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ASYMPTOTIC BEHAVIOR OF BOSON REGGE TRAJECTORIES

The asymptotic behavior of boson Regge trajectories is studied. Upper and lower bounds on the asymptotic growth of the trajectories are obtained using the phase representation for the trajectories and a number of physical requirements. It is shown that, within the assumptions made, the asymptotic behavior of the trajectories is a square root.

Keywords: asymptotics, Regge trajectories.

This is a reprint from the eponymous paper of Aleksandr A. Trushevsky, printed in English as a Bogolyubov ITP Preprint, ITP-75-81E (1975) and published (in Russian) in the Ukrainian Journal of Physics, V. 22, No. 3, pp. 353–361 (1977). In spite of the limited circulation of the preprint and language barriers of the journal publication, the paper became widely known, requested and cited by experts. Its value now is even increasing due to the interest in the analytic S-matrix theory as a possible way to resolve problems of the non-perturbed quantum chromodynamics and quark confinement. Trushevsky's paper contains a clear, self-consistent, and up-to-date presentation of non-linear complex Regge trajectories, the basic object in the Regge-pole theory. Its topicality has not devalued in 44 years after its first publication.

Our aim is to make A. Trushevsky's paper available to a wide audience as well as to commemorate his 70-th birthday. During his short but brilliant scientific carrier at the Bogolyubov Institute for Theoretical Physics of the NAS of Ukraine, A. Trushevsky made several important discoveries of international level in various fields of high-energy physics and cosmology. In particular, from the S-matrix formulation of statistical mechanics, combined with the Regge-pole theory, he derived an original equation of state of the hot and dense nuclear matter, that subsequently, well before A.H. Guth and A. Linde, led him, to the discovery the inflating universe, resolving or relating various problems in cosmology of those times, such as the horizon problem, flatness and homogeneity of the universe, baryon asymmetry, CPT invariance, role of monopoles.

With the present publication, we make A. Trushevsky's 1977 paper available to those interested in high-energy strong interaction theory. We have no doubt that the paper will be useful for many physicists interested in the ideas and methods of the analytic S-matrix.

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1. Introduction

While elucidating the dynamics of strong interactions, it is very important to study the behavior of Regge trajectories $\alpha(s)$. Although the trajectory form is more or less known for small argument values, its asymptotic behavior remains unclear.

As a result of the development of dual narrow-resonance models, there exists a widespread opinion that the Regge trajectories are characterized by the linear asymptotic behavior. However, it should be noted that duality does not require such an asymptotics at all [1]. Furthermore, it turns out that, in order to agree various properties of dual analytic models with one another [2], the application of trajectories with the asymptotics $\alpha(s) = -\text{const }(-s)^{1/2}$ is required.

Recently, a simple analytic model of Regge trajectories has been proposed [3] that satisfies those restrictions and describes well the available data on resonances [4].

Notice also that the asymptotic behavior of trajectories is closely associated with whether the number of resonances is finite or not.

In the recent years, a number of works [5–26] were devoted to the consideration of the asymptotic behavior of the rising Regge trajectories. They can be conventionally classified into the following five groups [for convenience, the following asymptotic trajectory parametrization is used: $\alpha(s) = -\gamma(-s)^{\nu}$].

1) The papers in which their authors attempt to impose the restriction $\nu \leq 1$: Ida [5], Childers [11], and Fleming [9, 10]. A brief analysis of those papers is given in Section 3.

2) The papers devoted to the restriction on the growth of the real part of the trajectory: Khuri [12], Jones and Teplitz [14], and Childers [13]. In the framework of some assumptions, Khuri [12] concluded that $\operatorname{Re}\alpha(s) \nrightarrow +\infty$ at $s \to +\infty$. But it was shown in work [14] that Khuri's requirements [12] do not agree with the Mandelstam symmetry $T\,(l,s) = T\,(-l-1,s)$, where $T\,(l,s)$ are the partial amplitudes. Later, Childers [13] showed that, within the requirements of work [12], the asymptotic behavior $\operatorname{Re}\alpha(s) \to +\infty$ at $s \to +\infty$ is still possible, but there is the restriction $\operatorname{Re}\alpha(s) \leq \text{const}\sqrt{s}$ with an accuracy to an arbitrary power exponent of logarithm.

