the new nonrelativistic global energy $E^{dfe}_{nr-nc}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ of some diatomic molecules such as N_2, I_2, HCl, CH, LiH, and CO in the NRNCQM symmetries as a sum of the nonrelativistic energy [see Eq. (69)] and the perturbative corrections $\Delta E^{nr}_{dfe}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ [see Eq. (68)]. The new relativistic and nonrelativistic energies appear as a function of the discrete atomic quantum numbers (n,j,l,s,m), the potential parameters (α,c,b,V_0,V_1,V_2) in addition to three noncommutativity parameters (Θ,σ,χ). This behavior under study is similar to that of a physical system that is affected by three infinitesimal external influences in comparison to the main potential effect (generalized Deng–Fan potential plus the deformed Eckart potential). But in our case, these effects appear automatically as a result of the new deformation of the space-space which is presented in Eqs. (3) and (4). Moreover, we have applied our results to composite systems such as molecules made of $N = 2$ particles of masses m_n $(n = 1, 2)$. We have also dealt with some related special cases in relativistic and nonrelativistic cases. We have observed that the DRKGE under the improved deformed generalized Deng–Fan potential plus the deformed Eckart potential model becomes similar to the Duffin–Kemmer equation for mesons with spin s. It can describe a dynamic state of the particle with spin one in the symmetries of RNCQM. It is worth mentioning that, in all cases where we apply the two simultaneous limits (Θ, σ, χ) and $(\Theta^c, \sigma^c, \chi^c) \rightarrow (0, 0, 0)$, the ordinary physical quantities are recovered. Furthermore, our research findings could also be applied to atomic physics, vibrational and rotational spectroscopies, mass spectra, nuclear physics, and other applications. Finally, given the effectiveness of the methods that we followed in achieving our goal in this research, we advise researchers to apply the same methods to delve more deeply into other potentials in the relativistic or nonrelativistic regime.

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ДВОАТОМНІ МОЛЕКУЛИ З ПОКРАЩЕНИМ ДЕФОРМОВАНИМ УЗАГАЛЬНЕНИМ ПОТЕНЦІАЛОМ ДЕНГА–ФАНА У МОДЕЛІ З ДЕФОРМОВАНИМ ПОТЕНЦІАЛОМ ЕКАРТА З ВИКОРИСТАННЯМ РОЗВ'ЯЗКІВ МОДИФІКОВАНИХ РІВНЯНЬ КЛЯЙНА–ГОРДОНА ТА ШРЬОДІНГЕРА З СИМЕТРІЯМИ НЕКОМУТАТИВНОЇ КВАНТОВОЇ МОДЕЛІ

Знайдено розв'язки модифікованих рівнянь Кляйна–Гордона та Шрьодінгера з покращеним узагальненим деформованим потенціалом Денга–Фана і в моделі з деформованим потенціалом Екарта, використовуючи метод Боппа зі зсувом та теорію збурень з урахуванням симетрій узагальненої квантової механіки. Використано покращене наближення для відцентрового доданка. Розраховано релятивістичні та нерелятивістичні енергії зв'язаних станів деяких двоатомних молекул, таких як N_2, I_2, HCl, CH, LiH і CO. Релятивістичний зсув енергії $\Delta E^{\rm tot}_{dfe}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ та пертурбативні нерелятивістичні поправки $\Delta E^{nr}_{dfe}(n,\alpha,c,b,V_0,V_1,V_2,\Theta,\sigma,\chi,j,l,s,m)$ знайдено як функції параметрів (α,c,b,V_0,V_1,V_2), параметрів некомутативності (Θ,σ,χ) та атомних квантових чисел (n,j,l,s,m). В нерелятивістичному і релятивістичному випадках показано, що поправки для спектра енергій менші, ніж у звичайних релятивістичній та нерелятивістичній квантових механіках. Граничний перехід до звичайної квантової механіки демонструє узгодження з результатами інших робіт. Для нових симетрій некомутативної квантової механіки немає точних аналітичних розв'язків для $l=0$ та $l\neq 0$, і можна отримати тільки наближені розв'язки. Чітко показано, що рівняння Шрьодінгера та Кляйна–Гордона з новими симетріями фізично описують обидва рівняння Дірака та рівняння Дафіна–Кеммера.

Ключові слова: рівняння Кляйна–Гордона, рівняння Шрьодінгера, узагальнений деформований потенціал Денга–Фана, деформований потенціал Екарта, двоатомні молекули, некомутативна геометрія, метод зсуву Боппа, зірчасті добутки.