3) Nevertheless, most papers are devoted to substantiating the linear growth of trajectories [15–18, 21, 24, 26]. The cited authors proceed from the expression for the trajectory written in the form

$$\alpha(s) = a + bs + \frac{s}{\pi}\int\limits_{s_0}^{\infty} \frac{\operatorname{Im}\alpha(s)}{s'(s'-s)}\,ds'$$

or represent the trajectory in the asymptotic form [15, 16]

$$\alpha(s) \approx s\ln^{\mu}(-s)$$

and either prove it by making use of somewhat formal, in our opinion, assumptions (for instance, the requirement of homogeneity for the limits [15, 16]) or use the assumption that $\operatorname{Re}\alpha(s) \sim s$ and construct integral equations for $\alpha(s)$ [18].

4) The works evaluating the trajectory asymptotics from various model concepts [19, 20, 22, 23]. It is interesting to note that, sometimes, the model constructions do not agree with the possibility of writting the trajectory at $|s| \to \infty$ in the form $\alpha(s) = -\gamma(-s)^{\nu}$. For instance, in work [19], a trajectory was constructed for $J \sim M_J \sim \Gamma_J$ at $|s| \to \infty$, where J, Γ_J, and M_J are the spin, width, and mass, respectively, of the resonance. However, in Section 4, it will be shown that $J \nsim M_J$ at $\nu > \frac{1}{2}$, and $\Gamma_J \nsim M_J$ at $\nu = \frac{1}{2}$.

5) This group of works includes Fleming's ones [6–8, 25] in which the conditions $\frac{1}{2} < \nu < 1$, $\frac{3}{2} < \nu < 2$, and so forth were obtained following the requirement that the resonance width is positive. However, for this purpose, the formula

$$\Gamma = \frac{\operatorname{Im}\alpha(M^2)}{M\ \operatorname{Re}\alpha'(M^2)}$$

was used, which is valid only if $\Gamma/M \to 0$, i.e., as is shown in Section 4, only at $\nu = 1$.

In this work, we analyze asymptotic constraints on the behavior of trajectories proceeding from the general concepts adopted in high energy physics

In Section 2, a convenient form for writing the trajectory in the asymptotic region is introduced. In Section 3, an upper bound for the behavior of trajectories is obtained, which is then used to obtain more rigorous constraints. In Sections 4 and 5, the lower and upper bounds for the asymptotic behavior of the trajectory are obtained, which fix the root asymptotics of the latter.

2. Asymptotic Representation of Trajectory

We will consider the boson trajectories assuming that

1) $\alpha(s)$ is an analytic function possessing only a physical cut from $s = s_0 = 4$ to ∞;

2) $\alpha(s)$ is polynomially bounded on the whole physical sheet;

3) there is a finite phase limit of the trajectory at $s \to +\infty$.

Let us demonstrate that, under the above assumptions, the trajectory at $|s| \to \infty$ can be considered in the form

$$\alpha(s) = -\gamma(-s)^{\nu}, \tag{1}$$

where $\gamma = \text{const} > 0$, on the whole physical sheet with an accuracy to the function increasing slower than any power exponent, with the phase of this function approaching zero at $|s| \to \infty$. Note that the signs in Eq. (1) are the only correct option, because $\alpha(s)$ is real at $s \to -\infty$ and satisfies the bound $\alpha(s \le 0) \le 1$, which follows from the Froissart theorem.

Following the method of works [9, 20], let us construct the entire function

$$F(s) = [\alpha(s) - N]\, e^{-h(s)}, \tag{2}$$

where

$$N > \alpha(4), \quad h(s) = \frac{s}{\pi} \int\limits_4^{\infty} ds' \frac{\psi(s')}{s'\,(s'-s)}, \tag{3}$$

$$\alpha(s) - N = [\alpha(s) - N]\, e^{i\psi(s)}. \tag{4}$$

According to Picard's little theorem, the entire function $F(s)$ can be written in the form

$$F(s) = P(s) e^{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials. With the help of Eq. (2), it is easy to find that $Q(s) = \text{const}$, i.e. $F(s)$ is polynomially bounded. Really, $\alpha(s)$ is polynomially bounded and, as was shown in works [11, 27],

$$e^{-h(s)} = \text{const} \times (-\text{s})^{-\frac{\psi(\infty)}{\pi}} e^{\lambda(s)} \text{ at } |s| \to \infty, \tag{5}$$

where $|\lambda(s)| < \delta \ln |s|$for any $\delta > 0$. Therefore,

$$\alpha(s) = P(s) e^{h(s)} + N. \tag{6}$$

From Eqs. (5) and (6), we obtain

$$\alpha(s) = -\gamma(-s)^{\nu} f(s) \text{ at } |s| \to \infty, \tag{7}$$

where $f(s)$ increases or decreases at $|s| \to \infty$ slower than any power exponent of s.

For the trajectory asymptotics to be completely determined by the factor $(-s)^{\nu}$, it is necessary that $\arg f(s) \to 0$ in expression (7) at $|s| \to \infty$. According to Eqs. (3) and (5)–(7),

$$\arg f(s) = \operatorname{Im} \lambda(s),$$

where

$$\lambda(s) = \int\limits_4^{\infty} ds' \frac{g(s')}{s'\,(s'-s)}, \tag{8}$$

$$g(s) = \psi(s) - \psi(\infty).$$

If $s \to \infty$ along the real axis, we obtain $\operatorname{Im} \lambda(s) = g(s) \to 0$. If $|s| \to \infty$ along a certain ray, then

$$\operatorname{Im} \lambda(s) = \frac{1}{\pi} \int\limits_{u_0}^{\infty} du \frac{g(u\,|\operatorname{Im} s| + \operatorname{Re} s)}{u^2+1} = \frac{1}{\pi} \left(\int\limits_{u_0}^{u_0+\delta} + \int\limits_{u_0+\delta}^{\infty} \right) du \frac{g(u\,|\operatorname{Im} s| + \operatorname{Re} s)}{u^2+1}, \tag{9}$$

where $u_0 = (4 - \operatorname{Re} s)/\,|\operatorname{Im} s|$, and $\delta > 0$.

According to the mean-value theorem, the first integral in Eq. (9) does not exceed $\text{const} \times \delta$. The argument of the function g in the second integral is larger than or equal to $4 + \delta\,|\operatorname{Im} s|$, i.e. it grows with the increase of $|s|$. As a result, both the integrand and the second integral are nondecreasing functions at $|s| \to \infty$. Hence, $\operatorname{Im} \lambda(s) \le \text{const} \times \delta$ at $|s| \to \infty$ for any $\delta > 0$, i.e. this quantity decreases along any ray in the s-plane. Therefore, while considering the asymptotic behavior of the Regge trajectory on a physical sheet, it is enough to confine the analysis to expression (1).

3. Upper Bound for the Trajectory Growth

This bound is dealt with in works [5, 9–11]. The author of work [5] expressed the condition $\nu \le 1$ in terms of the asymptotic behavior of the real and imaginary parts of the trajectory. The essence of the proofs presented in works [9–11] was expounded the most clearly in the last Fleming's work 24. It consists in that, according to Eq. (1), the trajectory phase acquires the values

$$\psi(-\infty) = \pi, \quad \psi(+\infty) = \pi(1-\nu)$$

at $s \to \pm\infty$. Demanding that $\operatorname{Im} \alpha(s)$ be positive at $s > 4$, Fleming assumes that $\psi(+\infty) > 0$, i.e. $\nu \le 1$. However, $\operatorname{Im} \alpha(s)$ is also positive at

$$2k \le \nu \le 1 + 2k \quad (k = 0, 1, 2, ...), \tag{10}$$

i.e. when the trajectory (as s passes from negative asymptotic values to positive ones) executes several rotations in the α-plane when returning to the upper half-plane. Fleming rejects the cases with $k \neq 0$ in Eq. (10) by claiming that the trajectory cannot circumvent zero in the α-plane. In our opinion, this assertion is somewhat formal.

Proceeding from the analyticity principles, according to which the amplitude has no singularities on the physical sheet of the s-plane, except for those following from unitarity, let us demonstrate that the trajectory cannot grow faster than a linear one. Further, it can be proved that, from the requirement that there are no poles in the amplitude $T(s,t)$ on the physical sheet, it follows that they are also absent from the partial amplitude $T(1,s)$ $(1 = 0,1,2,...)$.

Hence, if we exclude the case of imaginary trajectories from consideration, then the trajectory should not take positive integer values of proper signature on the physical sheet. Let us require the fulfillment of this condition at asymptotic s-values. For this purpose, let us present Eq. (1) in the form

$$\alpha(s) = \gamma\,|s|^{\nu}\, e^{i[\pi-(\pi-\varphi)\nu]}, \tag{11}$$

where φ is the s-phase equal to zero at the upper cut edge, and $\alpha(s)$ can acquire positive integer values on the ray defined by the condition

$$\pi - (\pi - \varphi)\nu = 0, \tag{12}$$

i.e. at

$$\varphi = \pi\left(1 - \frac{1}{\nu}\right). \tag{13}$$

The requirement that the corresponding ray should leave the physical sheet brings us to the condition

$$\nu < 1. \tag{14}$$

However, it is also possible to realize the case $\nu = 1$, if we take into account the function $f(s)$ in Eq. (7). Although the phase $f(s)$ tends to zero at $|s| \to \infty$, it nevertheless can differ from zero at any fixed s. As a result, the resonances become shifted to the unphysical sheet. Thus, the restriction can be written in the form

$$\nu \le 1. \tag{15}$$

Let us analyze some particular expressions for $f(s)$ at $\nu = 1$.

Case 1:

$$\alpha(s) = \gamma s \ln^{\mu}(-s) =$$
$$= \gamma\,|s| \ln^{\mu}|s| \exp\left\{i\varphi - \frac{i(\pi-\varphi)\mu}{\ln|s|}\right\}.$$

The poles in the asymptotic region lie at $\varphi = \pi\mu/\ln|s|$. From the requirement $\varphi < 0$, it follows that $\mu < 0$.

Case 2:

$$\alpha(s) = \gamma\,[s + a(-s)^{p}],$$

where $0 \le p < 1$. Then, from

$$\varphi \approx \frac{a \sin \pi p}{|s|^{1-p}},$$

it follows that $a < 0$.

Case 3:

$$\alpha(s) = \gamma\,[s + b \ln^{\mu}(-s)].$$

Analogously to the previous cases, we can obtain that

$$b\mu < 0.$$

In all those cases, it is possible to determine the ratio between the width and the mass of resonances at $M \to \infty$. For example, in case 1,

$$\frac{\Gamma}{M} = |\varphi| = \frac{\pi\,|\mu|}{\ln|s|} \sim \frac{1}{\ln M}.$$

4. Lower Bound for the Asymptotic Trajectory Behavior

Let us pass to the energy plane,

$$E = \sqrt{s} = |E|\, e^{i\varphi/2}.$$

We may assume that resonances at $E = M - \frac{i\Gamma}{2}$ are characterized by the wave function

$$\Psi(t) \sim e^{-i\left(M - \frac{i\Gamma}{2}\right)t}.$$

It is natural to impose the physical condition $M \ge 0$, i.e. the line $\operatorname{Im}\alpha(s) = 0$ in the E-plane must be in the right quadrant of the lower half-plane of the (nonphysical) E-sheet. From $\operatorname{Im}\alpha(s) = 0$, it follows that

$$\frac{\varphi}{2} = \frac{\pi}{2}\left(1 - \frac{1}{\nu}\right). \tag{16}$$

The condition $M \geq 0$ requires that $\frac{\varphi}{2} \geq -\frac{\pi}{2}$, i.e.

$$\nu \geq \frac{1}{2}. \tag{17}$$

The case $\nu = \frac{1}{2}$ is limiting. Here, it is of interest to obtain restrictions on various types of the function $f(s)$ in expression (7).

Case 1:

$$\alpha(s) = -\gamma(-s)^{\frac{1}{2}} \ln^{\mu}(-s).$$

From $M \geq 0$, it follows that

$$\mu \geq 0. \tag{18}$$

Case 2: If

$$\alpha(s) = \gamma \left[-(-s)^{\frac{1}{2}} + a(-s)^p\right],$$

where $0 \leq p < \frac{1}{2}$, it is necessary that

$$a \geq 0. \tag{19}$$

Case 3: If

$$\alpha(s) = \gamma \left[-(-s)^{\frac{1}{2}} + b \ln^q(-s)\right],$$

we have the restriction

$$bq \geq 0. \tag{20}$$

With the help of Eq. (16), it is easy to find that, at $\frac{1}{2} < \nu < 1$, the asymptotic ratio between the widths and masses of resonances is constant:

$$\frac{\Gamma}{M} = 2 \tan \left[\frac{\pi}{2}\left(1 - \frac{1}{\nu}\right)\right].$$

The ratio $\frac{\Gamma}{M} \to 0$ at $\nu = 1$, and $\frac{\Gamma}{M} \to \infty$ at $\nu = \frac{1}{2}$. The character of tending to zero or infinity in those cases is determined by the specific expression for $f(s)$.

Note that the spin of resonances is often written as follows:

$$J_M = \operatorname{Re} \alpha(M^2). \tag{21}$$

However, this formula is correct, only if $\nu = 1$. If $\frac{1}{2} < \nu < 1$, then

$$J_M = \frac{-\operatorname{Re} \alpha(M^2)}{\cos \pi\nu \left[\cos \frac{\pi}{2}\left(\frac{1}{\nu} - 1\right)\right]^{2\nu}}.$$

But if $\nu = \frac{1}{2}$, the resonances are close to the imaginary axis so that Eq. (21) becomes senseless.

In conclusion, it is interesting to compare the results of this Section with those of Fleming [6–8, 21] and Sivers [24]. The cited authors, using the expression

$$\Gamma = \frac{\operatorname{Im} \alpha(s)}{\sqrt{s} \operatorname{Re} \alpha(s)} \tag{22}$$

for the resonance width and demanding that $\Gamma < 0$, obtained the restrictions $\frac{1}{2} < \nu < 1$, $\frac{3}{2} < \nu < 2$, and so forth. However, expression (22) is valid only at $\Gamma/M \to 0$. In the framework of our approach, from the condition $\Gamma > 0$, i.e. $\frac{\varphi}{2} > -\pi$, it follows that $\nu > \frac{1}{3}$.

5. Bounds for the Growth of the Real Part of the Trajectory Along the Physical Axis

Childers in work [13], where Khuri's results [12] were specified, showed that

$$\operatorname{Re} \alpha(s \to +\infty) \leq \gamma \sqrt{s} \ln^{\mu} s, \tag{23}$$

where μ is an arbitrarily large constant. Below, we will strengthen result (23) and show that a number of requirements lead to the restriction

$$\operatorname{Re} \alpha(s) \leq \gamma \ln s.$$

Some considerations presented below bring us to the conclusion that

$$\operatorname{Re} \alpha(s \to +\infty) \nrightarrow +\infty.$$

Therefore, combining those results with the results of Sections 3 and 4, we come to the conclusion that the asymptotic behavior of the trajectory has a purely root character.

1) Let us make the following assumptions:

(a) the amplitude $T(s, t)$ is bounded by a polynomial in s at fixed non-physical $t > 0$;

(b) as s grows along the real positive semiaxis, the residue of the partial amplitude $T(j, s) - \beta(s)$ decreases not faster than a certain power of s;

(c) $\alpha(s)$ is the trajectory of the rightmost singularity in the j-plane;

(d) the partial amplitude $T(j, s)$ on the Sommerfeld–Watson contour is bounded by a polynomial in s as $s \to +\infty$;

(e) the background integral along a vertical line in the j-plane converges (it can be shown rigorously, only if the expression for t contains a complex addition).

Let us consider the amplitude analytically continued from the s-channel in the form of the contour integral

$$T(s,t) = \sum_{\sigma=\pm1} \int_{a-i\infty}^{a+i\infty} \frac{df}{2\pi i}\pi\left(j+\frac{1}{2}\right)\times$$
$$\times\,\eta_\sigma(j)T(j,s)P_j\left(1-\frac{2t}{s-4}\right),$$

where
$$\eta_\sigma(j) = \frac{\sigma + e^{-i\pi j}}{-\sin\pi j}$$

is the signature factor. Let $\operatorname{Re}\alpha(s)\to+\infty$ at $s\to\to+\infty$. Then, the j-pole intersects the contour, and its contribution can be determined:

$$T(s,t) = T_n + T_\Delta + T_b,$$

where
$$T_n = 2\pi\left[\alpha(s)+\frac{1}{2}\right]\left[\frac{\sigma+e^{-i\pi\alpha(s)}}{e^{-i\pi\alpha(s)}-e^{i\pi\alpha(s)}}\right]\times$$
$$\times\,\beta(s)P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right),$$

T_b is the contribution of the background integral, and T_Δ is the contribution of other features located to the left of the leading singularity.

In the framework of assumption (b), T_b is bounded as a polynomial in s as $s\to+\infty$. Furthermore, T_Δ makes a smaller contribution to the asymptotics than T_n does. Therefore, for the polynomial boundedness of the amplitude $T(s,t)$ in s, it is sufficient to require the polynomial boundedness of T_n.

At $s\to\infty$,

$$P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\approx$$
$$\approx\left(\frac{\sqrt{s}}{4\pi\alpha(s)\sqrt{t}}\right)^{\frac{1}{2}}\exp\left(\frac{2\sqrt{t}\alpha(s)}{\sqrt{s}}\right).$$

So, for the polynomial boundedness of $T(s,t)$, it is necessary that

$$\operatorname{Re}\alpha(s)\le\gamma\sqrt{s}\ln s. \quad (24)$$

From Eq. (24), taking Eqs. (1) and (15) into account, we obtain the restriction $\nu\le\frac{1}{2}$. But if

$$\alpha(s) = -\gamma(-s)^{\frac{1}{2}}\ln^\mu(-s),$$

it is necessary that $\mu\le 2$.

2) Let us impose a more strong condition and require the polynomial boundedness for the amplitude $T(s,z)$ with respect to s at fixed non-physical $z>2$. The assumptions are the same as assumptions (b), (c), (d), and (e) made in item 1). Then, it is again sufficient to require the polynomial boundedness for the pole contribution. Since

$$P_{\alpha(s)}(z)\sim\frac{e^{\alpha(s)\xi}}{\sqrt{\alpha(s)}}\ \text{at}\ |\alpha(s)|\to\infty,$$

where
$$\xi = \ln\left(z+\sqrt{z^2+1}\right),$$

then
$$T_n\sim\frac{\beta(s)}{\sqrt{\alpha(s)}}\,\alpha(s)e^{\alpha(s)\xi}\ \text{at}\ s\to+\infty.$$

For the polynomial boundedness of the amplitude, it is necessary that

$$\operatorname{Re}\alpha(s)\le\gamma\ln s. \quad (25)$$

This restriction is evidently not associated with a specific asymptotic representation of the trajectory in form (1). However, with its help and taking into account the results of Sections 3 and 4, it is easy to see that Eq. (25) leads to the equality

$$\nu = \frac{1}{2}. \quad (26)$$

Now, it is easy to obtain restrictions on the function $f(s)$ that specifies the $\alpha(s)$-asymptotics.

(a) Let

$$\alpha(s) = -\gamma(-s)^{\frac{1}{2}}\ln^\mu(-s).$$

Then,

$$\operatorname{Re}\alpha(s) = \gamma\pi\mu\,|s|^{\frac{1}{2}}\ln^{\mu-1}|s|\,.$$

From Eq. (25), it follows that $\mu\le 0$. Taking Eq. (18) into account, we obtain that $\mu = 0$.

(b) If

$$\alpha(s) = \gamma \left[-(-s)^{\frac{1}{2}} + a(-s)^p \right],$$

where $0 \le p \, \frac{1}{2}$ and $a = \text{const}$, then from Eq. (25) and (19), we obtain the condition $a = 0$.

(c) Finally, if

$$\alpha(s) = \gamma \left[-(-s)^{\frac{1}{2}} + b \ln^q(-s) \right],$$

where $q > 0$, then from Eq. (25) and (20), we obtain that $q \le 1$.

Hence, the requirement of the polynomial boundedness for the amplitude at fixed non-physical $z > 1$ together with the results of Sections 3 and 4 gives rise to a purely root asymptotic behavior of the trajectory to within the addition of a logarithm raised to a power exponent not exceeding unity,

$$\alpha(s) = \gamma \left[-(-s)^{\frac{1}{2}} + b \ln^q(-s) \right], \quad q \le 1. \tag{27}$$

3) In this item, we draw attention to the assumptions that can restrict the growth of the real part of the trajectory even more strongly, namely, $\mathrm{Re}\, \alpha(s) \nrightarrow +\infty$ at $s \to +\infty$ (although such assumptions cannot be convincingly substantiated).

Let us analyze the results of work [28]. The cited authors considered the amplitude representation in the form of an integral in the j-plane along a vertical line and a sum of the contributions made by the Regge poles located to the right of the integration contour. While studying the behavior of the amplitude in the region $(s > 4, t > 4)$, the continuation of the amplitude from the s-channel was used in one case, and from the t-channel in the other case. The incompatibility of the asymptotics of pole terms in those two representations of the amplitude was demonstrated, i.e., the requirement of crossing symmetry leads to the incompatibility of two assumptions:

(a) $\mathrm{Re}\, \alpha(s) \to +\infty$ at $s \to +\infty$;

(b) there is a certain fixed position of the vertical contour of integration in the Sommerfeld–Watson representation for which this integral gives an asymptotically smaller contribution than the poles located to the right of the contour.

Whence, it was concluded that assumption (b) was not valid and, as a consequence, there are no restrictions on the asymptotic growth of Regge trajectories.

However, the authors of work [28] proceeded from the narrow-resonance approximation and assumed that $\mathrm{Re}\, \alpha(s) \to +\infty$ at $s \to +\infty$ even in the case $\nu \le \frac{1}{2}$. But, as follows from our consideration, at $\nu < \frac{1}{2}$, as well as at $\nu = \frac{1}{2}$ and some additional conditions for the function $f(s)$, $\mathrm{Re}\, \alpha(s) \nrightarrow +\infty$ at $s \to +\infty$. Therefore, in those cases, the Regge poles do not go to the right of a certain vertical line in the j-plane. This fact testifies, in our opinion, that, instead of the negative conclusions of work [28], more preferable is the conclusion that

$$\mathrm{Re}\, \alpha(s) \nrightarrow +\infty \text{ at } s \to +\infty. \tag{28}$$

An additional argument in favor of the latter is the fact that, in this case, the amplitude can satisfy the Mandelstam representation with a finite number of subtractions.

Note that if Eq. (28) is satisfied, we have to put $q = 0$ in expression (27) (taking into account Eq. (20)) so that

$$\alpha(s) = -\gamma(-s)^{\frac{1}{2}} + \text{const}$$

in the asymptotic region.

Discussion of the Results

Thus, we came to the conclusion that the main asymptotic term of the trajectory has a purely root character. It corresponds to the fact that the resonances become infinitely wide and move away from the real axis in the asymptotic region. Therefore, one should better speak about a finite number of resonances in the corresponding amplitude.

It is interesting to note that result (28) can be interpreted in the framework of potential scattering. In particular, as was found in work [29], there is the trajectory $\alpha(s) = -\gamma(-s)^{\frac{1}{2}}$ at $s \to \infty$ for the Yukawa potential $V(r) = g^2 e^{-\lambda r}/r$, if the constant g is formally increased with the growth of energy provided that $\sqrt{s}/\left(\lambda g^2\right) = \text{const}$. A purely root asymptotics is obtained, if $g \sim s^{\frac{1}{4}}$.

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1. A.I. Bugrij, J. Cohen-Tannoudji, L.L. Jenkovszky, N.A. Kobylinsky. Dual amplitudes with Mandelstam analyticity. *Fortschr. Phys.* **21**, 427 (1973).

2. L.L. Jenkovszky. Dual properties of DAMA. *Yad. Fiz.* **21**, 645 (1975) (in Russian).
3. N.A. Kobylinsky. Properties of ρ-trajectories. *Ukr. Fiz. Zh.* **20**, 163 (1975) (in Russian); A.I. Bugrij, N.A. Kobylinsky. Are the ρ- and A_2-trajectories linear? *Ann. Phys.* **32**, 297 (1975); A.I. Bugrij, N.A. Kobylinsky. *Simple analytical model of boson Regge trajectories. Preprint ITF-74-53P* (Institute for Theoretical Physics, Kyiv, 1974) (in Russian).
4. T.A. Lasinski, A. Barbaro-Galtieri, R.L. Kelly *et al.* Review of particle properties. *Rev. Mod. Phys.* **45**, No. 2, part II, 1 (1973).
5. M. Ida. Asymptotic behaviour of infinitely rising meson trajectories. *Prog. Theor. Phys.* **40**, 901 (1968).
6. H. Fleming, T. Sawada. General results on the asymptotic behaviour of Regge trajectories. *Lett. Nuovo Cim.* **1**, 1045 (1971).
7. H. Fleming, E. Predazzi. Some speculations on the Pomeranchuk trajectory. *Lett. Nuovo Cim.* **4**, 556 (1970).
8. H. Fleming. Asymptotic behaviour of Regge trajectories and width. *Lett. Nuovo Cim.* **3**, 363 (1972).
9. H. Fleming, A.M. Filho. Upper bounds for the asymptotic behaviour of Regge trajectories. *Nuovo Cim.* **14A**, 215 (1973).
10. H. Fleming. Maximum behaviour of Regge trajectories at infinity. *Phys. Rev. D* **8**, 1256 (1973).
11. R.W. Childers. Second-sheet poles and maximum behaviour of Regge trajectories at infinity. *Phys. Rev. D* **2**, 1178 (1970).
12. N.N. Khuri. Possibility of an infinite sequence of Regge recurrences. *Phys. Rev. Lett.* **18**, 1094 (1967).
13. R.W. Childers. Asymptotic behaviour of infinitely rising trajectories. *Phys. Rev. Lett.* 21, 868 (1968).
14. C.E. Jones, V.L. Teplitz. Investigation of the hypotheses of Khuri's theorem of Regge-pole asymptotes. *Phys. Rev. Lett.* **19**, 185 (1967).
15. T. Becherrawy. Indefinitely rising trajectories. *Lett. Nuovo Cim.* **7**, 867 (1973).
16. D. Atkinson, K. Dietz. Indefinitely rising Regge trajectories and crossing symmetry. *Phys. Rev.* **177**, 2579 (1969).
17. S. Mandelstam. Rising Regge trajectories and dynamical calculations. *Comm. Nucl. Part. Phys.* **3**, No. 3, 65 (1969).
18. G. Epstein, P. Kaus. Rising meson trajectories. *Phys. Rev.* **166**, 1633 (1968).
19. R.L. Ingraham. Connection of "parabolic" mass and width trajectories with high-energy scattering. *Phys. Rev. D* **6**, 329 (1972).
20. J.C. Botke. Infinitely rising trajectories and existence of the Mandelstam representation. *Nucl. Phys. B* **40**, 141 (1972).
21. H. Meldner. On infinitely rising Regge trajectories. *Nuovo Cim.* **51A**, 882 (1967).
22. R.C. Brower, J. Harter. Kinematic constraints for infinitely rising Regge trajectories. *Phys. Rev.* **164**, 1841 (1967).
23. N.G. Antoniou, C.G. Georgalas, C.B. Kouris. On the decay of heavy Regge recurrences. *Nuovo Cim.* **16A**, 135 (1973).
24. D. Sivers. Characteristics of a Regge trajectory with a finite asymptotic phase. *Phys. Rev. D* **3**, 2275 (1971).
25. H. Fleming. Asymptotic behaviour of fermionic trajectories. *Lett. Nuovo Cim.* **7**, 209 (1973).
26. D.V. Shirkov. Properties of the Regge pole trajectories. *Usp. Fiz. Nauk* **102**, 87 (1970) (in Russian).
27. M. Sugawara, A. Tubis. Phase representation of analytic function. *Phys. Rev.* **130**, 2127 (1963).
28. S. Caser, J.M. Kaplan, R. Omnes. Rising Regge trajectories and symmetry. *Nuovo Cim.* **59A**, 571 (1969).
29. H. Cheng, T.S. Wu. Regge poles for large coupling constants. *Phys. Rev. D* **5**, 3189 (1972).

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АСИМПТОТИЧНА ПОВЕДІНКА
БОЗОННИХ ТРАЄКТОРІЙ РЕДЖЕ

Вивчено асимптотичну поведінку бозонних траєкторій Редже. Верхня та нижня межі асимптотичного зростання траєкторій отримуються з використанням фазового представлення для траєкторій та певних фізичних вимог. Показано, що в межах зроблених припущень асимптотична поведінка траєкторій є суто кореневою.

Ключові слова: асимптотика, траєкторії Редже.